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Fractality and Power Law Distributions: Shifting Perspectives in Educational Research

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Abstract

The dynamical character of education and the complexity of its constituent relationships have long been recognized, but the full appreciation of the implications of these insights for educational research is recent. Most educational research to this day tends to focus on outcomes rather than process, and rely on conventional cross-sectional designs and statistical inference methods that do not capture this complexity. This presentation focuses on two related aspects not well accommodated by conventional models, namely fractality (self-similarity, scale invariance) and power law distributions (an inverse relationship between frequency of occurrence and strength of response). Examples are presented of both phenomena based on my empirical work on of daily high school attendance rates over time. We will discuss how the statistical indicators are generated and interpreted and what they reveal about the underlying dynamics of school attendance behavior.

1. Introduction

Complex dynamical systems (CDS) theory has generated a major paradigm shift in many academic disciplines (Fleener & Merritt, 2007), but has been slower to catch on in education (Koopmans & Stamovlasis, 2016a). Yet, there has been significant theoretical work on the application of CDS to educational processes (Davis & Sumara, 2006; Osberg & Biesta, 2010), as well as some thorough reflections on how CDS can contribute to educational practice and policy (e.g., Lemke & Sabelli, 2008). Recent collections of empirical papers (Koopmans & Stamovlasis, 2016b; Stamovlasis & Koopmans, 2014) attest to the significance of the methodological advances that are at once compatible with the principles of CDS and specifically tailored to the information needs in the field. This paper is concerned with one aspect of these methodological developments, namely the use of time series data to generate models of time sensitive behavior in educational systems. Education lacks a tradition in time series analysis (Koopmans, 2011), although the groundwork for such as tradition was laid quite some time ago (Glass, 1972), and the longitudinal implications of the educational process have been clear for a long time (Dewey, 1929).

The CDS angle on the educational process provides a set of priorities and desiderata that differs from those forwarded by conventional research paradigms. For example, the latter conceptualize the causal process as a cause and effect sequence while CDS is more interested in the interactions between systems components and the dependencies of behaviors in a system on its previous behaviors. Such dependencies are the focus of the present paper. The analysis focuses on the detection of fractal patterns and self-similarity in time-dependent data. Fractal patterns and self-similarity are of interest in this context because they attest to the relative adaptability and cohesion of the system of interest (Mandelbrot, 1997; Stadnitski, 2012).

This presentation capitalizes on the opportunity that has been created for rigorous time series modeling by the availability of daily attendance data, collected since the 2003-04 school year up to the present day by the Department of Education in New York City for all of its schools. These attendance rates have been reported with a one-week lag on the websites for each of its schools as public information for parents and others who take an interest in the statistical summary of the school's effectiveness. When viewed over a time frame of several years, such analyses can reveal the possible dependency of these processes on system-environment interactions.

The analysis in this paper proceeds as follows. First a brief overview is provided of the aspects of the conventional scientific paradigm, referred to here as 'the linear model' with which CDS takes particular issue, and then the alternatives it proposes are briefly discussed. The analyses presented here describe a methodological strategy that allows for uncovering some aspects of the dynamical process of interest. The process of interest is the daily attendance rates in three public schools in an urban school district in the Northeast of the United States, observed over a long time period. The strategy consists of the use of conventional time series approaches to estimate outcomes in terms of previous occurrences, supplemented with power spectral density to decide to what extent we can conclude fractality in the time series of interest.

2. Linear Models

As a research paradigm, complexity theory sets itself apart from conventional research paradigms by the difference of its assumptions about the data we collect as well as the views it puts forward about cause and effect relationships. In the complexity literature, these conventional paradigms are usually referred to as the 'linear model'. Below is a discussion of some of the main assumptions of the linear model and the need they create for alternative conceptualizations and modeling strategies. The first assumption is the ergodic assumption, which postulates that every sampled unit of behavior across the temporal spectrum is equally representative of the entire time range of interest, as is every individual unit in the sample whose behavior is analyzed (Birkoff, 1931). The second assumption is that causality is a sequentially ordered relationship between antecedents and consequences (Pearl, 2009), which is to say that the cause comes first (e.g., an educational intervention) and the effect comes later (e.g., higher high school graduation rates). The third assumption is the notion that changes in outcomes are proportional to changes in the input condition, and that such outcomes are therefore fully predictable once the predictor values are known (West & Deering, 1995). Lastly, normal distributions are often assumed in educational outcomes, an assumption that relegates extreme observations to the tail end of the distribution, rather than purported distributions that incorporate rare events into their predictions, as for instance Bak (1996) does when correlating the severity of earthquakes on the Richter scale with the frequency with which they occur. While the four assumptions summarized above are frequently questioned, they tend to prevail in policy research in education and health (Murray, 1998; Shadish et al., 2002) and shape the way we design our research and predispose us to certain types of research questions at the expense of others. For instance, the preference of randomized control trial designs in education to address causality prompts toward questions of linear causality (Koopmans, 2014).

3. Complex Dynamical Systems

The assumptions of the linear model are not particularly well-suited to address the question of complex behavior in dynamical systems. A few terminological clarifications are in order here. There are many different ways of defining complexity (Alhadeff-Jones, 2008; Koopmans, 2017; Koopmans & Stamovlasis, 2016a). The two notions of complexity that are the focus in this paper are firstly the processes of *self-organized criticality*, which is the intermediate state between stability and turbulence that according to many is an indicator of healthy adaptive behavior in systems (e.g., Bak, 1996; Stadnitski, 2012), and secondly the notion of *irreducibility*, which is to say that the behavior of the system in its entirety is not reducible to that of its individual components (Ashby, 1957). Koopmans and Stamovlasis (2016a) call a system dynamical if its behavior at one point in time can be understood in terms of its deviations from past behavior. A dynamical conceptualization of what systems are brings the time aspect to the forefront in our approach to the system. And we call systems ‘systems’ if there is a coherent and knowable process of interaction between its elements that produces behavior at a higher systemic level. For example, the interaction between students and teachers produces a teaching and learning process in the classroom that goes over and above the segments of information exchange between individual students and their teacher.

The assumptions of the linear model outlined above are the ones that CDS tends to take issue with. Below is a brief synopsis of this discussion.

3.1 The ergodic assumption

Molenaar (2004; 2015) has argued that there is little reason to presume that the data we collect will be ergodic in the sense that samples consist of homogenous groups and in the sense that outcomes are randomly distributed across the time spectrum. Therefore, rigorous sampling of measurement occasions across subjects and across the time spectrum are both necessary. It is insufficient, then, to measure educational outcomes on a single occasion without considering the fluctuation patterns across the time spectrum. And while there is an extensive literature on sampling rigor across subjects (e.g., Cohen, 1988), no similar sampling rigor is found across the time spectrum where a small number of measurement occasions is generally deemed sufficient and no further interest is taken in how the behavior of the system evolves over time. Thus, any propensity toward change that the system may possess falls outside of the scope of the research (Koopmans, 2016). While the longitudinal perspective has long been part of the pantheon of educational research methods (e.g., Singer & Willett, 2003) and the applicability of time series analysis to education has been discussed in some detail (Glass, 1972), the use of time series approaches to address the influence of the passage of time on the behavior of systems is infrequent at best in education.

3.2 A sequential relationship between cause and effect

Few educators or educational researchers would dispute that understanding cause and effect relationships is central to the discipline. Teachers, parents and others need to understand the impact of their behavior on their interaction with children, and how they can help children grow and learn. The deliberate implementation of effective strategies in such situation is essential to educators' effectiveness, and therefore, modeling this effectiveness requires that we address the question of cause and effect. We need to know what works in education. Yet, the linear view of cause and effect, while important, ignores the contribution of recursive processes. In the social science context, such processes have been referred to as social causation (Sawyer, 2002; 2003), which is a feedback relationship between the system and its constituent components. Thus, for example, the ongoing student learning that occurs in exchanges between them and their teachers or parents is an example of such recursion. In the event that the effect of a specific intervention needs to be evaluated, a sequential causal model attributes outcomes to whether or not the intervention was implemented in a given setting, while a recursive causal model would describe the interactive processes between teachers and students to examine how this causal process is generated. Examples of such causal analyses can be found in Steenbeek et al. (2012).

3.3 Linear change

One of the defining characteristics of the linear model is that changes in outcomes are seen as being proportional to changes in input conditions such that those outcomes will be predictable when the input conditions are known. CDS considers linear change as a special case of a wide range of change scenarios that are sometimes gradual, sometimes qualitative and not always predictable. Watzlawick et al., (1974) provide the crucial distinction between gradual and qualitative change, referred to as *first order* and *second order change*, and Nicolis and Prigogine (1989) provide a comprehensive overview of the ways in which change can be nonlinear.

In this paper, we focus on the type of nonlinearity that Bak (1996) calls self-organized criticality, the idea that a gradual accumulation of inputs into the system brings the system in a critical state after which a qualitative transformation occurs. The prototypical example of this process is the sand pile on a flat surface over which new grains of sand are poured. The growing friction between the grains results in avalanches in the pile. Thus, the first order process of pouring grains over the pile triggers second order change, i.e., the avalanche that transforms the system qualitatively.

3.4 A normal distribution of outcomes

One of the central tenets of inferential statistics is that a distribution of outcomes can be characterized in terms of its central tendency and in terms of its variability, and that from observed sampling of frequency distributions we can generate a theoretical model of the population distribution. In the social sciences, it is often assumed that outcomes are normally distributed according to the well-known bell curve, which postulates that outcomes are symmetrically distributed around its mean and that many observations are close to the central tendency measure while few are far away from it. We use this distributional assumption for inferential purposes by defining the boundaries of our uncertainty at the tail end of this distribution, such that if an observed value surpasses a critical value, we reject the hypothesized central tendency value for the distribution. Thus, a buildup of extreme observations results in the rejection of the hypothesized distribution.

An alternative viewpoint is represented by power law distributions, which formulate the relationship between the extremity of events and their frequency of occurrence, such that non-extreme events are proposed to occur often while extreme ones occur rarely. The aforementioned earthquake example is an instance of this relationship. Thus, extreme and non-extreme values are incorporated into a single model. Obviously, power law distributions are but one of many examples of how distributions can be not normal. However, the power law distribution has particular significance in time series research because it suggests a dynamical interpretation of the ordering of observations over time, in the sense that a linear relationship between frequency and intensity is seen as an expression of *self-similarity* or *fractality* in the data. Below is an illustration of this idea based on real data.

4. Method

Two basic approaches have been outlined to time series analysis that would approach the same input information in a slightly different manner. One approach analyzes the data in what is called the *time domain*; the second type of analysis takes place in the *frequency domain*. In the time domain, observations in the series are regressed on previously occurring observations to model the time dependencies at given lag sizes (Box & Jenkins, 1970). This approach has been extended to enable the estimation of fractality in a time series as well (Beran, 1994; Granger & Joyeux, 1980). In brief, modeling proceeds as follows. Given a time series $x_1, x_2, \ldots, x_n$, and a backshift operator $Bx_t = x_{t-1}$, an ARIMA(p, d, q) process can be defined such that

$$\left(1 + \varphi_p B^p\right)(1 - B)^d x_t = (1 + \theta_q B^q) e_t. \tag{1.1}$$

The left-most term in this equation represents the sum of the sequence of autoregressive components at p lags, i.e., AR(p), and the term on the right side similarly represents a set of q moving average terms MA(q). Note that AR is defined in terms of x at previous lags, while MA(p) models the error variance at previous lags. It is assumed that

$$e_t(t = 1, 2, \ldots) \sim (0, \sigma^2). \tag{1.2}$$

The middle term in equation 1.1 represents the differencing parameter d. It can be seen in eq. 1.1 that, contrary to the AR and MA components, the estimation of d is not lag-specific and thus can be used to represent the long-range in the series, provided that $-0.5 < d < +0.5$ (Beran, 1994). The determination that $d \neq 0$ is taken as an indicator of fractality, in the sense that observations in the series are interdependent over the long range of the series.

The estimation of fractality in the frequency domain involves a re-structuring of the time series in terms of cyclical patterns that repeat at varying frequencies across the time spectrum (the relative frequency) and the amount of variance explained by these cycles (the spectral density). This transformation

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involves a mathematical operation called the Fourier transform, the explication of which can be found in many time series texts (e.g., Bloomfield, 1976; Shumway & Stoffer, 2011). Here, it is sufficient to note that the transformation to the frequency domain forms the basis for the generation of power spectra, which is based on the log relative frequency and the log spectral density as follows (see e.g., Delignières et al., 2005):

$$S(f) \propto 1/f^{\beta}. \tag{1.3}$$

In this equation, $S(f)$ represents the spectral density and β is the absolute value of the slope of the inverse relationship between log power and log frequency, also called the *power exponent*. We say that if a power spectrum yields a clear linear pattern in this relationship with a non-zero negative slope, it points to fractality, or self-similarity in the data (e.g., Delignières et al., 2005; Stadnitski, 2012).

5. Data Source

The research reported here is made possible by the fact that the New York City Department of Education (NYCDOE) started systematically collecting its daily attendance rates, starting in 2004. Out of a corpus of 40 schools whose daily attendance data were requested from the DOE, three urban high schools were selected for the present illustration, referred to here as School A, B and C. These schools provide typical examples of fractality and seasonality in their attendance patterns, hence their selection for this paper. The demographic characteristics of the students in these three schools is fairly typical for those served by public schools of New York City in general: the percent of non-white students was 81.8% in School A, 99% in School B and 98.1% in School C. The percent of students with disabilities was 3.3%, 16.3% and 31.2%, respectively in Schools A, B and C. The percent of English Language Learners (ELL) was 0.3% in School A, 9.6% in School B and 8.3% in School C. At least half of the students were eligible for free or reduced priced lunch in each of the three schools indicating that they served a student population predominantly from poor families. These three schools were relatively small by urban public school standards: 664, 508 and 157, respectively. The characteristics reported here are based on recordings from the 2013-14 school year and they are quite representative of the school demographics for the preceding seven years for which data were available.

Attendance recordings were used over a four-year period in School A (735 time points), a five-year period in School B (910 points) and a six-year period in School C (1,075 points). Median attendance rates in Schools A, B and C were 96.28, 90.41 and 84.78, respectively. In preparation for the time series analyses, missing values were removed, or replaced through imputation of the median of the series. Weekly subsets were preserved in the series to ensure an accurate estimation of the seasonal patterns in the data (i.e. the five days in a school week and the correlation between observations at a lag size of five). Stationarity tests were conducted to ensure the constancy of statistical properties, which is a necessary assumption for the fractional differencing analysis described above. For the stationarity tests, the augmented Dickey-Fuller (ADF) Unit Root test was conducted (Said & Dickey, 1984). The ADF values were -7.76, -7.73 and -5.05 for Schools A, B and C, respectively. These tests were performed at a lag order of 9, and the resulting values were significantly different from zero in all three instances, and thus grounds to reject the null hypothesis of non-stationarity.

6. Outline of the Analysis

The analysis proceeded as follows. I first obtained distributions of the data to conduct initial diagnostics of the data, including the generation of time series plots, autocorrelation function (ACF) plots and power function plots. Short range as well as seasonal AR and MA estimates were generated based on the information provided by the ACF plot about the dependencies within these data. Given that short range AR and MA estimates tend to be correlated with the differencing parameter, a stepwise model selection process was followed to estimate relative importance of each of these types of parameter over and above that of the others, and the goodness of fit indicators of these models were compared (Wagenmakers et al., 2004). The goodness of fit indicators considered were the Bayesian Information Criterion (BIC) and the Ljung-Box Portmanteau Q (LBQ) test. These two indicators address different aspects of the variance reduction that is attempted in these modeling efforts. As BIC gets lower, less variance in the data remains unexplained, while a lower LBQ values show a reduction in the autocorrelation remaining in the data. A significance test of LBQ values, tested under a chi-square distribution, indicates that no autocorrelation remains in the data if LBQ is not different from zero.

Note that extreme values were replaced by a linear combination of the preceding and subsequent neighboring value prior to the estimation of time-related dependencies, as the presence of those extreme values complicates efforts to distinguish short range and long range patterns. Three separate analyses were conducted, one for each school, as aggregation across schools tends to obfuscate the dependency patterns in the series, likewise making it harder to detect fractal patterns and distinguish them from the seasonal ones. In the confirmatory stage of these analyses, effective strategies will need to be developed to aggregate this information in such a way that the unique fractal features of these data are preserved. To my knowledge, the literature does not offer any general principles to follow.

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7. Results

Figure 1 shows the diagnostic plots for the daily attendance rates in Schools A, B and C. The panels on the left of the figure show the attendance rates as time series plots for each school. The straight lines superimpose the median of the series. It can be seen that there is less variability overall in School A than in Schools B and C, and that in all three schools, there are a considerable number of extreme values toward the bottom end of the range of attendance values. The plot for School C also shows the kind of undulating pattern that is typical of fractality, and may be related here also to the fact that this high school is very small and therefore more vulnerable to varying environmental conditions. As mentioned, the analyses were conducted on a cleaner version of these time series to bring out more clearly the seasonal and fractal patterns in the data.

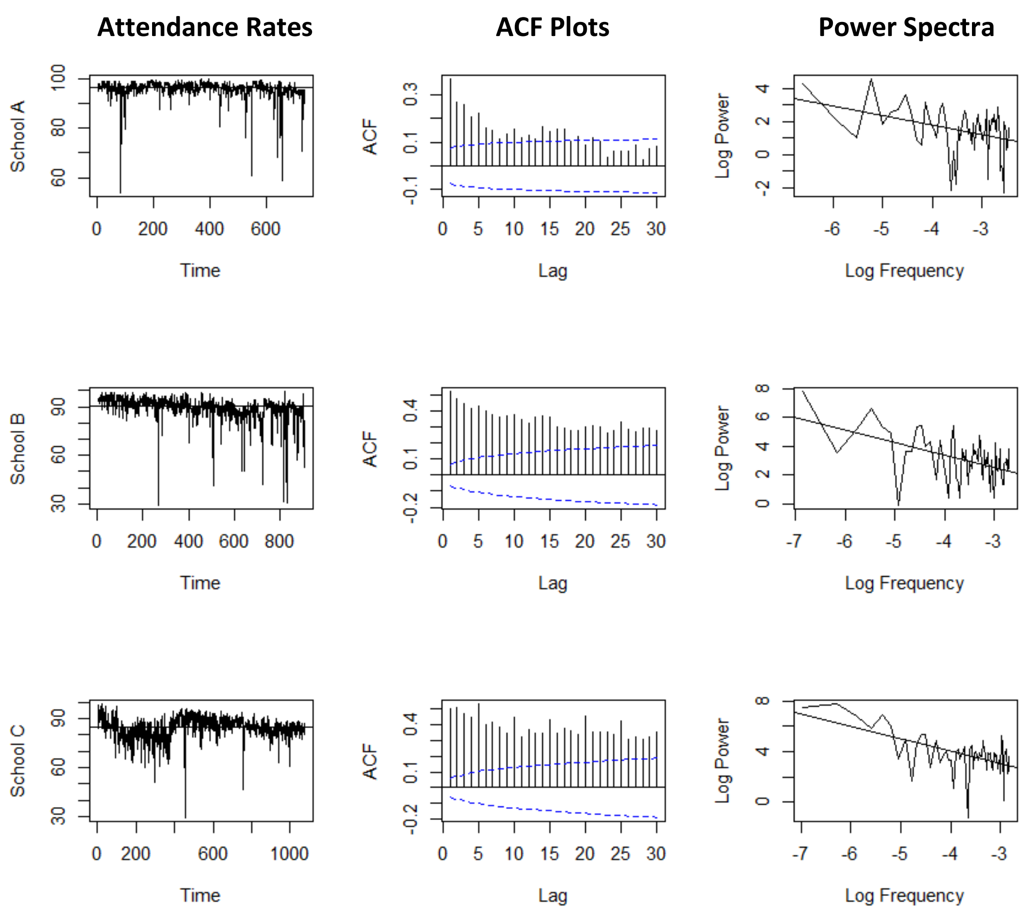

Figure 1. Daily Attendance in Three High Schools: Time Series Plots, Autocorrelation Function (ACF) Plots and Power Spectra.

The panels in the middle of Figure 1 show the autocorrelation function (ACF) plot, which plots the autocorrelations for the first 30 lags in the series (i.e., the 'cleaned version' of it). The blue dotted lines in the figure represent the confidence intervals. A slow recession to non-significance is seen in each of the three schools, a pattern that points to fractality and long-range dependence in the series. The spikes at lag 5 and its multiples are particularly pronounced in School C, but can also be observed in School A. This seasonal pattern is not as clearly visible in School B, although the spike at the tenth lag should be noted. On the right of Figure 1 are the power spectra, fitted over the first $2^6 = 64$ observations on the spectrum. This sampling is done in order to reduce the overwhelming influence of the short range cycles on the appearance of these plots. It can be seen that there are no clear signs of non-linearity in these spectra. The slope values are $\beta = -0.58$ (School A), $\beta = -0.89$ (School B) and $\beta = -0.98$ (School C), all of which are significantly different from zero.

Table 1. Stepwise Model Comparisons and Goodness of Fit Statistics: Schools A, B and C.

School	Model	ARIMA (p, d, q) Specification	d	σ^2	BIC	LBQ	p-value
A							
	1	(0, 0, 0)	--	11.90	3,003.34	371.66	.000
	2	(1, 0, 1)	--	8.01	2,871.22	7.41	.686
	3*	(0, d, 0)	.27	7.90	2,859.20	6.54	.886
	4	(1, d, 1)	.23	7.95	2,874.40	6,65	.880
B							
	1	(0, 0, 0)	--	21.98	5,401.43	1,827.70	.000
	2	(1, 0, 1)	--	13.26	4,956.30	12.58	.248
	3	(0, d, 0)	.33	13.48	4,963.63	19.72	.073
	4*	(1, d, 1)	.17	13.18	4.950.44	8.38	.755
C							
	1	(0, 0, 0)	--	32.04	6,784.68	2,428.10	.000
	2	(1, 0, 1)	--	19.01	6,238.74	99.21	.000
	3	(0, d, 0)	.32	19.07	6,269.96	127.95	.000
	4	(1, d, 1)	.12	19.19	6,253.92	92.41	.000
	5*	(1, 0, 1) X (1, 0, 1)$_5$	--	17.49	6,163.94	17.88	.119

* Marks the preferred model
BIC: Bayesian Information Criterion = -2*Log Likelihood + k (log(N)) (k: number of parameter estimates +1; N: Length of the Series)
LBQ: Ljung-Box Portmanteau (Q) Test (Cryer & Chan, 2008)

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Table 1 shows the results of the model comparisons made for each of the schools, including the ARIMA specifications. The models compared are a mean-based estimate (Model 1), short range ARIMA (1, 0, 1, Model 2), the estimate of the differencing parameter only (Model 3), ARIMA (1, d, 1, Model 4), and a multiplicative seasonal model, which includes lag 1 and lag 5 estimates. The periodicity of 5 lags stands for the days of the school week.

It can be seen in the table that the best fitting model for attendance at School A characterizes it as a fractal pattern with a differencing parameter of $d = .27$, although it should be added that Models 2, 3 and 4 all provide an acceptable fit in the sense that the LBQ statistics are not different from zero, indicating the that the autocorrelation has been effectively removed. Yet, comparison of the BIC measures indicates that the overall variance is reduced most effectively by only modeling long range fractality (Model 3). Attendance in School B is best described by Model 4, which includes both short range estimates at lag 1 and the differencing parameter estimate, which equals $d = .17$ for this model. The fact that this parameter takes on a much higher value in the Model 3 estimation ($d = .33$) attests to the correlation between the differencing parameter estimates and the lag 1 dependencies, which are absorbed in this higher value. Inspecting the results of the Ljung-Box Q test indicate again indicate that in terms of the removal of autocorrelation from the series, Models, 2, 3 and 4 are all acceptable, and the superiority of Model 4 is a matter of the degree to which overall variance is reduced by the models.

The seasonal factor plays a critical role in the modeling efforts for the School C data, where neither lag 1 nor differencing parameters, nor a combination of the two provide an acceptable fit, as can be seen in the difference of the LBQ statistic from zero in each of Models 1 through 4. Both in terms of overall variance reduction and removal of autocorrelation patterns, Model 5 is distinctly superior to the other four, indicating that a seasonal factor at five lags is an important source of variation over and above the short range and fractal dependencies in these data.

Table 2 shows the parameter estimates for the best fitting models. For Schools A and B, they shown the significance of the differencing parameter, as well as strong first order autocorrelation and moving average patterns in School B, and the prominence of the seasonal parameters in School C.

8. Discussion

The findings from this study show that in the daily high school attendance in Schools A, B and C, short range, seasonal and long range dependencies all have potential relevance and therefore need to be explicitly modeled. To the assumption that school attendance behavior may represent an ergodic system, these findings present two challenges. First of all, it is not sufficient to characterize daily

Table 2. . Parameter Estimates of the Fractionally Differenced and Seasonal Models: Schools A, B and C

Parameters	School A	School B	School C
Intercept (se)	96.02 (.39)	89.91 (4.35)	85.01 (1.04)
d (se)	0.27 (.00)	0.17 (.01)	--
ϕ_1 (se)	--	0.98 (.03)	0.88 (.03)
θ_1 (se)	--	-.92 (.01)	-.67 (.05)
$\Phi_{1,5}$ (se)	--	--	.93 (.02)
$\Theta_{1,5}$ (se)	--	--	-.80 (.04)

attendance in high schools without taking into account the interrelatedness of within subject observations across the time spectrum (Koopmans, 2011; 2015). This paper illustrates what can be learned in addition by studying daily high school attendance rates over a long period of time: there are seasonal as well as fractal dependencies to consider that are hidden in the averages that we would compute across the time spectrum. The second challenge we face if it cannot be assumed that attendance represents an ergodic system is that the across-subject variations need to be incorporated into our description of the system. The comparison of results for multiple schools is one aspect of this description, but if the number of cases being considered gets larger, more efficient ways of aggregating the data need to be found such that the unique aspects of the variability within cases is preserved when the data is summarized. Multivariate time series may provide some answers to this question, particularly if a structural equation framework is invoked (Molenaar et al., 2009) allowing for a systematic comparison between schools based on a

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relevant set of characteristics, such as perhaps location, school demographics, climate and leadership. In education, this type of research is in its infancy and urgently needs further development, because it permits for a systematic accounting of the variability across subjects as well as across the time spectrum.

The competitive modeling strategy illustrated here makes fractional differencing within the time domain a particularly useful approach for confirmatory studies about the significance of long range dependencies in observed time series. In addition, the power spectra help support a fractal interpretation of such dependencies if a downward nonzero slope can be fitted to characterize a linear trend in those spectra. This linearity in log-log scales indicates that there is scale invariance, and thus, fractality, in the fluctuation patterns. In other words, the patterns are the same, irrespective of the log relative frequency range in which they are observed. In the analyses presented here, the power spectra support the case for fractality in all three schools, as do the ACF plots which, in all three cases, show a gradual recession of the autocorrelations to statistical non-significance as lag size increases, another distinct characteristic of fractality. The results of the stepwise model comparisons are more equivocal. While none of the models for the three schools that include a differencing parameter are indispensable to the removal of autocorrelation from the series, they do improve the fit in Schools A and B. While the ACF plot for School C clearly shows long range dependencies, modeling the seasonal patterns that are much less prominently visible in those plots, ultimately absorbs much more of the autocorrelation in the data.

One of the major differences between linear and complexity paradigms lies in their conceptualizations of cause and effect relationships in the system of interest. CDS views these relationships as instances of adaptive behavior within the system in an ongoing interrelationship with its environment. Therefore, time series are an effective way of describing the time aspect of this relationship. Time series characterizes one important aspect of this adaptation, namely the endogenous process, i.e., the understanding of the behavior of a system in terms of its behavior on previous occasions. Ultimately, however, it is probably right to conclude that educational research often focuses on the effectiveness of educational interventions in terms of student learning behavior and outcomes. Such analyses require that the timing of these intervention gets correlated with the exogenous patterns that characterize the time series, as is done, for example in the single-case based behavior modification analysis.

Finding self-similarity and meta-stability in systems is an important motivator to conducting fractal analysis. We tend to assume that systems are stable, resulting in ergodic sampling spaces and representativeness over time and across subjects. If, however, we cannot make that assumption, the question then becomes what exactly the non-stability looks like. Self-similarity and meta-stability are complementary answers to this question. Systems can be stable and they can be turbulent, but dynamical scholars have argues that many systems fall in-between these two categories, where behavior is not quite predictable and the propensity of transformation exists, without there necessarily being heavy turbulence or complex attractor regimens for the behavior of the individual elements within the system (Goldstein, 1988; Waldrop, 1992). The analyses presented here attempt to describe this intermediate space between order and turbulence and the potential fruitfulness it derives from its irregular patterns of variability.

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