Introduction to Optimal Control and Estimation

The observer and state feedback combination was described in the last chapter as the most fundamental form of state space control system. However, as was mentioned in the summary of Chapter 10, it is not necessarily the most useful method. The distinction lies in the practicality of the feedback and observer gains. First, it was demonstrated that the feedback gains that place the eigenvalues of a multivariable system at specified locations are not unique. This difficulty is compounded because the designer does not actually know the best locations for closed-loop eigenvalues. Often, performance criteria are specified in terms of quantities that are of course related to, but not necessarily obvious from, the eigenvalue locations. How many of us can accurately predict the transient-response or frequency-response characteristics of a large system from a list of eigenvalues? Second, a similar problem occurs in the choice of the observer gain. Although we might agree that the observer eigenvalues should be “faster” than the controller eigenvalues, is there any other reason to prefer one location to another, and if there is, is any guidance available on the choice of observer gain which results in these eigenvalues?

These questions are addressed with the methods introduced in this chapter. First, the matter of feedback gains is approached by defining an optimization criterion that should be satisfied by the state feedback controller. By minimizing this criterion, the eigenvalues will be automatically placed and the feedback gains will be uniquely selected. The optimization criterion chosen is a functional of quadratic forms in the state vector and the input of the system. Such controller designs are known as linear-quadratic (LQ) controllers or linear-quadratic regulators (LQRs). We will see how such a criterion can be constructed, and how a choice of feedback gain can be made to minimize it.

In order to determine the gain of the observer, similar optimization criteria must be imposed. However, these criteria must be imposed not on the performance of the state trajectory but on the observation error of the system. In
order to “optimize” the state observation of a system, we will include explicit statistical models of the noise that might affect a state space description. Noise might appear as an input to the system or as a disturbance introduced by the sensors, modeled in the output equation. By minimizing the effects of this noise in the observed state vector, we will have created an optimal statistical estimate of the state, which can then be fed back through the state feedback gains. The estimator thus created is known as the Kalman filter.

When a Kalman filter estimates the state vector of a noisy plant and an LQ controller is used to compute state feedback, the combination is known as a linear-quadratic-gaussian (LQG) controller. It is the LQG controller that is regarded by many as the most useful state space controller. It is the simplest of many forms of optimal controllers. Optimal control is a broad and sometimes complicated topic that is only briefly introduced by this chapter.

11.1 The Principle of Optimality

In order to understand the idea of the optimization criterion in state variable systems, we will consider a discrete state space. We can imagine the state trajectory of a system making discrete transitions from one state to another under the influence of a continuous-valued input that is also applied at discrete-times. Thus, we can imagine that the goal of the control system is to force the transition of the plant from an initial state to a final state, with one or more intermediate states in between.

In the process of making the transition from one state to another, the system incurs a cost. The cost of a transition can be thought of as a penalty. A system might be penalized for being too far away from its final state, for staying away from the final state for too long a time, for requiring a large control signal $u$ in order to make the next transition, or for any other imaginable criterion. As the system moves from state to state, these costs add up until a total cost is accumulated at the end of the trajectory. If the final state of the trajectory is not at the desired goal state, further penalties can be assessed as well.

To simplify this concept, consider the set of states and possible transitions represented in the graph of Figure 11.1. In this figure, the initial state is denoted by “1”, and the final, desired state is denoted by “8”. The system jumps from state to state at each time $k$, as determined by the input $u(k)$ through the state equations $x(k+1) = Ax(k) + Bu(k)$. The possible transitions under the influence of the input are represented by the arcs that connect the initial state, through the intermediate states, to the final state. The cost associated with each transition is represented by the label on the arc, e.g., $J_{47}$ represents the cost of making the transition from state 4 to state 7. In real systems, this cost may be a function of the state vector, the input, and the time, but we represent the cost in Figure 11.1 as simply a real number. If it is assumed that costs accumulate additively, then the total cost incurred in any path is the sum of the costs incurred in each step.
Figure 11.1 Possible paths from an initial state 1 to a final state 8.

Because there are several alternate routes from the initial state to the final state, the total cost in reaching state 8 from state 1 will vary according to the path chosen:

\[
J_{18} = \begin{cases} 
J_{15} + J_{58} \\
J_{12} + J_{24} + J_{46} + J_{68} \\
\vdots
\end{cases}
\quad (11.1)
\]

If each segment results in a different cost, then it is clear that the cost of the whole trajectory might be minimized by the judicious selection of the path taken. Of course, it should not be assumed that the path with the fewest individual segments will be the one with the minimal cost. The input sequence \( u(k) \) applied such that \( J_{18} \) is minimized will be called the optimal policy \( u^*(k) \), and the resulting minimal cost that results from this input will be referred to as \( J_{18}^* \). The goal of an optimal controller is therefore to determine the input \( u^*(k) \). In continuous-time, as we will discuss in detail later, the accumulation of costs is represented by integration rather than by summation.

The mathematical tool by which we will find the optimal policy is called Bellman’s principle of optimality. This deceptively obvious principle states that at any intermediate point \( x_i \) in an optimal path from \( x_o \) to \( x_f \), the policy from \( x_i \) to the goal \( x_f \) must itself constitute optimal policy. This statement may seem too obvious to be useful, but it will enable us to solve, in closed form, for the optimal control in our problems. It is also used in recursive calculations, in a
procedure known as dynamic programming, to numerically compute optimal policy.

11.1.1 Discrete-Time LQ Control
Continuing with the discrete-time case so that the interpretation of incurred costs can be maintained, we will now consider the discrete-time state equation

\[ x(k + 1) = Ax(k) + Bu(k) \]  

and the problem of forcing the system to reach a final state \( x_f = x(N) \) from an initial state \( x_o = x(i) \). We wish to accomplish this task through full-state feedback, and we will specify a cost function to minimize in the process. The cost function we will use has the quadratic form:

\[ J_{i,N} = \frac{1}{2} x^T(N)Sx(N) + \frac{1}{2} \sum_{k=1}^{N-1} [x^T(k)Qx(k) + u^T(k)Ru(k)] \]

which may be interpreted as the total cost associated with the transition from state \( x(i) \) to the goal state \( x(N) \). In this expression, the term \( \frac{1}{2} x^T(N)Sx(N) \) gives the penalty for “missing” a desired goal state [if the desired goal of \( x(N) \) is zero]. The terms in the summation represent the penalties for excessive size of the state vector and for the size of the input. These terms are all quadratic forms, with weighting matrices \( S, Q, \) and \( R \) selected to penalize some state variables (inputs) more than others. It is necessary that the matrices \( S \) and \( Q \) be positive semidefinite, and \( R \) be positive definite, for reasons we will see later. As discussed in Section 5.3, this implies that they are, without loss of generality, considered to be symmetric. They can be functions of time \( k \) as well, but we will treat them as constants to simplify their notation. When selecting appropriate \( S, Q, \) and \( R \) matrices, it should be remembered that larger terms will penalize the corresponding variables more than smaller terms, and that it is only the relative size of the entries in this matrix that matter.

To find a control input function \( u(k) \) that should be applied to (11.2) such that (11.3) is minimized, we will begin at the last step and apply the principle of optimality as the state transitions are traced backward. Thus, according to (11.3), the cost incurred in the transition from step \( N-1 \) to step \( N \) is

\[ J_{N-1,N} = \frac{1}{2} \left[ x^T(N)Sx(N) + x^T(N-1)Qx(N-1) + u^T(N-1)Ru(N-1) \right] \]

We wish to minimize this function over all possible choices of \( u(N-1) \), but we
must first remember to express \(x(N)\) as a function of \(u(N-1)\) according to (11.2):

\[
J_{N-1,N} = \frac{1}{2} \left( (Ax(N-1) + Bu(N-1))^T S (Ax(N-1) + Bu(N-1)) 
+ x^T(N-1)Qx(N-1) + u^T(N-1)Ru(N-1) \right)
\]

(11.5)

Now, because \(J\) is continuous in the input \(u\), this function can be minimized over all inputs by differentiation:

\[
\frac{\partial^T J_{N-1,N}}{\partial u(N-1)} = 0
= B^T S [Ax(N-1) + Bu(N-1)] + Ru(N-1)
= \left[ R + B^T SB \right]u(N-1) + B^T SAx(N-1)
\]

(11.6)

(We use the transpose derivative in order to express the results as a column rather than a row.) Therefore,

\[
u^*(N-1) = -\left[ R + B^T SB \right]^{-1} B^T SAx(N-1)
\]

(11.7)

That this a minimizing solution rather than a maximizing solution may be checked with the second derivative:

\[
\frac{\partial^2 J_{N-1,N}}{\partial u^2(N-1)} = \left[ R + B^T SB \right] > 0
\]

(11.8)

Obviously, the expression (11.7) is in the form of state feedback, as we wished. Using a more compact and familiar notation, this expression can be written as

\[
u^*(N-1) = K_{N-1} x(N-1)
\]

(11.9)

with

\[
K_{N-1} = -\left[ R + B^T SB \right]^{-1} B^T SA
\]

(11.10)

Using (11.5) to express the value of \(J^*_{N-1,N}\):
Part II. Analysis and Control of State Space Systems

\[ J_{N-1,N}^* = \frac{1}{2} x^T (N-1) S x(N-1) \]

In order to simplify this notation, define

\[ S_{N-1}(A + BK_{N-1})^T S(A + BK_{N-1}) + Q + K_{N-1}^T R K_{N-1} \]

so that (11.11) becomes

\[ J_{N-1,N}^* = \frac{1}{2} x^T (N-1) S_{N-1} x(N-1) \]

The choice of the notation in (11.12) is motivated by the fact that if we set \( i = N \) in (11.3), then we get

\[ J_{N,N}^* = J_{N,N}^* = \frac{1}{2} x^T (N) S x(N) \]

which is the optimal \( J_{N,N}^* \) because it is the only \( J_{N,N} \). Therefore, comparing (11.14) and (11.13), we can make the notational identification that \( S = S_N \).

Now we take another step backward and compute the cost \( J_{N-2,N} \) in going from state \( N-2 \) to the goal state \( N \). First, we realize that

\[ J_{N-2,N} = J_{N-2,N-1} + J_{N-1,N} \]

as computed in (11.1) above. Therefore, to find the optimal policy in going from \( N-2 \) to \( N \), we use the principle of optimality to infer that

\[ J_{N-2,N}^* = J_{N-2,N-1}^* + J_{N-1,N}^* \]

So now step \( N-1 \) is the goal state, and we can find \( J_{N-2,N-1} \) using the same Equation (11.5), but substituting \( N-2 \) for \( N-1 \) and \( N-1 \) for \( N \):

\[ J_{N-2,N-1}^* = \frac{1}{2} x^T (N-2) Q x(N-2) + u^T (N-2) R u(N-2) \]
This can be minimized over all possible $u(N - 2)$ by differentiation just as in (11.6) and (11.7), but it should be apparent that the result will be the same as (11.7) except, again, for the substitutions of $N - 2$ for $N - 1$ and $N - 1$ for $N$.

$$u^*(N - 2) = K_{N-2} x(N - 2)$$  \hspace{1cm} (11.18)

where

$$K_{N-2} = -\left[R + B^T S_{N-1} B\right]^{-1} B^T S_{N-1} A$$  \hspace{1cm} (11.19)

Continuing these backward steps, we will get similar expressions for each time $k$. We can therefore summarize the results at each step with the set of equations

$$u^*(k) = K_k x(k)$$  \hspace{1cm} (11.20)

where

$$K_k = -\left[R + B^T S_{k+1} B\right]^{-1} B^T S_{k+1} A$$  \hspace{1cm} (11.21)

and

$$S_k = (A + BK_k)^T S_{k+1} (A + BK_k) + Q + K_k^T R K_k$$  \hspace{1cm} (11.22)

Note that (11.22) is a difference equation whose starting condition occurs at the final time and is computed toward earlier time. This starting condition is derived from (11.14) as

$$S_N = S$$  \hspace{1cm} (11.23)

At such time $k$, one can also compute the “cost-to-go,” i.e., the optimal cost to proceed from $x(k)$ to $x(N)$ as in (11.13):

$$J^*_k = \frac{1}{2} x^T(k) S_k x(k)$$  \hspace{1cm} (11.24)

The set of equations in (11.20) through (11.22) represents the complete LQ controller for a given discrete-time system. Equation (11.22) is known as the discrete-time matrix Riccati equation. The fact that it is a “backward-time” equation gives the system’s feedback gain $K_k$ the peculiar behavior that its transients appear at the end of the time interval rather than at the beginning.
Example 11.1: A Discrete-Time LQ Controller

Simulate a feedback controller for the system

$$x(k+1) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \quad x(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$ (11.25)

such that the cost criterion

$$J = \frac{1}{2} x^T(10) \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} x^T(10) + \frac{1}{2} \sum_{k=1}^{9} \left( x^T(k) \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} x(k) + 2u^2(k) \right)$$

(11.26)

is minimized.

Solution:

Given the cost criterion in (11.26), the only work necessary to solve this problem is to iterate the equations in (11.21) and (11.22) in backward time, saving the $S$-matrix and gain matrix at each step. When these values are computed, then the plant equation in (11.25) is simulated in forward time using the stored values of gain.

Plotted in Figure 11.2 below is the state variable sequence versus time. Although it should be obvious that the original plant is unstable, it is apparent from these results that the controlled system asymptotically approaches the origin of the two-dimensional space.

![Figure 11.2](image-url)
Figure 11.3 is a plot of the feedback gains versus time. From this plot we note the interesting feature that the gains change at the end of the time interval rather than at the beginning. This is because of the backward-time nature of the matrix equation in 11.22. This observation will become important in our discussion of infinite-time optimal control in Section 11.1.3.

![Figure 11.3 Feedback gains versus time for Example 11.1. Note that they are computed only up through \( k = N - 1 = 9 \) because \( K(10) \) is not needed to compute \( x(10) \).](image)

Figure 11.4 is a plot of the matrix \( S \) versus the step number. Normally, this matrix itself, the feedback gain matrix \( K_k \), need not be stored, but we wish to make the observation here that in the early stages of the time interval, the matrix appears to be constant or nearly constant. This observation will become important in Section 11.1.3.

### 11.1.2 Continuous-Time LQ Control

The development of the continuous-time LQ feedback controller follows the same principles as in the discrete-time situation, but the notion of “steps” is absent. Instead, we will first perform the derivation assuming small increments, then let these increments approach zero.
The continuous-time LQ cost function takes the form:

\[ J(x(t_0), u(t_0), t_f) = \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} [x^T(t) Q x(t) + u^T(t) R u(t)] dt \]  \hspace{1cm} (11.27)

This criterion can be interpreted in the same manner as the discrete-time cost function in (11.3): it is the cost in making the transition from the initial state \( x(t_0) \) to the final state \( x(t_f) \) using the control function \( u(t) \). It contains a final state cost and an accumulation term that penalizes deviation of the state and inputs from zero, as measured by the quadratic forms with user-defined weighting matrices. This accumulation of costs is now represented by the integration from initial time \( t_0 \) to final time \( t_f \). Once again, \( S \) and \( Q \) are symmetric, positive-semidefinite matrices and \( R \) is a symmetric, positive-definite matrix. It may be noted that in more general optimal control problems, \( J \) may also be an explicit function of time and of possible “cross terms” such as \( x^T N u \). We will make use of only the quadratic functions seen in (11.27).

For the continuous-time system

\[ \dot{x} = Ax + Bu \]  \hspace{1cm} (11.28)

the optimal control problem can thus be stated as the problem of finding the
optimal control function $u^*(t)$, where $t \in [t_o, t_f]$, and applying it to (11.28), such that the function (11.27) is minimized. Thus the optimal cost will be denoted

$$J^*(x, t) = \min_{u(t)} \left\{ \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2} \int_{t_o}^{t_f} \left[ x^T(t) Q x(t) + u^T(t) R u(t) \right] dt \right\} \quad (11.29)$$

Note that strictly, this notation removes from $J^*$ any explicit dependence on input $u(t)$. We will drop the arguments on $J^*$ in order to economize on the notation.

Suppose that we decompose the computation of the cost of moving from $t_o$ to $t_f$ into two stages: one from $t_o$ to an infinitesimal time $\delta t$ later and then from $t_o + \delta t$ to $t_f$. At time $t_o + \delta t$, we will assume that we have reached state $x(t_o + \delta t)$. We can then write (11.27) as:

$$J(x, u, t) = \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2} \int_{t_o}^{t_o + \delta t} \left[ x^T(t) Q x(t) + u^T(t) R u(t) \right] dt + \frac{1}{2} \int_{t_o + \delta t}^{t_f} \left[ x^T(t) Q x(t) + u^T(t) R u(t) \right] dt \quad (11.30)$$

Bellman’s principle of optimality would then imply that for an optimal path,

$$J^*(x(t_o + \delta t), t_o + \delta t, t_f) = \min_{u(t)} \left\{ \frac{1}{2} \int_{t_o}^{t_o + \delta t} \left[ x^T(t) Q x(t) + u^T(t) R u(t) \right] dt + J^*[x(t_o + \delta t), t_o + \delta t, t_f] \right\} \quad (11.31)$$

That is, the optimal cost in going from $t_o$ to $t_f$ is the cost of going from $t_o$ to $t_o + \delta t$ plus the optimal cost in going from $t_o + \delta t$ to $t_f$, which itself includes the terminal cost. In this expression, we will expand the term $J^*[x(t_o + \delta t), t_o + \delta t]$ into a Taylor series about the point $(x(t_o), t_o)$ as follows:

$$J^*(x(t_o + \delta t), t_o + \delta t) = J^*(x(t_o), t_o) + \frac{\partial J^*}{\partial t} \bigg|_{x(t_o), t_o} (t_o + \delta t - t_o) + \frac{\partial J^*}{\partial x} \bigg|_{x(t_o), t_o} [x(t_o + \delta t) - x(t_o)] + \text{h.o.t.} \quad (11.32)$$
Part II. Analysis and Control of State Space Systems

where “h.o.t.” stands for “higher-order terms.” Optimal cost, Equation (11.31), can then be written as:

\[
J^*(x,t) = \min_{u(t)} \left\{ \frac{1}{2} \int_{t_0}^{t_0+\delta t} \left[ x^T(t)Qx(t) + u^T(t)Ru(t) \right] dt + J^*(x(t_0),t_0) + \frac{\partial J^*}{\partial t} \bigg|_{(x(t_0),t_0)} \delta t + \frac{\partial J^*}{\partial x} \bigg|_{(x(t_0),t_0)} \left[ x(t_0 + \delta t) - x(t_0) \right] \right\}
\]

Within this expression, the terms \( J^*(x(t_0),t_0) \) and \( \left( \frac{\partial J^*}{\partial t} \right) \delta t \) do not depend on \( u(t) \) and can therefore be moved outside the min operator. Furthermore, assuming that \( \delta t \) is very small, we can make the approximations:

\[
\frac{1}{2} \int_{t_0}^{t_0+\delta t} \left[ x^T(t)Qx(t) + u^T(t)Ru(t) \right] dt \approx \frac{1}{2} \left[ x^T(t)Qx(t) + u^T(t)Ru(t) \right] \delta t \tag{11.34}
\]

and

\[
x(t_0 + \delta t) - x(t_0) = \delta x = \frac{\delta x}{\delta t} \delta t \approx \frac{dx}{dt} \delta t \tag{11.35}
\]

making (11.33) appear as

\[
J^*(x,t) = J^*(x,t) + \frac{\partial J^*}{\partial t} \delta t + \min_{u(t)} \left\{ \frac{1}{2} \left[ x^T(t)Qx(t) + u^T(t)Ru(t) \right] \delta t + \frac{\partial J^*}{\partial x} (Ax + Bu) \delta t \right\}
\]

or
\[
0 = \frac{\partial J^*}{\partial t} \delta t + \min_{u(t)} \left\{ \frac{1}{2} \left[ x^T(t)Qx(t) + u^T(t)Ru(t) \right] \delta t + \frac{\partial J^*}{\partial x} (Ax + Bu) \delta t \right\} 
\]

(11.36)

Dividing out the \( \delta t \) term from both sides:

\[
0 = \frac{\partial J^*}{\partial t} + \min_{u(t)} \left\{ \frac{1}{2} \left[ x^T(t)Qx(t) + u^T(t)Ru(t) \right] + \frac{\partial J^*}{\partial x} (Ax + Bu) \right\} 
\]

(11.37)

This equation is known as the Hamilton-Jacobi-Bellman equation. It is useful in a wide range of optimal control problems, even for nonlinear systems and nonquadratic cost functions. (Note that we have not yet exploited the linear system equations and the quadratic terms in \( J \).) It is a partial differential equation that can be numerically solved in the general case using a boundary condition derived by setting \( t_o = t_f \) in (11.27):

\[
J^*(x(t_f), t_f) = \frac{1}{2} \left[ x^T(t_f)Sx(t_f) \right] 
\]

(11.38)

The term in braces in (11.37) is known as the *hamiltonian*. Recall that we used the same terminology in Section 3.3.1, when we discussed solving underdetermined systems of equations. In that chapter, the hamiltonian had the same structure: the optimization criterion plus a multiple of the constraint expression. Thus, we define

\[
H(x, u, J^*, t) \overset{\Delta}{=} \frac{1}{2} \left[ x^T(t)Qx(t) + u^T(t)Ru(t) \right] + \frac{\partial J^*}{\partial x} (Ax + Bu) 
\]

(11.39)

If we can find the *optimal* control \( u^* \), then (11.39) will become (dropping the \( t \) arguments):

\[
H(x, u^*, J^*, t) \overset{\Delta}{=} \frac{1}{2} \left[ x^T Qx + u^T Ru^* \right] + \frac{\partial J^*}{\partial x} (Ax + Bu^*) 
\]

\[
= \min_u \left\{ \frac{1}{2} \left[ x^T Qx + u^T Ru \right] + \frac{\partial J^*}{\partial x} (Ax + Bu) \right\} 
\]

(11.40)

making the Hamilton-Jacobi-Bellman equation appear as
In order to minimize the Hamiltonian itself, over \( u \), we take the derivative:

\[
\frac{\partial \mathcal{H}}{\partial u} = 0
\]

0 = \frac{\partial \mathcal{H}}{\partial u} \left[ \frac{1}{2} (x^T Q x + u^T R u) + \frac{\partial J^*}{\partial x} (A x + B u) \right] + \frac{\partial J^*}{\partial x} (A x + B u)

(11.42)

0 = R u + B^T \frac{\partial J^*}{\partial x}

giving the solution

\[
u^* = -R^{-1} B^T \frac{\partial J^*}{\partial x}
\]

(hence the need for \( R \) to be positive definite). That this is a minimizing solution rather than a maximizing solution may be checked with a second derivative, similar to (11.8). This expression is still not suitable for use as an optimal control because of the partial derivative of \( J^* \). To solve for this term, substitute (11.43) into (11.40):

\[
H(x, u^*, J^*, t) = \frac{1}{2} (x^T Q x + u^*^T R u^*) + \frac{\partial J^*}{\partial x} (A x + B u^*)
\]

\[
= \frac{1}{2} x^T Q x + \frac{1}{2} \frac{\partial J^*}{\partial x} B R^{-1} B^T \frac{\partial J^*}{\partial x} + \frac{\partial J^*}{\partial x} (A x - B R^{-1} B^T \frac{\partial J^*}{\partial x})
\]

\[
= \frac{1}{2} x^T Q x + \frac{1}{2} \frac{\partial J^*}{\partial x} B R^{-1} B^T \frac{\partial J^*}{\partial x} + \frac{\partial J^*}{\partial x} A x - \frac{\partial J^*}{\partial x} B R^{-1} B^T \frac{\partial J^*}{\partial x}
\]

\[
= \frac{1}{2} x^T Q x - \frac{1}{2} \frac{\partial J^*}{\partial x} B R^{-1} B^T \frac{\partial J^*}{\partial x} + \frac{\partial J^*}{\partial x} A x
\]

(11.44)

Substituting this into the Hamilton-Jacobi-Bellman equation in (11.41):
The remaining task is to solve this equation. The solution is facilitated by the assumption that the optimal cost $J^*$ from any point until the final time can be represented as a quadratic form in terms of the state vector. We have already seen that this is true for the discrete-time system, Equation (11.24), and we know it is true for the end state, Equation (11.38), so it is a reasonable assumption here as well. We therefore denote

$$J^*(x,t) = \frac{1}{2} x^T(t)P(t)x(t)$$  \hspace{1cm} (11.46)

where the matrix $P(t)$ is presumed to be symmetric but as yet, unknown. Using this assumption,

$$\frac{\partial J^*(x,t)}{\partial x} = x^T(t)P^T(t) = x^T(t)P(t)$$  \hspace{1cm} (11.47)

and

$$\frac{\partial J^*(x,t)}{\partial t} = \frac{1}{2} x^T(t)\dot{P}(t)x(t)$$  \hspace{1cm} (11.48)

Substituting these relations into (11.45) gives

$$0 = \frac{1}{2} x^T \dot{P}x + \frac{1}{2} x^T Qx - \frac{1}{2} x^T PBR^{-1}B^T P x + x^T PAx$$  \hspace{1cm} (11.49)

In this expression, each term, except for the last, is a symmetric quadratic form. However, appealing to Equation (5.17), which allows us to treat any quadratic form as a symmetric one, we can express

$$x^T PAx = x^T \left( PA + \frac{1}{2} A^T P \right) x$$

$$= \frac{1}{2} x^T PAx + \frac{1}{2} x^T A^T P x$$

Using (11.50) in (11.49) gives

$$0 = \frac{1}{2} x^T \dot{P}x + \frac{1}{2} x^T Qx - \frac{1}{2} x^T PBR^{-1}B^T P x + \frac{1}{2} x^T PAx + \frac{1}{2} x^T A^T P x$$

$$= \frac{1}{2} x^T \left[ \dot{P} + Q - PBR^{-1}B^T P + PA + A^T P \right] x$$  \hspace{1cm} (11.51)
This equation must hold for all \( x \), so
\[
0 = \dot{P} + Q - PBR^{-1}B^TP + PA + A^TP
\]  
(11.52)
or equivalently,
\[
\dot{P} = PBR^{-1}B^TP - Q - PA - A^TP
\]  
(11.53)
This is a famous equation known as the differential matrix Riccati equation. It is a nonlinear differential equation whose boundary condition is given by (11.38):
\[
P(t_f) = S
\]  
(11.54)
where \( S \) is the end-state weighting matrix from the cost criterion in (11.27). Like the matrix equation in (11.22) for the discrete-time case, it is a backward-time equation whose solution can be obtained numerically. If we solve for \( P(t) \) from (11.53), then we have from (11.43) and (11.47):
\[
\begin{align*}
  u^*(t) &= -R^{-1}B^TP(t)x(t) \\
  \Delta &= K(t)x
\end{align*}
\]  
(11.55)
which is clearly a state variable feedback solution, albeit one with time-varying feedback gain. As in continuous-time, the backward-time nature of the matrix \( P(t) \) will give the feedback matrix a transient near \( t_f \) rather than \( t_o \), as is usually the case. Together, (11.53) and (11.55) represent the complete specification of the continuous-time LQ controller.

It should be noted that there are practical reasons for not solving the differential matrix Riccati equation, (11.53). Recall that in the discrete-time case, the corresponding equation in (11.22) was computed backward and the results at each step stored for application in forward time. If the solution of the differential Riccati equation is solved numerically, how can it be stored for use in the continuous-time feedback control (11.55)? If only discrete points of the solution can be stored, then it is actually the discrete-time equivalent of the system that is being controlled, and we are back to the discrete-time LQ controller. In scalar systems, the Riccati equation can be solved in closed form so that it can be used in (11.55) to generate the control signal \( u(t) \). If continuous-time optimal controllers are required for higher-order systems, they are most practical in the infinite-horizon case discussed in the next section.
11.1.3 Infinite-Horizon Control

We may wonder how to determine the final time as well as the penalty term for the final state. Usually in classical control systems, the time-domain characteristics such as rise time and settling time are important, but often a finite final time is not a factor in the design. Rather, it is intended that the regulating system operate continually after it is turned on. Yet in the LQ controllers discussed above, the final time is a critical piece of information. It is the starting point for the solution of the Riccati equations, for both discrete- and continuous-time, which then determine the values of the time-varying state feedback gains.

If we examine the plots in Figure 11.4, we might guess that if the final time were to approach infinity, the values for the elements of the $S$-matrix would remain constant for most of the operating interval (or at least until the state vector decays to a very small value). Then, the contribution to the total cost of the portion occurring during the end-time transient would be comparatively small, and we might justify using the constant values throughout. Figures 11.3 and 11.4 show Riccati equation solutions that hold steady at their initial horizontal values for a considerable simulation time.

For continuous-time systems, this might suggest solving the Riccati equation (11.53) by assuming that $P = 0$; this gives the algebraic Riccati equation:

$$0 = PBR^{-1}B^TP - Q - PA - A^TP$$

(11.56)

and the corresponding feedback function

$$u = Kx = -R^{-1}B^TPx$$

(11.57)

In discrete-time, the presumption of steady state would imply that $S_k = S_{k+1} = S$, giving

$$S = (A + BK)^TS(A + BK) + Q + K^TRK$$

(11.58)

and

$$K = -[R + B^TSB]^{-1}B^TS$$

(11.59)

which together can be combined into a decoupled equation:

$$S = A^TS - A^TS[R + B^TSB]^{-1}B^TS + Q$$

(11.60)

This would then imply a constant state feedback gain $K$ in each case, and, instead of the difference or differential equations in (11.22) or (11.53), we would have
Part II. Analysis and Control of State Space Systems

Instead algebraic Riccati equation\(^{M}\) (11.56) or (11.60). In essence, this is exactly what happens, but some complications may arise. In particular, we have not yet established that all optimal control solutions do reach a bounded steady state as \(t \to -\infty\) and that these solutions stabilize the system over infinite time. Nor have we considered whether these steady-state solutions are unique.

In order to investigate these issues briefly, we will consider the continuous-time situation in more detail. Consider the problem of controlling the time-invariant system

\[
\dot{x} = Ax + Bu
\]

with static state feedback

\[u = Kx\]  \quad (11.61)

such that the cost function

\[
J = \int_{0}^{\infty} \left[ x^T(t)Qx(t) + u^T(t)Ru(t) \right] dt \]  \quad (11.62)

is minimized. Using the feedback equation, (11.61), we obtain the closed-loop system

\[
\dot{x} = (A + BK)x
\]  \quad (11.63)

making the cost function, (11.62), appear as

\[
J = \int_{0}^{\infty} \left[ x^T(t)Qx(t) + x^T(t)K^T R Kx(t) \right] dt \]  \quad (11.64)

Knowing that without an exogenous input, (11.63) has the solution

\[x(t) = e^{(A+BK)t}x_0\]  \quad (11.65)

then (11.64) can be written as

\[
J = x_0^T \left[ \int_{0}^{\infty} e^{(A+BK)^T t} (Q + K^T R K) e^{(A+BK)t} dt \right] x_0 \]

\[= x_0^T Px_0\]  \quad (11.66)
We know from our study of Lyapunov stability in Chapter 7 (Theorem 7.32) that the system in (11.63) is asymptotically stable if and only if the matrix $P$ defined in (11.66) is the unique positive-definite solution of the Lyapunov equation

$$(A + BK)^T P + P(A + BK) + Q + K^T RK = 0$$

Therefore, for any constant feedback gain $K$ that asymptotically stabilizes the system, the solution $P$ of (11.67) will give a convergent cost integral computed by (11.66). [Also recall that if $Q + K^T RK$ is merely positive semi-definite, then we must also have observability of the pair $(A, Q + K^T RK)$.]

Now suppose that we hypothesize an optimal control input of

$$u = Kx = -R^{-1}B^T P_{ss} x$$

where $P_{ss}$ is the steady-state solution of the differential matrix Riccati equation solved for the same $Q$ and $R$. Substituting $K = -R^{-1}B^T P_{ss}$, (11.67) becomes

$$0 = (A - BR^{-1}B^T P_{ss})^T P_{ss} + P_{ss}(A - BR^{-1}B^T P_{ss}) + Q + P_{ss}BR^{-1}B^T P_{ss}$$

$$= A^T P_{ss} - P_{ss}BR^{-1}B^T P_{ss} + P_{ss}A - P_{ss}BR^{-1}B^T P_{ss} + Q + P_{ss}BR^{-1}B^T P_{ss}$$

$$= A^T P_{ss} + P_{ss}A + Q - P_{ss}BR^{-1}B^T P_{ss}$$

which we can see is exactly the differential matrix Riccati equation, (11.53), subject to the constraint that $P_{ss} = 0$. Therefore, the optimal solution of the infinite-time LQ problem is given by a solution to the algebraic Riccati equation, (11.56), when it exists. This does not necessarily mean that solutions to (11.56) are unique. If there is a unique stabilizing solution, it is optimal.

For discrete-time cases, exactly the same result holds: if a unique stabilizing solution to the steady-state algebraic Riccati equation, (11.60), exists, it solves the infinite-horizon LQ control problem via the feedback function in (11.21).

The question that remains in both cases is: When does such a suitable solution exist, and is it unique? The answer is that two theorems are required:

THEOREM: If $(A,B)$ is stabilizable, then regardless of the final state weighting matrix $S$, a finite steady-state solution $P_{ss}$ ($S_{ss}$) exists for the differential (difference) Riccati equation, (11.53) or (11.22). This solution will also be a solution to the corresponding algebraic Riccati equation, and will be the
optimizing solution of the infinite-time case; i.e., it will result in \( J = J' \). (11.70)

**THEOREM:** Let state weighting matrix \( Q \) be factored into its square roots as \( Q = T^T T \). Then the steady-state solutions \( P_{ss} \) (\( S_{ss} \)) of the differential (difference) Riccati equations are unique positive-definite solutions of their corresponding algebraic Riccati equations if and only if \((A, T)\) is detectable (see Section 8.3.3). (11.71)

In summary, if both of these conditions hold, i.e., if \((A, B)\) is stabilizable and \((A, T)\) is detectable, the solution to (11.56) or (11.60) exists, is unique, and provides the optimal static state feedback gain through (11.55) or (11.21). The reason for the first condition is fairly obvious given our infinite-length control interval. It guarantees that a static feedback gain will result in a convergent cost integral. When cost minimization over finite time was desired, there was actually no guarantee or assumption that the system was controllable or even stabilizable. If over finite time the integrand of cost function \( J \) remains bounded, then the optimal feedback control will minimize it, whether it stabilizes the system or not. If optimization of \( J \) over infinite time occurs, then the integrand of (11.64) must asymptotically approach zero in order for the costs to converge to a finite value and for a unique optimal solution to exist. By asking that the system be stabilizable, we are guaranteeing that a feedback \( K \) does exist that drives the state \( x(t) \) asymptotically to zero and gives a finite cost. This fact does not alone guarantee that the system is stable if \( J \) converges, or consequently, that the optimal feedback will be found by solving (11.56). For that guarantee we need the other condition, i.e., detectability.

Understanding the reason for detectability of the pair \((A, T)\) is slightly less obvious but just as reasonable. To justify this condition, consider the expression for cost as in (11.62):

\[
J = \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)] dt
\]

\[
= \int_0^\infty x^T(t)T^T Tx(t) dt + \int_0^\infty u^T(t)Ru(t) dt
\]

(11.72)

Note that the first term might converge even if the system is not stable. If, for example, \( x(t) \) is an eigenvector of \( A \) corresponding to an unstable eigenvalue and it is in the null space of \( T \) (\( Tx = 0 \)), it would not contribute to the cost \( J \). By the
results of Section 8.2.2, this is equivalent to \((A,T)\) being unobservable. Even if such modes were stable, two identical costs might result from two different feedback gains and, hence, two different \(P\)-matrices (\(S\)-matrices). If the system \((A,T)\) is observable, we can guarantee that no modes escape the integration of (11.72), and if the system \((A,T)\) is merely detectable, we guarantee that no unstable modes escape integration.

It should be remembered that multiple stabilizing solutions to (11.56) might exist if \((A,B)\) is stabilizable but \((A,T)\) is not detectable. The Riccati equation is, after all, a quadratic equation, which is easily seen if one considers scalar systems. Its solution would therefore not be unique, requiring us to find the “optimal” one, i.e., the one that is the asymptotic solution (as \(t \to -\infty\)) of the differential matrix Riccati equation. Rigorous proof of the theorems and methods for finding alternate solutions require an alternative formulation of the optimal control problem and are not given here (but can be found in [4]).

Example 11.2: Continuous-Time LQ Control

Consider the continuous-time system given by the equations:

\[
\dot{x} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
y = cx
\]  

(11.73)

Solve the differential matrix Riccati equation that results in the control signal that minimizes the cost function

\[
J = x^T(5)Sx(5) + \int_{0}^{5} \left[ y^T(t)y(t) + u^T(t)u(t) \right] dt
\]  

(11.74)

Use the two different final-state weighting matrices

\[
S_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad S_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}
\]  

(11.75)

Then find the solution to the algebraic Riccati equation that gives the optimal control for the cost function

\[
\int_{0}^{\infty} \left[ y^T(t)y(t) + u^T(t)u(t) \right] dt
\]  

(11.76)
Do this for both of the \( c \)-matrices, i.e., \( c_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \) and \( c_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \). Compare the results. Is the system asymptotically stabilized in each case?

**Solution:**

We should first analyze the problem to anticipate what results are expected. The system is clearly open-loop unstable, with eigenvalues of 0 and 1. It is easy to check that it is controllable (and hence, stabilizable). It is observable through output matrix \( c_1 \) but not through \( c_2 \). Because at least one of the modes is unobservable through \( c_2 \) but neither is asymptotically stable, we will expect that the infinite-time control of the system will then give an algebraic Riccati equation whose solution is unable to stabilize the system.

The second feature to notice from the problem statement is that the cost function is written in terms of the output signal \( y(t) \) instead of the state vector \( x(t) \). However because there is no feedthrough term, we can substitute \( y(t) = cx(t) \) to get the new integral

\[
J = x^T(5)Sx(5) + \int_0^5 [x^T(t)cc^T(t) + uu^T(t)]dt \tag{11.77}
\]

(and similarly for the infinite-time cost). This implies that the weighting matrices are \( Q = c^Tc \) and \( R = r = 1 \) (scalar). This provides a ready-made factorization for \( Q \).

For the case of output matrix \( c_1 \) and the finite-horizon cost function, numerical simulation of the differential matrix Riccati equation, (11.53) (see Problem 11.5) gives the solutions for matrix \( P \) as depicted in Figure 11.5. Note that in each case, the limiting solution (as \( t \to 0 \)) is the same, but that the simulation with the nonzero final state weighting matrix converges faster (as we consider traveling backward in time).

If we now seek the solution to the *algebraic* Riccati equation, we can guess that it will be equal to the limiting values of the curves in Figure 11.5, because the system is stabilizable and detectable when using \( c_1 \). This solution can be obtained using several numerical techniques, which are usually necessary because of the nonlinearity of the equations involved. However, for this particular system, the Riccati equation may be solved analytically:

\[
0 = A^TP + PA + c^TP - Pbr^{-1}b^TP
\]
This matrix equality represents a set of three simultaneous equations [because the (1,2) and (2,1) terms are the same]. They are:

\[ 2p_{11} - p_{11}^2 + 4p_{12} = 0 \]  \hspace{1cm} (11.79)

\[ p_{12} + 2p_{22} - p_{11}p_{12} = 0 \]  \hspace{1cm} (11.80)

\[ 1 - p_{12}^2 = 0 \]  \hspace{1cm} (11.81)

Solving (11.81) results in \( p_{12} = \pm 1 \). If we select \( p_{12} = -1 \), then Equation (11.79) will have complex roots, which is clearly undesirable. So instead we select \( p_{12} = +1 \), giving a quadratic equation for (11.79) that provides the two solutions \( p_{11} = 1 \pm \sqrt{5} \). If \( p_{11} = 1 - \sqrt{5} \), then the \( P \)-matrix cannot be positive definite. So instead take \( p_{11} = 1 + \sqrt{5} = 3.236 \). Finally, these two results substituted into (11.80) give the final element \( p_{22} = \sqrt{5}/2 = 1.118 \). Notice that these values are the steady-state solutions to the differential Riccati equations shown in Figure 11.5. Thus, the static state feedback result is

\[ u = -r^{-1}b^TPx = Kx = [-3.236 \quad -1]x \]  \hspace{1cm} (11.82)

When this feedback is used to compute the closed loop eigenvalues, we find that

\[ \sigma(A + bK) = -\sqrt{5}/2 \pm \sqrt{5}/2 \]

which indicates an asymptotically stable system.
For the second output matrix $c_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, the numerical simulations for the two cases are shown in Figure 11.6. Note that in this case, the simulation with the nonzero final-state matrix does not reach a steady state (again, backward) as fast as the $S = 0_{2x2}$ case does. In fact, the curves approach their steady state very slowly and are always poor approximations for constants. Note also that the limiting solutions are different from those of Figure 11.5.

To find the limiting solutions in Figure 11.6, we can solve the algebraic Riccati equations exactly as we did in (11.79) through (11.81), which results in

$$P = \begin{bmatrix} 1 + \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$$

(11.83)

This positive-semidefinite solution provides steady-state (i.e., static) feedback of

$$u = -r^{-1}b^TPx = Kx = \begin{bmatrix} -2.414 & 0 \end{bmatrix}$$

(11.84)

which results in closed-loop eigenvalues of $\sigma(A + bK) = \{0, 1\}$. The closed-loop system is therefore still not asymptotically stable. In fact, its eigenvalues have not changed at all.
Chapter 11. Introduction to Optimal Control and Estimation

11.2 Optimal Estimators

One of the most important but most often neglected factors in the performance of linear control systems is the influence of noise and disturbances. All systems are subject to noise, be it in the form of unmodeled input effects, unmodeled dynamics, or undesired signals that act as inputs to the system at different points. We have, in fact, ignored noise up to this point in our analysis and design of state space control systems. Yet, noise is known to have significant effects on the convergence of regulators and numerical design algorithms, excitation of unmodeled dynamics, and stability. A growing subdiscipline of control systems, robust control, is dedicated to the investigation of such influences and their rejection from the desired outputs of the system.

We will treat some simple noise models in this section and determine the ways we may deal with it in the context of our knowledge of observers and state feedback. First, we consider noise as being a corrupting influence in the process of observing the state of a system. With a state space model that includes noise, we will generate the best possible observer, i.e., the observer that most effectively rejects the effects of the noise. We call such observers estimators, and the particular estimator we develop is known as the Kalman filter.
11.2.1 Models of Systems with Noise

Consider the discrete-time linear system:

\[ x(k + 1) = Ax(k) + Bu(k) + Gv(k) \]
\[ y(k) = Cx(k) + w(k) \]

In this expression, two new terms appear, \( v(k) \) and \( w(k) \). These signals are considered to be random processes consisting of white, stationary noise, with zero mean, and uncorrelated with each other. Thus, they have the following properties:

\[
E[v(k)v^T(j)] = V \quad \text{if} \quad k \neq j
\]
\[
E[v(k)] = 0
\]

and

\[
E[w(k)w^T(j)] = W \quad \text{if} \quad k \neq j
\]
\[
E[w(k)] = 0
\]

as well as the joint property that \( E[v(j)w^T(k)] = 0 \) for all \( j \) and \( k \). Furthermore, we will consider the initial condition on the state to be a random variable \( x_0 \), because we seldom know it exactly. We also consider \( x_0 \) to be white noise, with

\[
E[(x_0 - E(x_0))(x_0 - E(x_0))^T] = S_0
\]

Furthermore, we assume that \( x_0 \) is uncorrelated with \( v(k) \) and \( w(k) \).

The plant input term \( v(k) \) is called the plant noise (sometimes called the process noise), and the noise term in the measurement equation \( w(k) \) is called the measurement noise. The plant noise models the effects of noise inputs on the state space variables themselves, and the measurement noise models such factors

\* Time-varying systems are easily included in the derivations of this section. However, the time argument on the system matrices, i.e., the \( k \) in \( A(k) \), as well as the \( d \) subscript we have been using to denote discrete-time quantities are omitted for brevity of notation.
as noisy sensors. The other input, \( u(k) \), is the conventional input term and is considered noise free, i.e., deterministic.

### 11.2.2 The Discrete-Time Kalman Filter

When the state space system is modeled as in (11.85), we must reconsider our derivation of the observer. In particular, we must consider the effects of the noise on the selection of the observer gain \( L \) (from Chapter 10). This gain should ideally be chosen to provide the best estimate of the system’s state while simultaneously rejecting any influence due to the two noise inputs. Such an estimator can therefore be thought of as a filter. For the model we have given, the most commonly used estimator for the state is known as the Kalman filter. We will present one of several possible approaches to the Kalman filter, one that follows our previous understanding of observers.

Our derivation of the Kalman filter begins with the formulation for the current estimator as seen in (10.98). Recall that in the current estimator, the observation process is separated into a time update and a measurement update. This is convenient for us now because it initially allows us to separate the effects of the two different noise terms.

For the time update of the current estimator, we will alter (10.97) by finding the best estimate for \( x(k + 1) \) given only the current best estimate \( \hat{x}(k) \) and the plant matrices (but not the output that is received at time \( k + 1 \)). The best estimate is the expected value

\[
\bar{x}(k + 1) = E[A\hat{x}(k) + Bu(k) + Gv(k)]
\]

\[
= A\hat{x}(k) + Bu(k) \tag{11.89}
\]

Now, we will propose an observer following (10.98) to update this estimate after the output at time \( k + 1 \) becomes available:

\[
\hat{x}(k + 1) = \bar{x}(k + 1) + L(k + 1)\left[y(k + 1) - C\bar{x}(k + 1)\right] \tag{11.90}
\]

Note that we have written the estimator gain \( L \) as a time-varying quantity. We will see that this is indeed necessary.

The Kalman filter is an optimal estimator in the sense that it provides the best estimate for the state while rejecting the noise. Therefore, it is derived through the minimization of an error criterion, which can be defined in two ways. The error in the a priori estimate [i.e., (11.89), before the measurement is considered] is

\[
\bar{e}(k) = x(k) - \bar{x}(k) \tag{11.91}
\]

while the error in the a posteriori estimate (i.e., after the measurement is considered) is:
Part II. Analysis and Control of State Space Systems

\[ e(k) = x(k) - \hat{x}(k) \]  

(11.92)

These errors are of course measures of the accuracy of the estimates. However, because they now depend on noise and are therefore random variables, it is necessary to also consider their noise statistical properties, for their sample values at any given time may not be indicative of their accuracy. Therefore we define

\[ \mathcal{S}(k) \triangleq E[\varepsilon(k)\varepsilon^T(k)] \]  

(11.93)

and

\[ S(k) \triangleq E[e(k)e^T(k)] \]  

(11.94)

These quantities are known as the a priori and a posteriori error covariances, respectively. Given that our guess at the initial condition of the system is a random one, we will therefore say that \( S(k) = S_0 \).

With the current estimator, we simply formed an error system, Equation (10.100), and selected \( L \) such that the error dynamics were asymptotically stable. With the inclusion of the noise terms, this process is insufficient. Instead, we will find a statistical error criterion and use it to minimize the effects of the noise. If we start by finding the error dynamics, we will later see how the error covariance is minimized:

\[
e(k + 1) = x(k + 1) - \hat{x}(k + 1) \\
= Ax(k) + Bu(k) + Gv(k) - A\hat{x}(k) - Bu(k) \\
= L(k + 1)\left[ Cx(k + 1) + w(k + 1) - C\hat{x}(k + 1) \right] \\
= Ax(k) - A\hat{x}(k) + Gv(k) \\
= L(k + 1)\left[ C\big[Ax(k) + Bu(k) + Gv(k)\big] + w(k + 1) - C\big[A\hat{x}(k) + Bu(k)\big] \right] \\
= \left[ A - L(k + 1)CA \right]e(k) + \left[ G - L(k + 1)CG \right]v(k) - L(k + 1)w(k + 1) \\
= \left[ I - L(k + 1)C \right] \big[ Ae(k) + Gv(k) \big] - L(k + 1)w(k + 1) \]

(11.95)

Using this measure, it is reasonable to optimize the estimator by finding

\[
\min_{L(k+1)} \mathbb{E}\left[ e(k+1) \right] \mathbb{E}\left[ e^T(k+1) \right] = \min_{L(k+1)} \mathbb{E}\left[ e^T(k+1)e(k+1) \right] \]

(11.96)

For such a minimization process, the scalar measure \( \mathbb{E}\left[ e(k+1) \right] \) must then be differentiated with respect to the matrix \( L(k + 1) \). Such differentiation is
undefined. Instead, we recognize that for any vector \( x \), \( x^T x = \text{tr}(xx^T) \), where \( \text{tr}(\cdot) \) is the trace operator. Therefore, (11.96) becomes

\[
\min_{L(k+1)} \|E[e(k+1)]\|^2 = \min_{L(k+1)} E\left[ e^T(k+1)e(k+1) \right] = \min_{L(k+1)} \text{tr}\left( E[e(k+1)e^T(k+1)] \right)
\]

where we have assumed the commutativity of the trace operator and the expectation operator.

This presents us with the need to compute \( S(k+1) \) from (11.95):

\[
S(k+1) = E[e(k+1)e^T(k+1)] = E\left[ (I - L(k+1)C)(Ae(k) + Gv(k)) - L(k+1)w(k+1) \right]
\]

\[
= E\left[ (I - L(k+1)C)(Ae(k) + Gv(k)) - L(k+1)w(k+1) \right]^T
\]

\[
= E\left[ (I - L(k+1)C)(Ae(k) + Gv(k)) - L(k+1)w(k+1) \right]^T (I - L(k+1)C)^T
\]

\[
- 2L(k+1)w(k+1)\left( e^T(k)A^T + v^T(k)G^T \right) (I - L(k+1)C)^T
\]

\[
+ L(k+1)w(k+1)w^T(k+1)L^T(k+1)
\]

(11.98)

Because of our assumptions on the statistics of the noise terms, we have that

\[
E\left[ w(k+1)w^T(k+1) \right] = W
\]

\[
E\left[ v(k)v^T(k) \right] = V
\]

(11.99)

\[
E\left[ w(k+1)v^T(k) \right] = 0
\]

It is also easy to compute that

\[
E\left[ w(k+1)e^T(k) \right] = E\left[ w(k+1)(x(k) - \hat{x}(k))^T \right]
\]

\[
= E\left[ w(k+1)x^T(k) \right] - E\left[ w(k+1)\hat{x}^T(k) \right]
\]

\[
= 0
\]
and

\[ E[v(k)e^T(k)] = E[v(k)(x(k) - \hat{x}(k))^T] \\
= E[v(k)x^T(k)] - E[v(k)\hat{x}^T(k)] \]  

(11.100)

[We should be careful to remember that \( x(k + 1) \) and \( v(k) \) are correlated through Equation (11.85).] Using these results and the fact that \( E[e(k)e^T(k)] = S(k) \), we may reduce (11.98) to

\[ S(k + 1) = [I - L(k + 1)C]^T AS(k)A^T + GVG^T]I - L(k + 1)C] \]

(11.101)

\[ + L(k + 1)WL^T(k + 1) \]

Note that this equation represents the “dynamics” of the error covariance \( S(k) \) in the sense that it is a recursion relationship for \( S(k + 1) \) in terms of \( S(k) \). However, at this point the equation is not entirely useful, as we do not yet know \( L(k + 1) \). To determine \( L(k + 1) \), we must find the trace of \( S(k + 1) \) as given in (11.101), then differentiate it with respect to \( L(k + 1) \):

\[ tr[S(k + 1)] = tr\{AS(k)A^T + GVG^T \}
+ L(k + 1)C[AS(k)A^T + GVG^T]C^T L^T(k + 1) \\
- 2L(k + 1)C[AS(k)A^T + GVG^T] + L(k + 1)WL^T(k + 1) \} \]

(11.102)

When we take the derivative with respect to \( L(k + 1) \), the first two terms will give zero because they do not depend on \( L(k + 1) \). The derivatives of the other terms are computed with the help of some interesting identities for derivatives of the trace of matrix quantities, see Appendix A, Equations (A.78) and (A.79):

\[ \frac{\partial}{\partial X} tr(XY^T) = 2XY \hspace{1cm} \frac{\partial}{\partial X} tr(YZX) = Y^T Z^T \]

Using these,
\[
\frac{\partial}{\partial L(k+1)} tr[S(k+1)] = 2L(k+1)C\left[AS(k)A^T + GVG^T\right]C^T
- 2\left[AS(k)A^T + GVG^T\right]C^T + 2L(k+1)W = 0
\]

which has the solution

\[
L(k+1) = \left[AS(k)A^T + GVG^T\right]C^T \left\{C\left[AS(k)A^T + GVG^T\right]C^T + W\right\}^{-1} \quad (11.103)
\]

This is the best gain for the estimator given in Equation (11.90). It is known as the Kalman gain. Note that even if all the system matrices are time-invariant, this gain will vary with time because of its dependence on \( S(k) \), which evolves according to (11.101).

Together with (11.89), (11.90) and (11.101), the Kalman gain given in (11.103) completes the specification of the Kalman filter. To summarize, the following sequence of computations may be performed:

1. The “time” update can be computed with (11.89):

\[
\bar{x}(k+1) = A\hat{x}(k) + Bu(k) \quad (11.104)
\]

which is initialized with the initial guess \( x(k_0) \).

2. The Kalman gain is computed according to (11.103):

\[
L(k+1) = \left[AS(k)A^T + GVG^T\right]C^T \left\{C\left[AS(k)A^T + GVG^T\right]C^T + W\right\}^{-1} \quad (11.105)
\]

which may be initialized with the original covariance \( S(k_0) = S_0 \).

3. The measurement update can then be applied to the time update according to (11.90):

\[
\hat{x}(k+1) = \bar{x}(k+1) + L(k+1)\left[ y(k+1) - C\bar{x}(k+1) \right] \quad (11.106)
\]

where \( y(k+1) \) is the measurement from the plant. This can be further simplified as
\[
\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(k+1)\{y(k+1) - C[A\hat{x}(k) + Bu(k)]\}
\]
\[
= (I - L(k+1)C)A\hat{x}(k) + [I - L(k+1)C]Bu(k) + L(k+1)y(k+1)
\]
\[
= (I - L(k+1)C)[A\hat{x}(k) + Bu(k)] + L(k+1)\gamma(k+1)
\]

(11.107)

4. The accuracy of the estimate is gauged and the next iteration of (11.103) is prepared for by calculating the error covariance update according to (11.101):

\[
S(k+1) = [I - L(k+1)C][A\hat{x}(k) + Bu(k)] + L(k+1)\gamma(k+1)
\]

(11.108)

The process then repeats itself at the next time step.

**Simplifications of the Kalman Filter**

The Kalman filter can be written in a number of different ways, some for simplicity, some for numerical properties, and some for physical insight. For example, the calculation of the Kalman gain in (11.103) can be simplified if we investigate the significance of the term that appears twice in that relationship, \(AS(k)A^T + GVGT\). Consider taking the “time” update of the error covariance \(S\), just as we did with the state estimate. Then from (11.93),

\[
\tilde{S}(k+1) = E\left[\tilde{e}(k+1)\tilde{e}^T(k+1)\right]
\]

\[
= E\left[[x(k+1) - \bar{x}(k+1)][x(k+1) - \bar{x}(k+1)]^T\right]
\]

\[
= E\left[[Ax(k) + Bu(k) + Gv(k) - [A\hat{x}(k) + Bu(k)]]^T\right]
\]

\[
= E\left[[Ae(k) + Gv(k)][Ae(k) + Gv(k)]^T\right]
\]

\[
= AS(k)A^T + GVGT
\]

(11.109)

Thus, the Kalman gain can be more efficiently expressed as

\[
L(k+1) = \tilde{S}(k+1)C^T[C\tilde{S}(k+1)C^T + W]^{-1}
\]

(11.110)
if Equation (11.109) is first used to compute $\bar{S}(k+1)$.
Likewise, (11.101) could then be written

$$S(k+1) = \left[ I - L(k+1)C \right] \bar{S}(k+1) \left[ I - L(k+1)C \right]^T + L(k+1)WL^T (k+1)$$ (11.111)

This, in fact, can be further simplified:

$$S(k+1) = \left[ I - L(k+1)C \right] \bar{S}(k+1) \left[ I - L(k+1)C \right]^T + L(k+1)WL^T (k+1)
= \left[ I - L(k+1)C \right] \bar{S}(k+1) + L(k+1)WL^T (k+1)
= \left[ I - L(k+1)C \right] \bar{S}(k+1) - \bar{S}(k+1)C^T L^T (k+1)
+ L(k+1)C \bar{S}(k+1)C^T + W \right]^T (k+1)
= \left[ I - L(k+1)C \right] \bar{S}(k+1)
$$ (11.112)

which, if (11.109) is used to find $\bar{S}(k+1)$, can be readily substituted for (11.101).

Now, combining (11.112) and (11.110),

$$S(k+1) = \left[ I - L(k+1)C \right] \bar{S}(k+1)
= \left[ I - \bar{S}(k+1)C \right] \left[ C \bar{S}(k+1)C^T + W \right]^{-1} C \bar{S}(k+1)
= \bar{S}(k+1) - \bar{S}(k+1)C \left[ C \bar{S}(k+1)C^T + W \right]^{-1} C \bar{S}(k+1)
$$
or

$$S(k) = \bar{S}(k) - \bar{S}(k)C \left[ C \bar{S}(k)C^T + W \right]^{-1} C \bar{S}(k)$$ (11.113)

Equation (11.113) represents the effect of the measurement on the a priori error covariance. Comparing this formula to the matrix inversion lemma given in Appendix A results in
Part II. Analysis and Control of State Space Systems

\[ S(k) = \bar{S}(k) - \bar{S}(k)C^T\left[C\bar{S}(k) + W\right]^{-1}C\bar{S}(k) \]
\[ = \left[\bar{S}(k) + C^TW^{-1}C\right]^{-1} \]

(11.114)

This equation can also be used as an update for the error covariance. Inverting it gives the backward relationship:

\[ \bar{S}(k) = \left[S^{-1}(k) - C^TW^{-1}C\right]^{-1} \]

or

\[ \bar{S}(k + 1) = \left[S^{-1}(k + 1) - C^TW^{-1}C\right]^{-1} \]

(11.115)

To further simplify the expression for the Kalman gain, take (11.110) and again use the matrix inversion lemma:

\[ L(k + 1) = \bar{S}(k + 1)C^T\left[C\bar{S}(k + 1)C^T + W\right]^{-1} \]
\[ = \bar{S}(k + 1)C^T\left[W^{-1} - W^{-1}C\left[C^TW^{-1}C + \bar{S}^{-1}(k + 1)\right]^{-1}C^TW^{-1}\right] \]
\[ = \bar{S}(k + 1)\left[I - C^TW^{-1}C\left[C^TW^{-1}C + \bar{S}^{-1}(k + 1)\right]^{-1}\right]C^TW^{-1} \]

(11.116)

Now substitute (11.115) into (11.116):

\[ L(k + 1) = \left[S^{-1}(k + 1) - C^TW^{-1}C\right]^{-1} \cdot \left[I - C^TW^{-1}C\left[C^TW^{-1}C + S^{-1}(k + 1) - C^TW^{-1}C\right]^{-1}\right]C^TW^{-1} \]
\[ = \left[S^{-1}(k + 1) - C^TW^{-1}C\right]^{-1}\left[I - C^TW^{-1}CS(k + 1)\right]C^TW^{-1} \]
\[ = \left[S^{-1}(k + 1) - C^TW^{-1}C\right]^{-1}\left[S^{-1}(k + 1) - C^TW^{-1}C\right]S(k + 1)C^TW^{-1} \]
\[ = S(k + 1)C^TW^{-1} \]

(11.117)

This will further reduce (11.90) to the form:

\[ \hat{x}(k + 1) = \bar{x}(k + 1) + S(k + 1)C^TW^{-1}[y(k + 1) - C\bar{x}(k + 1)] \]

(11.118)
Note the similarity between (11.117) and the formula for state feedback gain $K$ from Equation (11.55), particularly its dependence on the matrix $W$ and the time-varying matrix quantity $S(k+1)$. To develop this similarity further, one additional manipulation is in order. Inserting (11.113) into (11.109) gives

$$
\bar{S}(k+1) = A\bar{S}(k)A^T + GV^T \\
= A\left\{ \bar{S}(k) - (\alpha(k) + \beta(k))C^T\bar{S}(k)C + W \right\} A^T + GV^T \\
= A\bar{S}(k)A^T - A\bar{S}(k)C^T\left\{ \bar{S}(k)C + W \right\}^{-1} C\bar{S}(k)A^T + GV^T \\
$$

(11.119)

This can be compared to (11.22) to realize that it is a difference (discrete-time) Riccati equation that governs the time update of the error covariance. This is a remarkable and important fact and again illustrates the inherent duality seen so often in linear systems. More will be said concerning this duality following the next section, in which we derive the continuous-time counterpart to this Kalman filter [8].

### 11.2.3 The Continuous-Time Kalman Filter

The discrete-time Kalman filter was derived from the formulation of the current estimator, as presented in Section 10.3.2. We will begin the derivation of the continuous-time Kalman filter in the same way, but we will then proceed somewhat differently. In practice, the continuous-time Kalman filter is rarely used because it is almost always implemented on a computer, in which case the discrete-time version is more natural. However, it has some interesting parallels to the continuous-time LQ controller and has some steady-state properties that are important to the theory of analog signal processing.

Assume that the continuous-time system equations are of the form

$$
\dot{x}(t) = Ax(t) + Bu(t) + Gv(t) \\
y(t) = Cx(t) + w(t) \\
$$

(11.120)

where, as with the discrete-time case, the system could be time varying, although we will drop the $t$ arguments for simplicity. Analogously to (11.85), we have included in this model the two white, zero-mean, mutually uncorrelated noise signals $v(t)$ and $w(t)$. Following (11.86) and (11.87), we assume that

$$
E[v(t)v^T(\tau)] = V\delta(t - \tau) \\
E[v(t)] = 0 \\
$$

(11.121)

and
Part II. Analysis and Control of State Space Systems

\[ E[w(t)w^T(\tau)] = W\delta(t - \tau) \]
\[ E[w(t)] = 0 \]  
(11.122)

where \( \delta(t) \) is the Dirac delta and \( E[w(t)w^T(\tau)] = 0 \). Furthermore, we assume the initial guess at the system state to be denoted by \( x(t_0) = \hat{x}_0 \) and let it be uncorrelated with the plant and measurement noise:

\[ E[x_0v^T(\tau)] = 0 \quad E[x_0w^T(\tau)] = 0 \]

The covariance of the initial guess is defined as

\[ E\left\{ \left[ x_0 - E(x_0) \right] \left[ x_0 - E(x_0) \right]^T \right\} = P_0 \]
(11.123)

Again, we need to determine the estimator that best estimates the state of (11.120) while rejecting the influence of the noisy inputs and random initial condition. As before, we will pose this problem as an observer design, where the observer gain will be chosen to minimize an error criterion. For the observer, we cannot use the separate “time” and “measurement” updates, because time is continuous. The estimator is therefore proposed in the structure

\[ \hat{x}(t) = A\hat{x}(t) + Bu(t) + L(t)\left[ y(t) - C\hat{x}(t) \right] \]  
(11.124)

Having discovered that the discrete-time Kalman gain \( L(k + 1) \) was time-varying even for time-invariant plants, we should be prepared for the gain \( L(t) \) to be a function of time as well.

First, we construct the estimation error signal \( e(t) = x(t) - \hat{x}(t) \) and then find its derivative to determine the error dynamics:

\[ \dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) \]

\[ = Ax + Bu + Gv - A\hat{x} - Bu - L(Cx + w - C\hat{x}) \]
(11.125)

\[ \quad = (A - LC)e(t) + Gv(t) - Lw(t) \]

Because we are not interested in the covariance of \( \dot{e} \), we will solve (11.125) according to the methods of Chapter 6. First, let \( \Phi(t, t_0) \) denote the state-transition matrix for the error system in (11.125). Then the complete solution of (11.125) can be written as
Chapter 11. Introduction to Optimal Control and Estimation

\[ e(t) = \Phi(t, t_0) e(t_0) + \int_{t_0}^{t} \Phi(t, \tau) \left[ G v(\tau) - L w(\tau) \right] d\tau \] (11.126)

Now finding the error covariance from this expression:

\[ P(t) = E \left[ e(t) e^T(\tau) \right] \]
\[ = E \left[ \Phi(t, t_0) e(t_0) + \int_{t_0}^{t} \Phi(t, \tau) \left[ G v(\tau) - L w(\tau) \right] d\tau \right] \cdot (11.127) \]
\[ = \left[ \Phi(t, t_0) e(t_0) + \int_{t_0}^{t} \Phi(t, s) \left[ G v(s) - L w(s) \right] ds \right]^T \]

In simplifying this expression, we immediately recognize that because of the mutual uncorrelatedness of the signals \( x(t), v(t), \) and \( w(t), \) the cross-terms within the braces of (11.127) will have an expected value of zero. Therefore, (11.127) reduces to

\[ P(t) = E \left[ \Phi(t, t_0) e(t_0) e^T(t_0) \Phi^T(t, t_0) \right. \]
\[ + \left. \int_{t_0}^{t} \int_{t_0}^{s} \Phi(t, \tau) \left[ G v(\tau) v^T(s) G^T + L w(\tau) w^T(s) L^T \right] \Phi^T(t, s) d\tau ds \right] \]
\[ = \Phi(t, t_0) P_0 \Phi^T(t, t_0) + \int_{t_0}^{t} \int_{t_0}^{s} \Phi(t, \tau) \left[ G v(\tau) v^T(s) G^T + L w(\tau) w^T(s) L^T \right] \Phi^T(t, s) d\tau ds \]
\[ = \Phi(t, t_0) P_0 \Phi^T(t, t_0) + \int_{t_0}^{t} \Phi(t, s) \left[ G v^T + L w^T \right] \Phi^T(t, s) ds \] (11.128)

Next we take the derivative of (11.128) with respect to time \( t \) in order to determine the dynamics of the error covariance.*

* See Appendix A, Equation (A.94), for the formula for finding the derivative of an integral with respect to one of its arguments.
We can immediately recognize this as a differential matrix Riccati equation for the error covariance $P(t)$ whose initial condition is $P(t_0) = P_0$. However, we have not yet optimized the norm of the error over all possible gains $L(t)$.

To perform the optimization, we will attempt to minimize the squared error at any time $t$. This squared error may be expressed as

$$E[e^T(t)e(t)] = tr\left\{E[e^T(t)e(t)]\right\}$$

To do this, we ask that the negative change in $P$ be as large as possible at any given instant, i.e., if the derivative of $P$ is as large (and negative) as possible at every instant, then $P$ is decreasing at a maximal rate, and $P$ itself will be minimized over time. Therefore we will seek to find

$$\max_{L(t)} \left\{tr\left[\dot{P}(t)\right]\right\} = \max_{L(t)} \left\{tr\left[(A-LC)P + P(A-LC)^T + GV^T + LWL^T\right]\right\}$$

Proceeding with this maximization,
\[
\frac{\partial}{\partial L} \left[ \text{tr} \left( \dot{P} \right) \right] = \frac{\partial}{\partial L} \left[ \text{tr} \left[ (A - LC)P + P(A - LC)^T + GV^T + LW^T \right] \right] \\
= \frac{\partial}{\partial L} \text{tr} (-LCP) + \frac{\partial}{\partial L} \text{tr} \left(-PC^T L^T\right) + \frac{\partial}{\partial L} \text{tr} \left(LW^T\right) \\
= -2P(t)C^T + 2L(t)W \\
= 0
\]
giving

\[
L(t) = P(t)C^TW^{-1}
\]  

This is the Kalman gain for the continuous-time Kalman filter.\(^M\) Note the similarity between it and the discrete-time Kalman gain in (11.117).

Using (11.133) for the gain, the error covariance dynamics simplify as well:

\[
\dot{P}(t) = (A - LC)P(t) + P(t)(A - LC)^T + GV^T + LW^T \\
= AP(t) - P(t)C^TW^{-1}CP(t) + P(t)A - P(t)C^TW^{-1}CP(t) \\
 \quad + GV^T + P(t)C^TW^{-1}WW^{-1}CP(t) \\
= AP(t) + P(t)A^T - P(t)C^TW^{-1}CP(t) + GV^T
\]  

This is recognizable as a differential matrix Riccati equation, independent of the gain \(L\).

### 11.2.4 Properties of Kalman Filters

**Optimality of Kalman Filters**

In each of the derivations above, an error criterion was minimized in order to arrive at an “optimal” estimator gain. The error criterion was chosen as the squared error, and it was used to optimize the behavior of the observer structures that were proposed, Equations (11.90) and (11.124). However, these observers were chosen merely because of their familiarity. There is, in most cases, no guarantee that the estimator equations, (11.90) or (11.124), are the best possible estimator structures for a given linear system. The exception is when the noise signals \(vu\) and \(w\) are gaussian. If the plant and measurement noise are both gaussian, then the Kalman filters we have presented are indeed the best possible estimators of the plant’s state. If the noise signals follow a different probability distribution, then the Kalman filters are the best linear estimators of the state. To show this would require a derivation from statistical principles that we will not explore.
Steady-State Behavior

It may be noticed that the discrete and continuous-time Kalman filters above have strong similarities to the LQ controllers derived in Section 11.1. In particular, each system required the solution of a Riccati equation in parallel with the computation of the optimal gain. In the LQ controller, this was a backward time Riccati equation, and in the Kalman filter, it is a forward-time equation. When discussing the Riccati equations for the LQ controllers, we took some time to investigate their steady-state behavior in the hopes that a constant solution would lead to an approximately optimal system that is easier to implement numerically. Clearly, the same questions can now be asked regarding the Kalman filter. Thus, when does the Riccati equation that governs the Kalman filter have a stabilizing steady state that is also a solution of the corresponding algebraic Riccati equation? We will briefly answer this question by appealing to the duality of the equations.

Consider the continuous-time case, i.e., the LQ controller’s Riccati equation, (11.53), and the Kalman filter’s Riccati equation, (11.134). For convenience, they are repeated here:

LQ controller:
\[ \dot{P} = PBR^{-1}B^T P - Q - PA - A^T P \]
\[ K = -R^{-1}B^T P \]  \hspace{1cm} (11.135)

Kalman filter:
\[ \dot{P} = -PC^T W^{-1} CP + GVG^T + AP + PA^T \]
\[ L = PC^T W^{-1} \]  \hspace{1cm} (11.136)

In the case of the LQ controller, the algebraic version of (11.135) needed the following conditions to provide a unique positive-definite stabilizing solution that corresponds to the steady-state value of the differential equations: \((A,B)\) must be stabilizable, and \((A,T)\) must be detectable, where \(Q = T^T T\). By simply comparing Equations (11.135) and (11.136) above, the following can be said about the steady-state behavior of the Kalman filter:

THEOREM: The algebraic Riccati equation has a unique, stabilizing, positive-definite solution \(P\) that is the steady-state solution of the differential equation in (11.136) if and only if \((A^T, C^T)\) is stabilizable, and \((A^T, T^T)\) is detectable, where \(GVG^T = TT^T\).  \hspace{1cm} (11.137)
Chapter 11. Introduction to Optimal Control and Estimation

If these conditions are met, then the error dynamics, (11.125), will be stable and the Kalman filter will converge. The meanings of these conditions can be better understood by their stabilizability and detectability duality. First, stabilizability of \((A^T, C^T)\) is equivalent to detectability of \((A, C)\). This is the expected necessity for the unstable modes of the system to be observable. This condition is fairly obvious. Second, detectability of \((A^T, T^T)\) is equivalent to stabilizability of \((A, T)\). This condition is less obvious. In essence it dictates that the noise term \(v(t)\) must somehow affect each unstable mode of the system. The system is therefore not stabilizable unless it is sufficiently corrupted by noise! This will prevent the error covariance for this mode from approaching zero, which would thereby keep such a mode from affecting the error dynamics. Naturally, similar analogies can be drawn for the discrete-time LQ controller and Kalman filters.

Example 11.3: A Continuous-Time Kalman Filter

Generate and simulate a continuous-time Kalman filter to estimate the state variables of the system:

\[
\dot{x} = \begin{bmatrix} -4 & 2 \\ -2 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ -1 \end{bmatrix} v
\]

\[
y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + w
\]  

(11.138)

where the noise term \(v(t)\) has zero mean and covariance \(V = 0.09\). The measurement noise term is assumed to have zero mean and covariance \(W = 0.25\). Use as an input \(u(t) = \sin t\) over a period of \(t \in (0, 10)\) s. Guess that the initial state of the plant is \(x(0) = [0.5 \ -0.5]^T\), with a covariance of this initial estimate of \(P_0 = I_{2\times2}\).

Solution:

Because Equations (11.124), (11.133), and (11.134) completely describe the Kalman filter, there is very little analysis necessary before simulating the system. The result is shown in Figure 11.7. In the plot, the noisy state variables are clearly seen to be filtered by the estimator, giving smoothed versions. In Figure 11.7, the time-varying Kalman filter is used with the Riccati equation solved explicitly using MATLAB. The resulting error covariance is plotted in Figure 11.8. Note that the elements of \(P(t)\) reach a steady state value relatively quickly. Note also that the time at which the error covariance reaches its approximate steady-state (\(\sim 0.5\) s) is also approximately the time at which the estimates approach the true states.
Figure 11.7  True and estimated state variables using the continuous-time Kalman filter for the system of Example 11.3. Shown are the filtered estimates using time-varying gains.

Figure 11.8  Elements of the error covariance matrix $P(t)$ from numerical solution of the differential matrix Riccati equation.

Because of the difficulty of solving the Riccati equation (11.134), we will also consider the steady-state Kalman gain formulation. The relatively flat values
Chapter 11. Introduction to Optimal Control and Estimation

for $P(t)$ in Figure 11.8 suggest that this will be a good approximation. To do this, we must first verify that the conditions on $(A^T, C^T)$ and $(A^T, T^T)$ are true, according to the theorem above (they are). When the corresponding algebraic Riccati equation is solved, we get

$$P(t) = \begin{bmatrix} 0.0224 & -0.0299 \\ -0.0299 & 0.0522 \end{bmatrix} \quad (11.139)$$

giving a constant Kalman gain of

$$L = \begin{bmatrix} -0.0549 \\ 0.0598 \end{bmatrix} \quad (11.140)$$

The result of applying this constant gain is shown in Figure 11.9. Comparison of this result with that of Figure 11.7 indicates that the approximation of the Kalman gain as constant is a reasonable one.

![Figure 11.9](image)

*Figure 11.9* True and estimated state variables when the algebraic Riccati equation is used to compute a constant Kalman gain. Shown are the filtered estimates with steady-state gains.

The open-loop system given in (11.138) happens to be asymptotically stable. Appropriate choice of the estimator gain $L$ ensures that the estimator itself is stable, so we have the estimate tracking seen in the figures. Note however that we
are still simulating an open-loop system, i.e., one without a controller. If the plant itself were unstable, we could still design a stable Kalman filter that would track the unstable plant, even while the plant state variables diverge to infinity. To stabilize the plant itself, the Kalman filter must be combined with a controller, such as the LQ regulator. This is our next topic.

### 11.3 LQG Control

The next logical step in controller design is to combine the LQ regulator with the Kalman filter so that a system with noise can be controlled to minimize a cost criterion. Of course if the system contains plant and/or measurement noise, then the cost to be minimized must in fact be an expected value. For continuous time, we can therefore state the LQG control problem for the state space equations as follows:

\[
\begin{align*}
\dot{x} &= Ax + Bu + Gv \\
y &= Cx + w
\end{align*}
\]

(11.141)

find the control input \( u^*(t) \) that minimizes the cost criterion

\[
\tilde{J}(x(t_0), t_0) = E\left\{ \frac{1}{2} x^T(t_0)Sx(t_0) + \frac{1}{2} \int_{t_0}^{t} [x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau)] d\tau \right\}
\]

(11.142)

The complete derivation of the solution to this problem is beyond the scope of this book. However, we are already prepared to use the solution based on our investigations of the deterministic LQ control problem and the Kalman filter. This is because the solution of the LQ problem for stochastic systems (i.e., with the noise models we have introduced) is exactly the same as the deterministic LQ problem. Thus, the optimal feedback control is of the form (11.55), where the matrix \( P(t) \) is the solution of the differential matrix Riccati equation, (11.53).

This solution will be truly optimal when the noise is gaussian. The steady-state solution with constant \( P \) may of course be used when the infinite-horizon approximation is made.

There is a distinction between the deterministic and stochastic problems, but it occurs in the computation of the overall cost of a control, i.e., \( \tilde{J} \). If the feedback in (11.55) is applied to a system with plant noise in the form of (11.120), the cost \( \tilde{J} \) will include a term for the effect of the noise. This is unavoidable as the noise affects the plant states. However, in practice the cost is not usually computed. The cost criterion is merely the optimization guideline, and its numerical value is not very meaningful.

The feedback solution (11.55) presumes that the state \( x(t) \) is available for computation of \( u(t) \). This is sometimes referred to as the “complete information”
LQG problem. As we are well aware, though, the state is not always available, and must often be estimated. This is the “incomplete information” LQG problem. To solve it, we simply employ the Kalman filter in order to generate the “optimal” state estimate, then use this estimate to compute the state feedback. Of course, the Kalman filter estimates the state variables from noisy measurements, and as a result, this measurement noise appears (though filtered) in the estimates themselves, which are then included in the state feedback. Therefore, some additional noise will appear in the overall closed-loop cost, just as the plant noise resulted in a term in this cost. However, the extra cost due to the plant noise is independent of the extra cost due to the measurement noise. This fact implies the separation principle for LQG control: The best linear controller/estimator for the stochastic system, \((11.120)\), consists of the LQ controller, \((11.135)\), and the Kalman filter, \((11.136)\). The gains of each can be computed independently. The optimal control will therefore be

\[
 u^*(t) = -R^{-1}B^TP(t)\hat{x}(t) \tag{11.143}
\]

where \(P(t)\) is the solution to \((11.135)\) and \(\hat{x}(t)\) is the state vector estimate provided by the Kalman filter. All of these results apply, with only a change to the corresponding discrete-time governing equations, to discrete-time systems as well as continuous-time systems.

**Example 11.4: LQG Controller for Disturbance Rejection**

We will end this chapter with an example illustrating the LQG controller. Consider a plant with state equations:

\[
\dot{x} = Ax + bu = \begin{bmatrix} 0 & 1 \\ -50 & -10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u 
\]

\[
y = cx + w = \begin{bmatrix} 20 & 0 \end{bmatrix} x + w 
\]

where \(w(t)\) is zero-mean white noise with covariance \(W = 0.01\). It is observed that, in operation, the state vector of this plant is corrupted by a noisy signal of approximately 4 Hz, i.e., a narrow-band noise signal. The goal of the control problem is to reject this 4-Hz disturbance.

To model the disturbance, we will assume that there is a dynamic system of the form

\[
\dot{x}_n = A_n x_n + g_n v = \begin{bmatrix} 0 & 4(2\pi) \\ -4(2\pi) & 0 \end{bmatrix} x_n + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v 
\]

\[
\zeta = c_n x_n = [100 \ 0] x_n 
\]

(11.145)
where $v(t)$ is a zero-mean white noise with covariance $V = 0.01$. This noise therefore excites the system to produce a signal of 4 Hz, scaled by the factor of 100. We will then treat the noisy output of (11.145), i.e., $\zeta$, as a noise-source disturbance for (11.144):

$$
\begin{align*}
\dot{x} &= Ax + bu + g\zeta \\
y &= cx + w
\end{align*}
$$

(11.146)

where $g = [0 \ 1]^T$. Combining the two systems, (11.145) and (11.146), together gives the coupled (“augmented”) system

$$
\begin{align*}
\dot{\xi} &= A_{aug} \xi + b_{aug} u + g_{aug} v \\
y &= c_{aug} \xi + w = [c \ 0] \xi + w
\end{align*}
$$

(11.147)

where $\xi = [x \ x_n]^T$.

To control this system, we will compute state feedback of the form $u = K\hat{\xi}$, where $\hat{\xi}$ is an optimal estimate of the true state $\xi$ (which is clearly not available, because we have modeled the disturbance $\zeta$ as the output of a hypothetical plant). First, we will generate an optimal feedback gain $K$ by minimizing the cost criterion

$$
J = \int_0^\infty \left( x^T Q x + u^T R u \right) dt
$$

(11.148)

This is the infinite-time LQ criterion for which we must solve the algebraic Riccati equation, (11.135). We will choose weighting matrices as

$$
Q = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \quad R \ \text{variable}
$$

(11.149)

This form of $Q$ penalizes the “position” coordinate of the state vector (note that the original plant is in “phase-variable” form) but not the velocity coordinate. It does not penalize the “noise” states, which are not controllable anyway. We will
use two different values for \( r \) to see the effect of each. Finding the solution of the Riccati equation, and using (11.55) to form state-feedback matrix \( K \), we now set 
\[ u = K \hat{\xi} , \]
where \( \hat{\xi} \) is the output of an estimator.

To construct the estimator, assume the form of the Kalman filter,
\[ \dot{\hat{\xi}} = A_{\text{aug}} \hat{\xi} + b_{\text{aug}} u + L (y - c_{\text{aug}} \hat{\xi}) \]  
(11.150)

and, with the known noise covariances, solve the algebraic Riccati equation necessary to compute the optimal Kalman gain \( L \) from (11.133). With this gain, an overall composite system can be constructed as
\[
\begin{align*}
\begin{bmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2
\end{bmatrix} &= 
\begin{bmatrix}
A_{\text{aug}} & b_{\text{aug}} K \\
L c_{\text{aug}} & A_{\text{aug}} - L c_{\text{aug}} + b_{\text{aug}} K
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\
\xi_2
\end{bmatrix} + 
\begin{bmatrix}
0 & 0 \\
g_{\hat{\xi}} & 0
\end{bmatrix}
\begin{bmatrix}
v \\
w
\end{bmatrix}, \\
y &= c_{\text{aug}} \begin{bmatrix}
\xi_1 \\
\xi_2
\end{bmatrix} + 0 \begin{bmatrix}
v \\
w
\end{bmatrix}
\end{align*}
\]  
(11.151)

This system is in a form suitable for simulating in MATLAB, where the noisy input signal is created with the random number generator. The resulting output signal \( y(t) \) is shown in Figure 11.10 below for the open-loop case (i.e., setting \( K = 0 \)) and for \( R = 10^{-6} \) and \( 10^{-8} \).

The important feature to note is that as \( R \) gets smaller, the rejection of the noisy disturbance improves. This is because by penalizing the inputs less, the feedback signal is allowed to grow larger, thus allowing more effective, higher gains [as seen by the inverse effect of \( R \) on feedback in (11.55)]. With \( R = 10^{-6} \), the controller results in moderate improvement, while with \( R = 10^{-8} \), the disturbance has all but disappeared. With a further decrease in the input weighting \( R \), not much further improvement will result because of the “noise floor” of the system.

To get a comparative illustration of the effects of the weighting factor \( R \), we can plot the magnitude of the frequency response function from the disturbance input \( \zeta \) (Equation 11.145) to the output \( y \). These are shown in Figure 11.11, and the block diagram representing such a system is shown in Figure 11.12. Note that the dynamic compensator that we have implicitly created becomes a notch filter at 4 Hz as \( R \) grows larger.
Figure 11.10  Output signal of the system in (11.146), with no feedback (open loop), and with optimal feedback computed using \( R = 10^{-6} \) and \( 10^{-8} \).

Figure 11.11  Frequency responses for the signal path from disturbance \( \zeta \), [Equation (11.145)], to the output \( y \), for the open-loop case (i.e., \( K = 0 \)) and for two different values of \( R \).
This example presents not only an illustration of the use of the LQG controller, but also a method by which “colored” noise may be handled by the Kalman filter. Although we have assumed in the development of the filter that the plant noise was white, colored noise may be modeled with a state space system that is driven by white noise. The “noise system” can then be appended onto the plant, and the Kalman filter may be designed based on the augmented model.

\[ A \quad b \quad c \quad K \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \quad A \quad g \quad n \n
\textbf{Figure 11.12}  Block diagram for the plant and observer, with the noise model included. This diagram can be used to find the frequency response functions shown in Figure 11.11.

\section{11.4 Summary}

As we indicated at the beginning of the chapter, the LQG controller is one of the most basic and important tools in the control engineer’s toolbox. For state space models of systems, it is the first choice control approach, and in most cases it will provide acceptable performance. Certainly, it can be criticized on the basis of
various criteria, such as robustness, but for an accurately known plant it almost always works well.

We have chosen to present the LQG controller here because it also culminates the book well by tying together the concepts of stability, controllability, observability, state feedback, and observers (estimators), as well as introducing practical applications of matrix calculus in the optimization of the various cost and error criteria. While drawing all these skills into a single methodology for control design, the LQG controller illustrates the duality of a linear system even more strongly than in previous chapters. It is quite remarkable that the LQ regulator and the Kalman filter are so similar, both being governed by an optimal gain matrix that depends on the solution of a (difference or differential) matrix Riccati equation. The Riccati equation itself is an interesting mathematical relationship that is still the subject of active research.

To summarize the important topics covered throughout the course of this chapter:

- Discrete- and continuous-time LQ regulators were derived with similar procedures, each based on Bellman’s optimality criterion. This criterion (that an optimal journey consists of a first step plus an optimal journey thereafter) is both elegant and powerful. It was used here to generate optimal gains but is also the basis for other optimization techniques, such as dynamic programming [2] and modern machine learning [12]. We should point out at this point, though, that while the equations for optimal control that we derived suffice for many cases, alternative formulations based on variational calculus can provide results with somewhat different, though ultimately equivalent, formats. Different formulations can provide better insight into a broader class of problems than the simple regulators discussed here. Furthermore, there are many kinds of optimal control that we have not discussed.

- Discrete- and continuous-time Kalman filters were derived using significantly different methodologies. The discrete-time Kalman filter was based on the current estimator approach to state observation, while the continuous-time version had no such starting point and therefore had to be derived differently. In fact, although the Kalman filters we present here are indeed accurate and equivalent to the filters derived from any other method, we have completely ignored many of the statistical technicalities that must be considered to prove the filter “optimal.” Like the LQ regulator, Kalman filtering is a discipline in itself, and we urge caution in the use of this chapter. We have presented only the most basic formulation of the filter, with the most common model from which to start. Kalman filter behavior is complex and deserves a more thorough treatment if the filters are to be thoroughly appreciated.

- The LQ controller and Kalman filter both depend on a matrix Riccati equation, which is either a difference equation (discrete-time) or a
differential equation (continuous-time). These equations will reach an optimizing steady state under certain stabilizability and detectability conditions. Such a steady-state result will provide a suboptimal but often acceptable approximation to the optimal controller (estimator). In practice, it is usually the steady-state solution that is used because of the difficulty in solving the dynamic Riccati equations.

- We have unapologetically given the definition of the LQG controller as the simple sum of an LQ controller and a Kalman filter. This is the practical definition of an LQG controller. However, the justification for this claim was not given here as it requires too lengthy a derivation and too much statistical analysis to justify in a one-chapter treatment. Nevertheless, for a state space model of a plant, if a reader designs an LQ controller and a Kalman filter from the basic techniques presented here, a good LQG controller design will result.

It would seem that this chapter raises as many unresolved questions about controller and observer design as it answers. A good chapter summary always points toward the next unsolved problem, and this one is no exception. However, as this concludes the basic information needed for competency in linear control systems, we simply invite the reader to pursue additional reading from the many excellent references in linear systems, control systems, and optimization theory.

### 11.5 Problems

11.1 Write the MATLAB program necessary to solve the discrete-time LQ problem given in Example 11.1.

11.2 For the discrete-time system

\[
x(k + 1) = \begin{bmatrix} 0.6 & 0.3 \\ 0.1 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 0.03 \\ 0.1 \end{bmatrix} u(k)
\]

Find the control sequence that minimizes the cost function

\[
J = \frac{1}{2} \sum_{k=1}^{19} \left( x(k)^T \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} x(k) + u^2(k) \right)
\]

11.3 For the discrete-time system given as
Part II. Analysis and Control of State Space Systems

\[ x(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \quad x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

determine the feedback control that minimizes the cost criterion

\[ J = \frac{1}{2} \sum_{i=1}^{\infty} [x_{1}^{2}(i) + u_{1}^{2}(i)] \]

11.4 Show how Equations (11.58) and (11.59) can be combined to form (11.60).

11.5 Consider the definition of the Kronecker product for two matrices \( A \) and \( B \), see Equation (A.10):

\[ A \otimes B = \begin{bmatrix} a_{ij}B \end{bmatrix} \]

[This means that if \( A \) is \( m \times n \), and \( B \) is \( p \times q \), then \( A \otimes B \) is \( mp \times nq \) and the \( (i, j)^{th} \) \( p \times q \) block of \( A \otimes B \) is \( a_{ij}B \).] Use the stacking operator also defined in Appendix A, Equation (A.10), and devise a method by which the differential matrix Riccati equation, (11.53), can be transformed into a vector differential Riccati equation and thereby numerically solved with MATLAB's ODE routines [9].

11.6 For the system

\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ -10 & -7 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \]

Find a control \( u(t) \) that minimizes the performance measure

\[ J[x(t), t] = 10x_{1}^{2}(5) + \frac{1}{2} \int_{0}^{5} [5x_{1}^{2}(t) + x_{2}^{2}(t) + 0.25u^{2}(t)] dt \]

Then discretize the system using a step size of \( T = 0.05 \) and find a control that minimizes the performance measure

\[ J[x(k), k] = 10x_{1}^{2}(100) + \frac{1}{2} \sum_{k=1}^{9} [5x_{1}^{2}(k) + x_{2}^{2}(k) + 0.25u^{2}(k)] \]
11.7 For a given single-input continuous-time system, show how the modified LQ cost criterion

\[ J = \int_0^\infty e^{2s} \left[ x^T(t)Qx(t) + ru^2(t) \right] dt \]

can be minimized. Show that the resulting closed-loop poles will be to the left of \( s = -\lambda \).

11.8 A scalar system has the equations

\[ \dot{x} = -x + u \quad x(0) = 1 \]

Find the feedback control that minimizes the \textit{minimum energy} criterion

\[ J = x(1) + \int_0^1 u^2(t) \, dt \]

11.9 Point-masses under the influence of a force are described by Newton’s law \( F = mx'' \), and are known as “double integrators.” Consider the double-integrator system given by:

\[ \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
 y = cx \]

Find and simulate the optimal control system that minimizes the cost functional

\[ J = \int_0^\infty \left( y^T y + u^T u \right) dt \]

given the two \( c \)-matrices \( c_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \). Determine all solutions that exist if \( c_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \).

11.10 Consider the system given by
Part II. Analysis and Control of State Space Systems

\[
\dot{x} = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
\]
\[
y = \begin{bmatrix} 0 & 1 \end{bmatrix} x
\]

Suppose we desire to use output feedback of the form \( u = ky \) to control the system such that the cost criterion

\[
J = \int_{0}^{\infty} \left( y^2 + u^2 \right) dt
\]

is minimized. Find the cost \( J \) in terms of feedback gain \( k \) by solving a Lyapunov equation similar to Equation (11.67). Plot this cost versus \( k \) and determine the minimum \( J \) and the \( k \) at which it occurs. Then plot the root locus of the system and show where the chosen \( k \) places the poles.

11.11 A two-link robot arm shown in Figure P11.11 is underactuated in the sense that it has a motor at the elbow joint but not at the base [11].

![Two-link robot arm diagram]

\( P11.11 \)

The equations of motion for the robot in terms of the two angles \( \theta_1 \) and \( \theta_2 \) and the input torque \( \tau \) are

\[
(2.66 + 2 \cos \theta_2) \ddot{\theta}_1 + (1.33 + \cos \theta_2) \ddot{\theta}_2 \\
-(\sin \theta_2) \dot{\theta}_1^2 - 2(\sin \theta_2) \dot{\theta}_1 \dot{\theta}_2 + 24.5 \cos \theta_1 + 9.8 \cos(\theta_1 + \theta_2) = 0
\]

\[
(1.33 + \cos \theta_2) \dot{\theta}_1 + 1.33 \dot{\theta}_2 + (\sin \theta_2) \dot{\theta}_1^2 + 9.8 \cos(\theta_1 + \theta_2) = \tau
\]

Let the state variables be
and linearize the equations about the equilibrium point 
\( x_0 = [0 \ 0 \ 0 \ 0]^T \). Then design an LQ controller that minimizes the cost function

\[
J = \int_0^\infty (x^T Q x + u^T R u) dt
\]

where \( Q = I_{4 \times 4} \) and \( R = 1 \). Simulate the controller using initial conditions 
\( x(0) = [0.01 \ -0.01 \ 0 \ 0]^T \). The system is judged to have exceeded its linear region if any of the state variables exceed 0.5 rad (or 0.5 rad/s for the velocities). How far off can the initial angles be before the transients in the controller exceed the linear bounds?

11.12 Consider the problem of fitting a straight line \( y = mt + b \) to a set of points given by the data pairs \([t(i), y(i)]\) for \( i = 1,2,3, \ldots \), obtained sequentially. Model the problem as one of estimating a constant plant

\[
x(k + 1) = x(k) + \begin{bmatrix} m \\ b \end{bmatrix}
\]

with the noisy measurements

\[
y(k) = [t(k) \ 1] x(k)
\]

Formulate a discrete-time Kalman filter to recursively provide the best fit for the line parameters \( m \) and \( b \). Simulate the filter with some corrupted test data, and plot the error covariance and the squared errors \( \|m - \hat{m}\|^2 \) and \( \|b - \hat{b}\|^2 \).

11.13 Consider the scalar Kalman filtering problem presented by the plant

\[
\dot{x} = -2x + u + v \\
y = x + w
\]
where \( v(t) \sim (0.1) \) and \( w(t) \sim (0, 1) \). Solve the differential matrix Riccati equation analytically and use it to simulate the Kalman filter. Use zero input and plot the state variable \( x(t) \), estimate \( \hat{x}(t) \), and Kalman gain \( L(t) \) over the interval \( t \in [0, 5] \) s.

11.14 In the underactuated robot problem, Problem 11.11, suppose that only the joint angles \( \theta_1 \) and \( \theta_2 \) are measured, but that their measurements are given by sensors that also add zero-mean white noise \( w \) with covariance \( W = 0.01 \) to each reading. Suppose further that the actuator at the elbow provides noise \( v \) of covariance \( V = 0.25 \). Design and simulate an LQG controller using the same weighting matrices as in Problem 11.11.

11.15 The continuous-time system

\[
\begin{align*}
\dot{x} &= x + u - 2d \\
y &= 2x + d
\end{align*}
\]

represents a first-order linear system that is subject to a deterministic disturbance signal \( d \). If \( d(t) = d_0 e^{0.1t} \), find a feedback controller and observer system that minimizes the effect of the disturbance on the output.
11.6 References and Further Reading

While the material presented in this chapter is considerably more advanced than the state space fundamentals in previous chapters, it is still only a taste of linear state-feedback control theory. Our derivations may not be the same as the original papers, the most elegant, nor the most theoretically general. Rather, all the presentations given here are simply the most direct results from the viewpoints presented in previous chapters. Alternative derivations can be enlightening and instructive. For example, our use of the current estimator as the motivation for the Kalman filter is quite dissimilar from the optimal projection development given in [6]. Other derivations, discussions, and applications of Kalman filters may be found in [1], [3], and [10].

Likewise, the optimal LQR can be developed from different viewpoints, such as variational calculus, as opposed to the dynamic programming approach [2] we use. Our use of dynamic programming was simply expedient given the contents of the rest of the book. See [4], [7], [8], and [9] for other presentations. The steady-state behavior of the Riccati equations is particularly well addressed in [4] and [5]. The LQG controller, i.e., the LQR in conjunction with the Kalman filter, is specifically addressed in [4] and [10].


