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Aristotle's Philosophical Principles of Mathematics

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The definition of mathematics as the science which investigates the properties of quantities was first conceived by the ancient Greeks and was formulated later both philosophically and scientifically by Aristotle in his various works. Later mathematicians accepted this definition tacitly or by habit, for philosophical problems about mathematics as a whole were not much of a concern to them nor was the field of mathematics far advanced in research to suggest alternative definitions. Early last century Gauss reaffirmed this definition in his treatise, The Foundations of Mathematics.

From the latter part of last century until today, however, there has been a tendency away from this definition and in the direction of what was thought to be a better and more general definition; and in the opinion of most modern mathematicians and philosophers the old definition is too limited to cover modern mathematical research. The introduction of the so-called "non-Euclidean geometries" and of transfinite numbers, too, contributed somewhat to this tendency; for, it was thought, if the parallel postulate did not possess the absolute truth which was once attributed to it, it could have only hypothetical truth, if any truth at all, and today we find many if not most mathematicians and even philosophers taking the position that mathematics is not interested in the truth or falsity of its principles but only in consistent sets of postulates and the deduction of theorems from those postulates.

As a consequence of such thinking there arose a number of new definitions which give the appearance of being general enough to include all actual and perhaps possible mathematical research. Peirce regarded mathematics as the science which draws necessary conclusions, Russell identified mathematics with logic, Hilbert emphasized the symbolic nature of mathematics, and others posited such wide concepts for mathematical objects as order, intuition, relations, and the like.

Now a fair criticism and evaluation of the old definition presupposes an understanding of the terms in that definition and the principles according to which the formulation was made. Unfortunately, however, the critics failed on both counts; for they knew neither the meaning of those terms nor the principles according to which the formulation was made.

First, I shall present Aristotle's definition of mathematics and the principles which he uses in formulating it; second, I shall discuss some definitions which have been given lately and their difficulties; also, ~~and third,~~ I shall show that Aristotle's definition best fits modern mathematical research.

> DEFINITION: Mathematics is the science which investigates generically, specifically, and analogically the properties of quantities and whatever belongs to quantities.

Aristotle did not give expressly this definition, but it can be gathered from what he says in his various works. Whether he wrote a work on mathematics or not is not known. Now the key terms in this definition are "science," "quantity," "generically," "specifically," "analogically," "belonging," and "property."

First, let us turn to the term "science". It has two senses for Aristotle. Its main and narrow sense is: necessary knowledge of what exists through its cause. The other sense includes the principles and the logical proofs of theorems from the principles. So the definition of science in this sense would be: universal knowledge of principles and demonstrations of properties from those principles under one genus of existing things or under one aim; and the definition of a theorem would be: a demonstrated statement which signifies an attribute as belonging to a subject through the cause. In the case of mathematics, quantity is the genus.

Now the principles in a science are four in kind. They are (1) the indefinable concepts, (2) the definitions, (3) the hypotheses, and (4) the axioms. The premises come from the definitions and the hypotheses; and as for the axioms, they are not premises but what some moderns call "directive" or "regulatory" principles which are used to demonstrate conclusions from premises.

Second, the meaning of the term "quantity" is clear to those who have read carefully Aristotle's Categories and Book Delta of the Metaphysics. 'Quantity' is a category, and its two immediate species are 'number' and 'magnitude', and by "number" Aristotle means what nowadays call "a natural number which is greater than 2". Bertrand Russell, whose ignorance of the history of mathematics surpasses even his ignorance of Aristotle's logic, chooses to reject the ancient definition in his Introduction to Mathematical Philosophy. I quote:

> "It used to be said that mathematics is the science of 'quantity'. 'Quantity' is a vague word, but for the sake of argument we may replace it by the word 'number'. The statement that mathematics is the science of number would be untrue in two different ways. On the one hand, there are recognized branches of mathematics which have nothing to do with number - all geometry that does not use coordinates or measurement, for example; projective geometry and descriptive geometry, down to the point at which coordinates are introduced, does not have to do with number, or even with quantity in the sense of greater and less ... "

In this passage, Mr. Russell considers the word "quantity" as vague; and this shows that he did not do his homework. Again, he replaces the word "quantity" by the word "number" and then tries to refute the ancient definition; and this is like setting up a straw man and then knocking him down. Again, the expression "mathematics is the science of quantity" was given as a definition and not as a fact, but truth and falsity do not apply to definitions; yet he regards the expression as being untrue, that is, as false. Again, he is unaware of the fact that the relations of greater and less apply even to magnitudes in geometry which does not use coordinates, as in the case of plane geometry. There are other errors, but these should be enough for a logician.

Third, let us turn to the term "property". It means an attribute which is not the essence of a subject but belongs to that subject and to no other subject. The reason why the definition of mathematics uses the term "property" instead of the term "attribute" is that the demonstration of a property shows the cause and uses no extraneous information, whereas a demonstration of an attribute may use extraneous information and so fail to use only the cause. For example, the concurrence of the medians is demonstrated as a property of a triangle because it does not belong to a genus higher than the triangle, but it is demonstrated as an attribute of a right isosceles triangle because it belongs to other kinds of triangles also. In the latter demonstration, the right angle and the equality of two sides are irrelevant in proving the concurrence, but in the former demonstration the definition of a triangle is both ~~they are~~ necessary and sufficient. Moreover, the first demonstration is most general and has the widest applications, but the second demonstration is limited to one kind of triangles only. To take another example, the reason why the function x^2 is integrable between the limits, say, $x = 2$ and $x = 4$ is not because it is continuous, for some discontinuous functions too are integrable, but because all the points of discontinuity have a zero measure. Evidently, then, Aristotle's insistence on properties includes what mathematicians call "necessary and sufficient conditions." But Aristotle imposes a further condition, namely, that there is a difference between an essence and a property, and that one should not define a quantitative object in terms of a property.

Fourth, the expression "whatever belongs to quantities" is included in the definition of mathematics because properties under the genus of quantity are not limited to quantities as subjects only but extend to attributes as subjects also. Thus a point is not a quantity but a limit of a magnitude, and similarly for betweenness, separation, equality, and some other attributes of quantities, and these attributes can become subjects with properties, as in the case of the properties of equality, of an equation, of correspondence and the like.

Fifth, the properties of a genus are not properties of any species of the genus, and the properties of a species are not properties of any genus of it; so in investigating generic properties one does not investigate any specific properties, and conversely. The same applies to analogical properties relative to generic or specific properties. For example, the number system has analogical properties; for here the unit taken as a principle may be any magnitude, such as a line or a surface or a volume or an angle, and what we call "the origin" or "zero" serves as an initial principle if a magnitude is considered without direction, but as an initial principle of direction if a magnitude is considered with two directions.

The principles above with other details are sufficient to set up analytic geometry in three spacial dimensions, for there are three such dimensions, each of which has two directions relative to a principle of direction, and what is now called the "origin" is what Aristotle would call the initial principle of direction for all six directions. Further, alalytic geometry would be considered by Aristotle as a mixed science, with magnitude as the subject but with divisible numbers as attributes, for the unit in this geometry, unlike the unit in the theory of natural numbers, is infinitely divisible and not indivisible. Calculus would be a brance of mathematics whose subject is studied analogically, but we are omitting the details here.

As for Russell's assertion that quantity does not apply to projective and descriptive geometries, it is false, as we have indicated, for both those geometries have magnitudes as subjects, and magnitudes are quantities as already stated. Topology, too, would come under geometry for the same reason, and so would algebra; but the foundations of these have to be laid down not in the hypothetical manner in which it is done nowadays, but in a manner in which the existence principles are introduced. And this brings us to the problems of existence in mathematics.

As we have stated at the start, mathematicians nowadays are inclined to bypass the problem whether the postulates laid down are ture or not, as long as all the postulates as a set are consistent - so to aay. But they assume truth, which signifies existence, even if they do not mention it. For a set of postulates is said to be consistent if from them and all the demonstrable theorems which follow, it is impossible to choose two statements which are contradictories. But why avoid contradictories? The only reasonable answer is that mathematicians are anxious to keep the law of contradiction, and in keeping it, they believe that it is true, otherwise there would be no point in insisting on consistency. In other words, they believe in the truth of the law of contradiction, which is one of their postulates and a postulate without which they cannot proceed. But if they so believe, it can be shown that they have to believe in other truths which they use. So why believe in the truths of some postulates and omit their belief in others.

There is another difficulty. What is the point of deducing theorems if we are not concerned with their truth? It is said that science seeks truth, and if truth is omitted, deduction loses its dignity and worth which is usually attributed to it. Are we then investigating things or playing games? Paying mathematicians to play games is certainly strange. If, on the other hand, we attend to mathematical results rather than to what is said about them, we find overwhelming evidence that those results are truly applicable and are therefore true insofar as they are applicable. In other words, two and two make four, and there is no doubt about it; and the universal use of arithmetic confirms without doubt the accuracy and truth of arithmetic. Again, the use of geometry, too, is not regarded as faulty, otherwise there would have been a change in its postulates and in rulers and compasses and the like.

A theoretical problem arose last century, however, when the so-called "non-Euclidean geometries" were put forward as alternatives to Euclidean geometry; for doubt arose as to the absolute truth of the Euclidean postulates, especially the parallel postulate. Do the non-Euclidean geometries contradict Euclidean geometry? In words, they do; for the parallel postulate is different in each case in such a way that each of them contradicts the other two. It is agreed, however, that each of these geometries is consistent within itself, and further, that either all of them are consistent or all of them are inconsistent because transformations exist which transform each of them into the others. Consequently, it appears that either all are true, or all are false whether partly or wholly.

Now it is easy to define a straight line so as to mean not what Euclid meant by it but something else and then show that straight lines always meet in a plane or that many straight lines can be drawn through an external point parallel to a given line, and this is exactly the situation. But doing so is like calling a man "a cat" and then proving that many American cats have received the Nobel Prize. First, all models offered of non-Euclidean straight lines are either arcs of great circles on a sphere or circular arcs within a plane circle and perpendicular to that circle or some other kinds of lines which are not straight in Euclid's sense. Second, because of the transformations stated above, the theorems of Euclidean geometry are exactly the same as those of each of the other geometries, except for words, and this shows, as Poincare has shown in his Science and Hypothesis, that, for example, the Riemannean triangle has the same properties as the Euclidean spherical triangle. This indicates that Riemann equivocates when he uses the term "straight line" since what he means or should mean is an arc of a great circle. Third, let us consider the projective plane and its postulates. The term "straight line" here is used as a genus of the so-called "straight lines" in each of the different geometries, and its properties are those which are common to the so-called "straight lines" in each of those geometries. So this is a further equivocation of the term "straight line." Moreover, a postulate is added to the postulates of the projective plane to give the Euclidean plane, and another and different postulate is added likewise to give each of the other geometries. The upshot of all this is the following: (1), the method of studying properties in geometry by beginning with the projective plane is sound and even approved by Aristotle as being better than that of Euclid, since to begin with the more universal is to be more scientific because this method is better by nature, and (2), this study can be done within Euclidean geometry, and (3) there will be clarity in meaning and the absence of equivocation if it is made plain that the term "straight line" is used in different ways, one of them being generic and the others specifically different, and that each of those lines can be understood and intuited only in the Euclidean manner, seeing that all models presented are Euclidean. We may add, some of the difficulties faced by modern mathematicians with respect to geometry are philosophic, others are logical, others are semantic, and others are considered in Aristotle's On the Soul.

Another concern is the treatment of the infinite. The main difficulty here is that something is presented as actually infinite but what we are given is a process. For example, is the following equality, namely,

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + 1/8 + \ldots = 2,$$

true? The left side looks like a process because it has no end, whereas the right side is actual. The following proof for its truth ~~may~~ might be given. Either it is true or false. If not true, let the left side ~~by~~ be 2 - R, where R must be positive, however small. Then it can be shown that the left side is greater than 2 - R. Hence the equality is true. But this reasoning is faulty; for if the left side is not true, then it may be either actual, or not a number, or potential and changing and not a definite number. Of course, the last alternative is the case, and so the equality is not true.

Another example which depends on potentiality in a certain sense is the following. The points on two lines are said to be equal if there exists a rule whereby their points can be put into one-to-one correspondence; and then it is proved that any two lines are equal. Now if we define two lines as unequal if there exists a rule whereby all their points cannot be put into one-to-one correspondence, it can also be proved that the same two lines are unequal. So the same two lines are equal and unequal. Here, ~~took~~ too, potentiality is confused with actuality. The philosophical mistake, on the other hand, rests on the assumption that a line is made out of points; but a point is incapable of existing apart from a line just as a surface is incapable of existing apart from a body; so a point must be a limit and an attribute of a line

I have given, then, a brief discussion of how modern mathematics can come under the ancient definition of mathematics, but I have left out Aristotle's discussion, given in the Metaphysics, of how mathematical objects exist since I assume you are familiar with it. As for the modern belief that Aristotle's logic is inadequate for mathematical demonstration, it suffers from a mistake similar to that about Aristotle's definition of mathematics. Aristotle's logic is usually identified with the Prior Analytics, but it should include the Posterior Analytics; and if this is done, it can be shown that his logic is adequate for mathematical demonstration. An example of such a demonstration is given on pages 250 to 260 in my Aristotle's Posterior Analytics. We may now turn to the recent definitions and indicate briefly some of their difficulties.

According to the Logistic School, mathematics is the class of all propositions of the form P implies Q, where the propositions contain variables and only logical constants.

This definition, briefly stated, is in accord with the opinion of most modern mathematicians who assert that mathematics is not concerned with the truth or falsity of its postulates but only with their consistency. First, the definition of implication is of no help to mathematicians because they do not use it in their demonstrations. Second, the definition hardly differs from the ancient definition of logic, except in philosophy and details, and as such it is too wide for mathematics. Third, although implications are used in mathematics, most or all the theorems are or can be turned into single statements, such as "the derivative of x^2 with respect to x is 2x" and "the angle bisectors of a triangle are concurrent." Fourth, mathematics loses its dignity and worth if truth is not attributed to its postulates and conclusions. Fifth, the definition is too wide. Sixth, the objects of mathematics exist outside of the mind and are not propositions, for these exist primarily in the mind, whereas numbers and lines do not. Seventh, it is said that from few postulates the whole of mathematics can be derived; but we observe that thousands and thousands of new principles are introduced as the Principia Mathematica advances without any mention of this fact.

According to the School of Formalism, pure mathematics is the science of the formal structure of symbols and hence, indirectly, of the structure of objects. The terms "formal" and "without reference to meaning" are taken as synonyms, and so are the terms "a symbol" and "a mark with meaning." Accordingly, the formal structure or properties of symbols would be the formal structure or properties of symbols qua marks and not qua marks with meaning.

First, every discipline uses symbols, which have structures qua marks. It would then follow that pure mathematics is applicable to all disciplines or else includes all disciplines. Second, the structures of objects do not follow from the structure of symbols. In the statement "John has sickness" the term "sickness" is to the right of the term "John". But it will take an impossible imagination to conceive that sickness as being to the right of John. Third, if symbols are investigated qua marks, whatever is used as a premise in mathematics would be neither a premise nor true nor false nor inconsistent nor have any of the usual logical attributes which belong to mathematical expressions. Fourth, if only some of the symbols in an expression are devoid of meaning, the expression itself is devoid of meaning. For example, the expression "P is higher than Q" has meaning only if the symbol "P" signifies something to which place or being higher or lower can belong.

The definition of mathematics according to the Intuitionist School may be expressed by a number of statements. Mathematics is founded on a basic intuition of the possibility of constructing a series of numbers or objects; it is thus founded on thought and not on a symbolis~~t~~m of a particular language, which is only a means to thought; it is not timeless or static or dogmatic but growing and dynamic and fallible and always in process, and it can never be completely symbolized; and it is the product of social activity by fallible minds and so subject to revision and development.

The above definition applies to every science, and mathematics is not every science. Besides, most mathematical research, or at least some of it, is true and not subject to change; and that which is subject to change or improvement is bad or vaguely formulated and not good mathematics. Again, the definition restricts intuition to numbers, but much if not most of mathematics is not reducible to numbers.

Some other definitions may be mentioned. Benjamin Peirce defines mathematics as the science which draws necessary conclusions. This sounds like the definition of logic, and, besides, there are other sciences which draw necessary conclusions.

Auguste Comte defines mathematics as the science of indirect measurement. But this is a very limited definition, and, besides, direct measurement is more mathematical than indirect measurement, and it is presupposed by indirect measurement.

Maxime Bôcher regards mathematics as follows: If we have a certain class of objects and a certain class of relations, and if the only questions which we investigate are whether those objects do or do not satisfy those relations, the results of the investigation are called "mathematics." This statement applies to every discipline, and not every discipline is mathematics.

The definition by A. B. Kempe differs very little in principle from that of Maxime Bôcher.

In conclusion, I may say that all the recent thinkers who undertook to define mathematics could have heeded the words of Aristotle who said in his *Metaphysics*:

> "We must examine what has been said by others; for one should be pleased to state some things better than one's predecessors, and the rest not worse."