

March 2021

Emergent Hierarchy Through Conductance-based Degree Constraints

Christopher Tyler Diggans

Air Force Research Laboratory / Clarkson University, digganct@clarkson.edu

Jeremie Fish

Clarkson University, jafish@clarkson.edu

Erik M. Bolt

Clarkson University, ebollt@clarkson.edu

Follow this and additional works at: <https://orb.binghamton.edu/nejcs>



Part of the [Discrete Mathematics and Combinatorics Commons](#), [Non-linear Dynamics Commons](#), [Numerical Analysis and Computation Commons](#), and the [Organizational Behavior and Theory Commons](#)

Recommended Citation

Diggans, Christopher Tyler; Fish, Jeremie; and Bolt, Erik M. (2021) "Emergent Hierarchy Through Conductance-based Degree Constraints," *Northeast Journal of Complex Systems (NEJCS)*: Vol. 3 : No. 1 , Article 4.

DOI: [10.22191/nejcs/vol3/iss1/4](https://doi.org/10.22191/nejcs/vol3/iss1/4)

Available at: <https://orb.binghamton.edu/nejcs/vol3/iss1/4>

This Article is brought to you for free and open access by The Open Repository @ Binghamton (The ORB). It has been accepted for inclusion in Northeast Journal of Complex Systems (NEJCS) by an authorized editor of The Open Repository @ Binghamton (The ORB). For more information, please contact ORB@binghamton.edu.

Emergent hierarchy through conductance-based degree constraints

C. Tyler Diggans^{124*}, Jeremie Fish²³, and Erik M. Bollt²³

¹Air Force Research Laboratory: Information Directorate, Rome, NY, USA

² Clarkson Center for Complex Systems Science (C³S²), Clarkson University, Potsdam, NY

³ Department of Electrical and Computer Engineering, Clarkson University, Potsdam, NY, USA

⁴ Department of Physics, Clarkson University, Potsdam, NY, USA

* digganct@clarkson.edu

Abstract

The presence of hierarchy in many real-world networks is not yet fully understood. We observe that complex interaction networks are often coarse-grain models of vast modular networks, where tightly connected subgraphs are agglomerated into nodes for simplicity of representation and computational feasibility. The emergence of hierarchy in such growing complex networks may stem from one particular property of these ignored subgraphs: their graph conductance. Being a quantification of the main bottleneck of flow through the coarse-grain node, this scalar quantity implies a structural limitation and supports the consideration of heterogeneous degree constraints. The internal conductance values of the subgraphs are mapped onto integer degree restrictions, which we call *internal bottlenecks*, by using inverse cumulative density functions. This leads to a hidden variable model based on the rich-get-richer scheme. It is shown that imposing such restrictions generally leads to an increased measure of hierarchy and alters the tail of the degree distribution in a predictable way. Thus, a mechanism is provided whereby inherent limitations on network flow leads to hierarchical self-organization.

1 Introduction

Many real-world networks display both a measure of hierarchy [1] and a scale-free (SF) powerlaw form for some portion of the degree distribution. However, recent work [2] has reminded us that the rich-get-richer SF model was only meant as a first approximation [3]. In particular, the SF portion of the degree distribution does

not necessarily extend into the region where the hubs are found [4], indicating a potential relationship between the failings of the SF model and the observation of hierarchy. We shed light on this relationship by considering the consequences of the fact that many complex networks are coarse-grain models of more intricate, yet highly modular interaction networks, where the finer grained details are either computationally prohibitive or unknown.

To illustrate, we consider the extreme case of a social network, where each node represents a person. In turn, each person could be represented as a complex brain network, making the social network such a coarse-grain model. The properties of the highly connected subgraphs that are being agglomerated into nodes (brain networks in this case) are ignored due to ignorance, however, one property of these subgraphs that should not be overlooked is their graph *conductance*, especially in applications of transport and information flow. The conductance of a network, G , is a scalar value, denoted by $\Phi = \Phi(G) \in [0, 1]$; it serves as a quantification of the main bottleneck to the flow over the graph [5], where $\Phi \sim 0$ implies an extreme bottleneck, while $\Phi \sim 1$ implies essentially no bottleneck. An adaptation of this metric applied to subgraphs was defined as cluster conductance in [6], but we refer to this feature as the agglomerated node's internal conductance. While these internal conductance values may not be known, there are often proxy measurements that can quantify the effects produced by this fundamental restriction. For example, extrapolating from data on primate societies and the sizes of their neocortex, Robin Dunbar posited that human beings are likely capable of on average tracking a network of approximately 150 human relationships [7]. This result was supported by archaeological evidence of human tribe sizes along with the natural formation of divisions in large corporations. This provides a clear example of the type of mapping to proxy values that might associate the conductance of the neural network of the neocortex to a person's maximum allowable degree in a social network.

A hidden variable model for producing a randomly growing network (RGN) is presented. The node attribute is called the node's *internal bottleneck*, and its value is assumed to be a function of that node's internal conductance (as a subgraph within the coarse-grain model). The internal bottleneck, which we denote with \mathfrak{B} , defines a limit for the number of links to a node, and we are interested in showing that such restrictions lead to an increased measure of hierarchy as measured by common measures such as the Global Reaching Centrality (GRC) [8] and the Random Walk Hierarchy (H_{RW}) measure [9]. A survey of many early models, some of which impose degree restrictions in various ways, can be found in [10], and other work has considered degree limitations, but usually by setting a global degree limit, often through a probabilistic activation function as in [11]. To the authors' knowledge, fixed heterogeneous degree limits have not been considered for an RGN model, especially in the context of their influence on the development of

hierarchy. Peer-to-Peer (P2P) networks have been studied using a similar approach to the one presented here, except that all nodes share a common degree limit. And in that case, it was shown to lead to an exponential cutoff in the otherwise scale-free degree distribution [12]. Since that model is in fact a limiting case of the present model, we explore the effects on the tail of the degree distribution from allowing heterogeneity in these imposed limits.

We begin by reviewing two measures of hierarchy. This is followed by a description of a hidden variable RGN model that incorporates a review of the classic SF rich-get-richer model of Barabási-Albert [13]. Since measuring hierarchy for undirected networks is limited to a single metric, we consider a directed variant of this original model and explain why the particular choice in direction was made. The assumptions chosen for the unknown internal conductance values are described, for which various inverse Cumulative Density Functions (CDF) are then used to map these values onto the node's \mathfrak{B} values. We then present our results and provide explanation of both how the restrictions imposed generally lead to higher measures of hierarchy and how the tail of the degree distribution is altered by choices made in the model. We conclude with a summary and provide some interesting future work in this direction.

2 Hierarchy

The historical definition and understanding of the term hierarchy is complicated, and may best be described in terms of category theory; but, the measurement of hierarchy in complex networks has recently reached a reasonable, albeit fractured, consensus [8], [9], [14], [15], [16]. Of the widely known measures of hierarchy, two are considered in this paper: the Global Reaching Centrality (GRC) [8] and the Random Walk Hierarchy measure (H_{RW}) [9]. Perhaps the most versatile measure of hierarchy, presented in 2012, is the GRC. It is intuitively based on the definition of hierarchy as a *heterogeneous distribution of reach centrality*, where the ideal network would have few nodes with large centrality values and relatively many nodes with smaller values. The main advantage of this measure is the inclusion of undirected and weighted networks with simple alterations to the formulae, although this flexibility is not utilized here. We use directed links in the presented model to enable the comparison of different measures. The GRC is described in terms of the set of reaching centralities, denoted C_R . More specifically, the GRC is defined as the average distance of these centrality values from the maximum reaching centrality, denoted C_{Rmax} , i.e.:

$$GRC = \frac{\sum_i^N C_{Rmax} - C_R(i)}{N - 1}. \quad (1)$$

This same formula can be used for different graph schemes by altering how the C_R values are computed. In the presently considered case of directed unweighted graphs, $C_R(i)$ is simply the proportion of nodes that are reachable along directed edges from node i .

It is clear from (1) that having few nodes with large centrality will result in many large contributions to the overall sum. In this way, the GRC measures the spread of the distribution of C_R , in particular promoting distributions with exponentially decreasing histograms of reach centrality values. This captures the essence of tree-like networks, without explicitly requiring tree-like structure. For example, a flower graph [17], which has no discernible tree-like structure and many cycles, is still very hierarchical by the GRC measure.

Having identified a potential issue with the widely accepted GRC measure, the Random Walk Hierarchy measure (H_{RW}) was introduced in 2015 in order to penalized structures that were technically tree-like, though not noticeably hierarchical, such as chains or star graphs. This alternate measure is formulated through simulated backward diffusion of decaying random walkers. It is a measure of the spread of the stationary distribution of a particular random walk (\mathbf{p}^{stat}) and can be computed in closed form by

$$H_{RW} = \sqrt{N \sum_{i=1}^N (p_i^{stat})^2 - 1}; \quad (2)$$

with the i^{th} element of the stationary distribution being given by

$$\begin{aligned} p_i^{stat} &= \frac{e^{1/\lambda} - 1}{N} \left(e^{1/\lambda} \mathbb{I}_N - \hat{\mathbf{T}} \right)^{-1} \hat{\mathbf{T}} \cdot \hat{\mathbf{1}} \\ &= \frac{e^{1/\lambda} - 1}{N} \sum_{n=1}^{\infty} \left(e^{-1/\lambda} \hat{\mathbf{T}} \right)^n \hat{\mathbf{1}}; \end{aligned}$$

where $\hat{\mathbf{1}}$ is the vector of ones, $\hat{\mathbf{T}}$ is a stochastic transition matrix computed by setting the probability of transition from node j to node i such that

$$Prob(j \rightarrow i) \propto \begin{cases} \frac{1}{k_j^{in} k_i^{out}} & , \quad \text{if } i \sim j \\ 0 & , \quad \text{o.w.} \end{cases} ,$$

and λ is a parameter that represents the characteristic distance for which the weight of the random walker is decreased to e^{-1} . This parameter was shown to be optimally set to $\lambda = 4$ in the case of large normal branching trees, balancing the benefits of exploring more of the local neighborhood, while remaining well-defined

in the thermodynamic limit of an infinite graph. In the absence of more thorough analysis of this parameter for non-tree networks, we retain $\lambda = 4$, where the optimum branching number for tree-like growth is near 3.84. The series representation provided is used for approximations when matrix inversion becomes too computationally expensive. This measure targets a heterogeneity in the distances between pairs of points, and we chose it to provide a counter to certain results for the GRC, which do not involve the distances between nodes in the directed case.

3 The Internal Bottleneck Model

In order to focus our attention on the main concepts presented, we chose to alter the basic Albert-Barabási *rich-get-richer* model using $m = 2$, which was shown to create Scale-Free (SF) powerlaw degree distributions [13]. In this model, one node is added in each generation and that node is connected to m previously generated nodes; these nodes are chosen at random but are weighted by their current degrees. One specific network that might be generated by this model is shown in Figure 1 for reference.

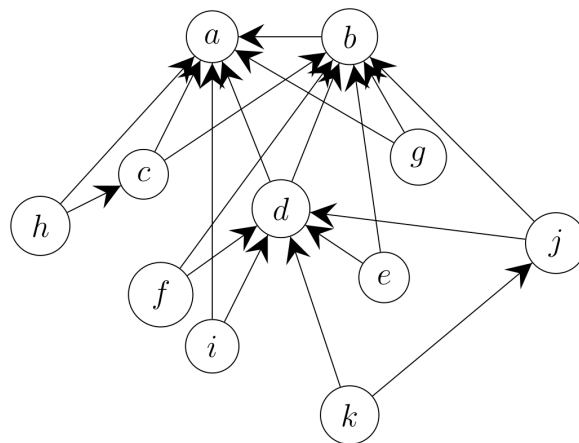


Figure 1: An example of a directed Barabási-Albert SF net using the parameter $m = 2$. The proposed internal bottleneck model follows this construction, except that each node has a specific degree restriction, which is theoretically obtained from a transformation of the hidden subgraph's conductance value. Nodes are taken out of the pool of potential links from new nodes once their degree reaches this *internal bottleneck* state.

As stated in the introduction, we consider a directed version of this basic model since only one metric for hierarchy can be generalized well to undirected networks (GRC). As such, there are essentially two options for how direction can be assigned. The more natural choice may be to have links directed from older nodes to the new

node, however, we found that this choice leads to uninteresting results. Particularly for the GRC metric, we find that the node with maximum reach centrality is just the oldest node. Additionally, as the network grows in size, the GRC value converges to 1 in the infinite limit for all choices of bottleneck values. Thus, our model uses edges directed away from the newly added nodes, meaning older nodes are considered offspring of the new nodes. This version has the more interesting results where the overall size of the network has less of an influence as the age of the node is not the only determining factor in reach centrality or hub formation.

We alter this basic SF model further by arguing that Φ should naturally correspond to a limit on the potential degree of the agglomerated node through some monotonically increasing function $f : [0, 1] \rightarrow \mathbb{Z}^+$, which is used to assign what we call an *internal bottleneck*, \mathfrak{B} , to each node. Furthermore, being a structural property of a functioning unit in the coarse-grain network, we assume a truncated normal distribution on the underlying internal conductance values to ensure that $\Phi \in [0, 1]$. We then need only define the mapping f from these internal conductance values to their corresponding \mathfrak{B} values, for which inverse CDF provide an obvious choice. More specifically, if we define $F : [a, \infty) \rightarrow [0, 1]$ as the definite integral

$$y = F(x) = \int_a^x p(s) ds,$$

for some probability distribution p , then F is the CDF of p and we can say that

$$\mathfrak{B} = f(\Phi) = F^{-1}(\Phi). \quad (3)$$

In this way, we consider the results from four common distributions for comparison: binomial ($p = 0.5$), poisson, negative binomial ($p = 0.5$), and powerlaw ($\alpha = 2.5$); altering the parameters n , λ , r , and x_{min} respectively in order to shift the average bottleneck values.

This hidden variable could also very well be assigned arbitrarily using any probability distribution in general, but the present model is based on a theoretical argument involving graph conductance. Taking this into consideration, an arbitrary distribution may still be utilized by assuming a uniform distribution of internal conductance values, which is then mapped through the inverse CDF of the desired distribution. However, the fact that conductance is a structural property and these values represent subgraphs that are functional units of some more complex system, we assumed a truncated normal distribution for Φ in order to obtain a localized distribution with compact support on $[0, 1]$ that tends toward better flow characteristics (e.g. mean $\mu = 0.75$ and standard deviation $\sigma = 0.1$).

As many measures of human potential actually follow a heavy tailed distribution [18], there is no reason to expect this normality to be retained through the

mapping to \mathfrak{B} , i.e. $f : [0, 1] \rightarrow \mathbb{Z}^+$ may incorporate time dependent growth processes that lead to heavy tailed distributions. Thus, using the same distribution of underlying conductance values for all simulations, we compare the resulting networks from assigning \mathfrak{B} values using inverse CDF of both localized (binomial and Poisson) and heavy tail (powerlaw and negative binomial) distributions. It is also worth mentioning that the P2P network case described in the introduction is incorporated into the present model through the use of the inverse CDF of the degenerate distribution. Figure 2 shows examples of the inverse CDF functions used (with parameter choices that lead to comparable $\langle \mathfrak{B} \rangle$), together with a superimposed sketch of the truncated normal distribution used for the conductance values that make up the input to these functions. This shows how our sampling strategy will disproportionately utilize the tail of the distributions.

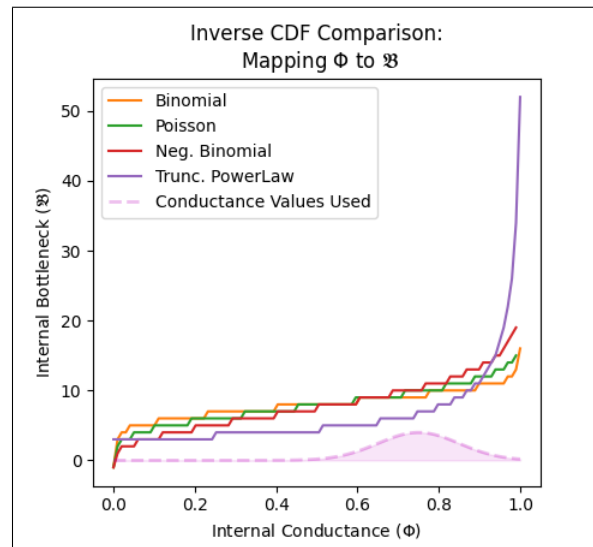


Figure 2: Examples of the inverse CDF functions (3) and a sketch of the truncated normal distribution used for the conductance values. We assume normality with $\bar{\Phi} = 0.75$ meaning the tails of the distributions are more heavily sampled.

Having described the assignment of the variable, we now focus on its function. During the random growth process, once a node's degree meets its assigned internal bottleneck value, that node is no longer considered in the selection of children for new nodes. For example, referring back to Figure 1, if $\mathfrak{B}(a) = 5$ upon creation, after the current generation it would no longer be eligible to link to newly added nodes. However, it is important to note that node a would still likely be a more distant descendent of many additional nodes due to its high relative number of links at this point in the process. This feature of increased likelihood of becoming a de-

scendent but not a direct child of new nodes is important to showing why measures of hierarchy increase as the restrictions are imposed.

4 Results

In considering the two options for assigning direction to the links in the original model, one choice proved to be problematic for measuring hierarchy, and we now take the time to explain this issue before presenting the results for our chosen model. For the directed model in which links are directed from old nodes to the newly added ones, we find misleading results. In this case, the oldest nodes are by definition going to obtain the maximum reaching centrality, and thus, C_{Rmax} is too closely tied to the age of the node. In fact, we find that the general trends of the GRC and H_{RW} disagree, as can be seen in Figure 3.

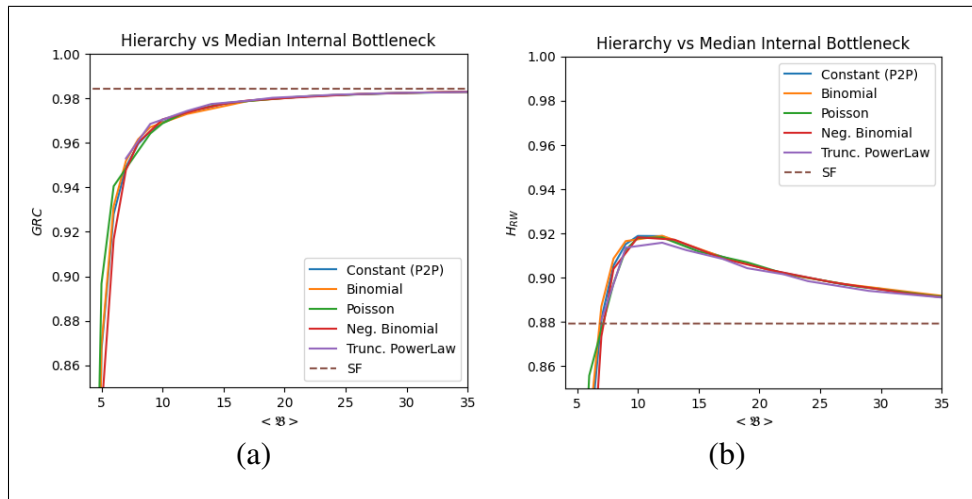


Figure 3: Measures of (a) GRC and (b) H_{RW} against the sample median $\langle \mathfrak{B} \rangle$ for the incorrectly assigned directed edge model. Results are averaged over 1000 trials for graphs of size $N = 1000$ nodes, i.e. 1 million nodes. All networks were created using the python Networkx package. Curves represent the same inverse CDF maps through different parameter choices. In this directed case, while H_{RW} increases as the average restriction imposed becomes more severe, the GRC has the opposite trend. This is due to C_{Rmax} being a function of node age and thus directly related to the network size; this is seen as a problem with the model and not reflective of an accurate appraisal of the hierarchy of the network. The original expected Scale Free values are included as a dashed line to which all measures converge for sufficiently large bottleneck values.

Though it is interesting to find a RGN model for which the GRC trend fails to coincide with that of H_{RW} , it should be made clear why this occurs. For a

directed graph, the path lengths are not relevant to the definition of the C_R values in calculating the GRC. Instead, it is only the number of nodes that can be reached that matters. Thus, the alteration from hub formation to more chain-like growth does not affect the trend as it does for H_{RW} . This chain-like growth actually leads to a more evenly spread out distribution of reaching centralities, which results in lower contributions to the overall GRC sum. While the results for H_{RW} for this case are more in line with what is expected, one may note that the choice of distribution used does not make much of a difference here either. Instead, the measure is again highly dependent on the size of the graph and is a function of how chain-like the network is made by the restrictions. Thus, the location of the hump in this case is due to the choice of $\lambda = 4$ and the reduction afterward is due to the general prevention of hub formation.

For these reasons, we define our model in what may be thought of as the reversed direction, having links directed from newly added nodes to existing nodes. This choice leads to quite low GRC values in general, but we find that the two metrics chosen for assessing hierarchy at least have similar trends under this model. The low GRC values are simply due to younger nodes being more likely to attain the maximum reaching centrality, meaning the finite simulations lead to a somewhat artificial reduction in GRC. Regardless, we find that distributions resulting in lower $\langle \mathfrak{B} \rangle$ values lead to the prevention of hub nodes. As mentioned previously, those nodes, which reach their maximum degrees, will not be direct descendents of any additional new nodes. They will, however, have a much higher likelihood of becoming distant descendents of many future nodes, allowing its ancestors to attain larger C_R values, even if they reach their own \mathfrak{B} values. With respect to H_{RW} , the prevention of hub formation particularly leads to more clearly defined levels of hierarchy emerging, which drives the measure up until there are too many levels for the number of nodes and we get chain-like networks.

Figure 4 (a) and (b) show the change in GRC and H_{RW} , respectively, as a function of the sample median of \mathfrak{B} values for the proposed model under various choices for the mapping f . Medians were used to mitigate comparison issues that arise with heavy tailed distribution means, and sample means were shifted for the various f through a single distribution parameter for each case.

As might be expected, the influence of the imposed limits are subtle in all cases where $\langle \mathfrak{B} \rangle \sim N$. However, as the restrictions get more severe, i.e. $\langle \mathfrak{B} \rangle \ll N$, both measures of hierarchy increase with an acceleration that is dependent on the variation in the distributions. A break down in the trend is noticeable for H_{RW} when the average bottleneck value gets smaller than $k = 5.84$, since each node has $m = 2$ out-links and the optimal branching for the choice of $\lambda = 4$ is 3.84 [9]. Any additional restriction then leads to lower measures of H_{RW} .

We also use this model to explore changes to the tail of the degree distribution.

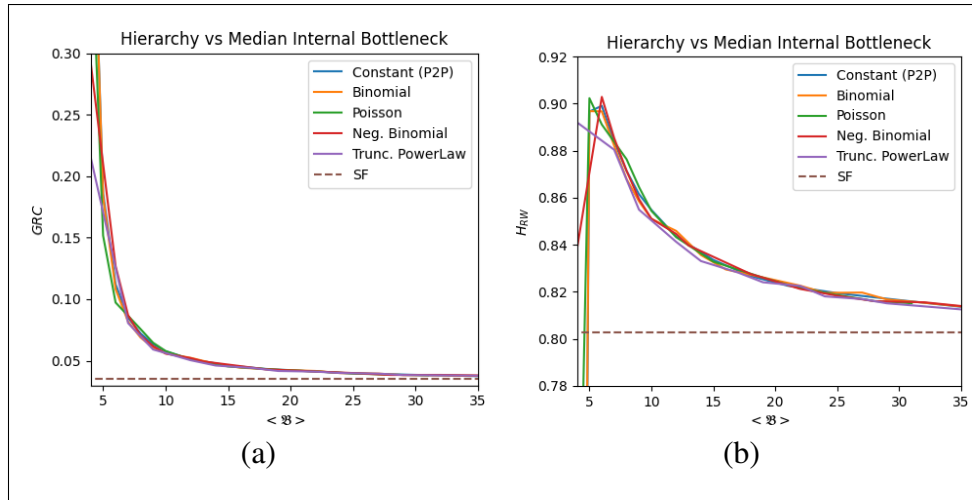


Figure 4: Measures of (a) GRC and (b) H_{RW} against the sample median of bottleneck values for networks created by the internal bottleneck model. Results are averaged over 1000 trials with graph sizes of $N = 1000$ nodes, i.e. 1 million nodes. All networks were created using the python Networkx package. Curves represent the same inverse CDF maps through different parameter choices. In all cases, both measures of hierarchy increase as the average restriction imposed becomes more severe. This general trend breaks down in the case of H_{RW} , as the restrictions lead to more chain-like networks. Maps associated with heavy tail distributions have the potential to lead to slightly less hierarchical networks on average due to an increased likelihood of hub formation. The values obtained by the original SF model are included as dashed lines, to which all measures converge for sufficiently large $\langle \mathfrak{B} \rangle$ values.

Again, the extreme case of using a degenerate CDF leads to a constant degree limit, which was shown to result in an exponential cut-off of the otherwise scale-free degree distribution. In general, for all choices of f , the rich-get-richer growth process is essentially unaltered for nodes with small degrees, until the internal limits are attained. This quenching of degree growth leads to two general properties of the resulting degree distributions: there is a bunching up of degrees near or before $\langle \mathfrak{B} \rangle$, followed by a drop-off in the otherwise SF degree distribution. We find that using the inverse of CDF for localized distributions as f leads to similar results to that of the degenerate distribution (i.e. $\mathfrak{B} = \text{constant}$). Though, as seen in Figure 5, the bunching up of the degree count before $\langle \mathfrak{B} \rangle$ is more spread out and is followed by a slower drop-off whose slope depends on the variance of the distribution, which controls hub formation.

These localized distributions can be contrasted with the results from using the inverse CDF for heavy tail distributions. The curves for both the negative bino-

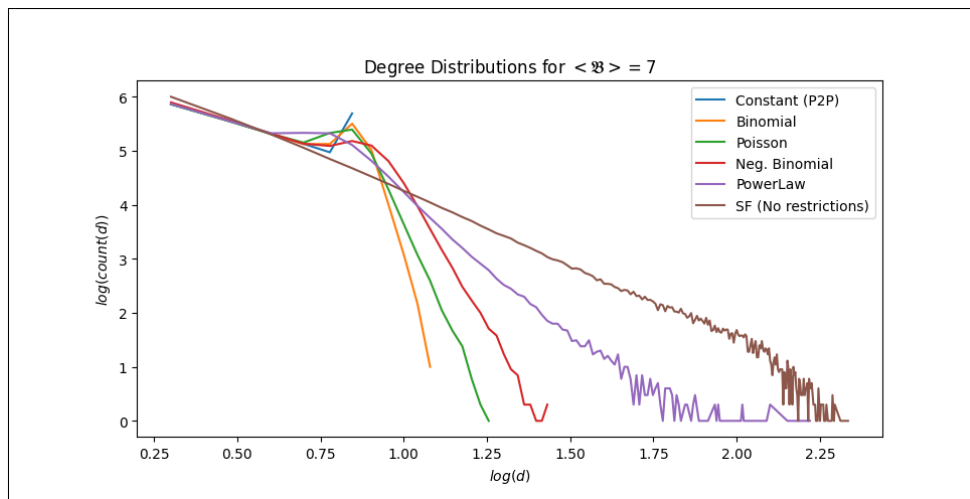


Figure 5: A log-log plot of the degree distributions for 1000 trials of graph sizes $N = 2000$ nodes, i.e. 2 million nodes, resulting from a localized distribution of conductance values being mapped through various inverse CDF, i.e. $f : [0, 1] \rightarrow \mathbb{Z}^+$, with parameters chosen so that $\langle \mathfrak{B} \rangle = 7$ (near the maximum H_{RW} value). As parameters are shifted so that $\langle \mathfrak{B} \rangle$ gets larger, all degree distributions converge to the SF degree distribution regardless of f .

mial distribution and a powerlaw distribution, included in Figure 5, have a less pronounced drop-off in the degree distribution due to higher probability of hub formation. Of interest is the noticeable hump representing an increased number of smaller degree nodes from the skewness of the powerlaw distribution. It appears that the negative binomial distribution provides the most realistic choice for obtaining networks whose degree distribution follow the SF form more closely, except in the region of hub formation, as expected in real-world networks.

5 Conclusions

Nothing in the physical world has true limitless potential. Networks of information flow are restricted by the processing speed of each node, as rivers are restricted by the height of their banks. We have presented a paradigm for understanding why real-world networks tend to display increased hierarchical structure over what is expected from purely scale-free growth by assuming that many networks are coarse-grain models of more intricate networks, and that the ignored properties of the underlying structure impact the growth process. Link constraints can increase hierarchy, but using heavy tailed distributions for the hidden variable can reduce the observed hierarchy to an extent due to increased hub formation. We also find the change in the tail of the degree distribution is associated with the increased measure

of hierarchy. Future work will explore how the growth process itself may impact the development of the underlying conductance values either through a multiscale model or under a *use-it-or-lose-it* growth assumption in an effort to identify the correct distribution to use for this model to effectively reproduce real-world network structures.

Acknowledgments

CTD gratefully acknowledges funding from the Air Force Office of Scientific Research in support of this work.

References

- [1] A. Clauset, C. Moore, and M. E. J. Newman. Hierarchical structure and the prediction of missing links in networks. *Nature*, 453(1):98–101, 2008.
- [2] A.D. Broido and A. Clauset. Scale-free networks are rare. *Nat Commun*, 10, 2019.
- [3] Albert-László Barabási. Scale-Free Networks: A Decade and Beyond. *Science*, 325(5939):412–413, 2009.
- [4] P. Holme. Rare and everywhere: Perspectives on scale-free networks. *Nat Commun*, 10, 2019.
- [5] F.R.K. Chung. *Spectral Graph Theory*. Number 92 in CBMS Regional Conference Series. 1997.
- [6] Ravi Kannan, Santosh Vempala, and Adrian Vetta. On clusterings: Good, bad and spectral. *J. ACM*, 51(3):497–515, May 2004.
- [7] R. I. M. Dunbar. Neocortex size as a constraint on group size in primates. *Journal of Human Evolution*, 22(6):469–493, 1992.
- [8] Enys Mones, Lilla Vicsek, and Tamás’ Vicsek. Hierarchy measure for complex networks. *PLoS ONE*, 7, 2012.
- [9] Dániel Czégel and Gergely Palla. Random walk hierarchy measure: What is more hierarchical, a chain, a tree or a star? *Nature Scientific Reports*, 5(17994), 2015.
- [10] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang. Complex networks: Structure and dynamics. *Physics Reports*, 424(4):175–308, 2006.

- [11] Emily M. Jin, Michelle Girvan, and M. E. J. Newman. Structure of growing social networks. *Phys. Rev. E*, 64:046132, Sep 2001.
- [12] Hasan Guclu and Murat Yuksel. Limited Scale-Free Overlay Topologies for Unstructured Peer-to-Peer Networks. *IEEE trans. parallel and distr. systems*, 20(5):667–679, 2009.
- [13] Albert-László Barabási and Réka Albert. Emergence of Scaling in Random Networks. *Science*, 286:509–512, 1999.
- [14] Krackhardt D. Graph theoretical dimensions of informal organizations. In K. Carley and M. Prietula, editors, *Computational organization theory*, pages 89–111. Lawrence Erlbaum Associates, Hillsdale, NJ, 1994.
- [15] Jianxi Luo and Christopher Magee. Detecting evolving patterns of self-organizing networks by flow hierarchy measurement. *Complexity*, 16:53–61, 07 2011.
- [16] Bernat Coromina-Murtra, Joaquín Goñi, Ricard Solé, and Carlos Rodríguez-Caso'. On the origins of hierarchy in complex networks. *PNAS*, 110(33):13316–13321, 2013.
- [17] Hernán D. Rozenfeld, Shlomo Havlin, and Daniel ben-Avraham. Fractal and transfractal recursive scale-free nets. *New J. Phys.*, 9, 12 2006.
- [18] Ernest O'Boyle. The best and the rest: Revisiting the norm of normality of individual performance. *Personnel Psychology*, 65, 03 2012.