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On a pseudodifferential calculus with modest boundary decay condition

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ON A PSEUDODIFFERENTIAL CALCULUS WITH MODEST BOUNDARY DECAY CONDITION

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Abstract

A boundary decay condition, called vanishing to infinite logarithmic order is introduced. A pseudodifferential calculus, extending the b-calculus of Melrose, is proposed based on this modest decay condition. The mapping properties, composition rule, and normal operators are studied. Instead of functional analytic methods, a geometric approach is invoked in pursuing the Fredholm criterion. As an application, a detailed proof of the Atiyah-Patodi-Singer index theorem, including a review of Dirac operators of product type and construction of the heat kernel, is presented.

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Introduction

In 1975, M. Atiyah, V. Patodi and I. Singer succeeded in extending the celebrated index theorem([4]) to the setting of a manifold with boundary. A Dirac type operator of product type, $\mathcal{D}$, namely, a first-order elliptic differential operator such that the principal symbol of $\mathcal{D}^*\mathcal{D}$ is the metric for all cotangent vectors, and over a color neighborhood of the boundary,

$$\mathcal{D} = \Gamma(\partial_s + \mathcal{D}_0),$$

where $\mathcal{D}_0$ is a self-adjoint Dirac type operator on the boundary, and where Γ is a unitary isomorphism of the the vector bundles in question, is never Fredholm on its natural (Sobelev) domain([5]). To obtain a good index theory, a non-local boundary condition

$$Pf(\cdot\ ,0) = 0,$$

where P is the spectral projection of $\mathcal{D}_0$ corresponding to eigenvalues $\geqslant 0$, was introduced in [1], and the following index formula for the boundary condition problem was proved

$$\mathrm{Ind}\mathcal{D} = \int K_{AS} - \frac{\dim\ker\mathcal{D}_0 + \eta(\mathcal{D}_0)}{2}, \tag{1}$$

where K_{AS} is the Atiyah-Singer integrand as in the local index formula([8]), and $\eta(\mathcal{D}_0)$ the η-invariant of D_0: let $\{\lambda_j\} \subset \mathbb{R}$ be the (discrete) spectrum of $\mathcal{D}_0$,

$$\eta(\mathcal{D}_0) := \lim_{z \to 0} \sum_{\lambda_j \neq 0} \frac{\operatorname{sign} \lambda_j}{|\lambda_j|^z} = \lim_{z \to 0} \eta(z).$$

Note that (1) can be viewed as providing information about the function $\eta(z)$; in particular that $\eta(\mathcal{D}_0)$ is finite and measures the "spectral asymmetry" of $\mathcal{D}_0$. For applications in Riemannian geometry, see [1] and [2]; For more general operators, see [3].

To give an alternative description of the boundary value problem, Atiyah, Patodi and Singer also considered a related non-compact manifold with an infinite cylindrical end, and the non-local boundary value condition was interpreted as L^2-integrability conditions on the non-compact manifold. In 1993, R. Melrose picked up this underdeveloped idea, and extended it into a systematic approach, called b-calculus, to study and generalize the index problems on manifolds with boundaries or corners([25]). Further developments in this direction were obtained in [29], [12], and [16]. The b-calculus can be further embedded into a more general framework called boundary-fibration structure([26]). Various analytic problems on spaces with singular structures can be studied under this framework, for example, see [23], [6], [11] and [28] and many others.

In the b-calculus, the pseudodifferential operators are described by their Schwartz kernels as conormal distributions living in the stretched double product of the original manifold, which is obtained by "blowing up" the product of the boundary in the (ordinary) product of the manifold. To get a good theory, some decaying properties have to be imposed near the left and right boundary of the stretched product. In the classical setting, the condition was specified to be *vanishing in Taylor series*. This choice of decay condition is natural in that it induces nice mapping properties on

(b-)Sobolev spaces, and techniques from functional analysis can be applied.

In this work, inspired by a more geometric approach manifested in [17], [12], [22], or the unpublished manuscript [20], we propose a calculus, called bl-calculus, by replacing the vanishing in Taylor series condition by the *vanishing to infinite logarithmic order* condition. The major attention was paid to the discussion of the Fredholm property: On one hand, the bl-calculus is larger than the b-calculus, inclusive enough to incorporate the process of inverting normal operators (when possible); on the other hand, the function spaces involved in the mapping properties are neither Hilbert nor Banach spaces, hence the usage of functional analysis is required to be minimized. We adapt the "finite rank remainder" method initiated in [19] and developed in [20] to tackle the Fredholm problem. To illustrate the application of this calculus, we include a heat kernel proof of the Atiyah-Patodi-Singer index theorem for a Dirac operator.

Outline

In Chapter 2, we develop the bl-calculus in detail. In Section 1.1, we study the notion of vanishing to infinite logarithmic order. In Section 1.2, we describe the properties of the local bl-symbols and give the definition of the calculus. In Section1.3 - Section 1.4, we describe the mapping properties and the composition rules. Detailed computations are given in a fashion mixing up the geometric and analytic aspects of Melrose's blow-up techniques. In Section 1.5, explicit computation is demonstrated to establish the algebra homomorphism from bl-pseudodifferential operators to their normal operators. In Section 1.6, a substitute of the analytic Fredholm theory is constructed and applied to study the Fredholm problem.

In Chapter 3, we review the Dirac operator of product type on a manifold with cylindrical end. In Chapter 4, we construct the heat kernel of a Dirac operator via the heat calculus. A brief comparison of b-calculus and bl-calculus is included in Section 3.2.

In the appendices, we collect some technical facts needed. In Appendix A, basic notions of conormal distributions with bl-symbols are recovered. This formulation is required in our definition of the calculus. In Appendix B, we describe the b-geometry and the blow-up techniques of Melrose's, on which the numerous calculi associated to different boundary-fibration structures rely heavily. In Appendix C, we introduce a unconventional approach to establishing Fredholm property observed by P. Loya. In Appendix D, we review the fundamental properties of the η-invariant that we need in the proof of the index theorem. We also present some useful, well-known results in spectral theory, which are not readily available in the literature though. To enhance readability, most of the proofs are included.

Chapter 1

bl-pseudodifferential operators

1.1 The logarithmic decay

We fix the notion of logarithmic decay at boundary hypersurfaces of a manifold with corners in this section. We also familiarize the readers with the various coordinate systems we use when studying the stretched double product X_b^2 of X, that is, the blown-up space of X^2 along $\partial X \times \partial X$. Note that X_b^2 is a typical example of manifolds with corners in Melrose's sense ([25]).

Recall that $\mathbb{R}^{n,1} := [0, \infty) \times \mathbb{R}^{n-1}$. Denote the interior of $\mathbb{R}^{n,1}$ by $\mathring{\mathbb{R}}^{n,1}$. A function $u(r, s, y) = (r, s, y_1, \ldots, y_{n-2}) \in C^\infty(\mathring{\mathbb{R}}^{n,1})$ is said to be (b-)*Schwartz in s* within $\{r < a\}$, if given any (multi-)indexes α, β, γ and $\ell \in \mathbb{N}$,

$$\sup_{r<a} \left|(1 + |s|)^\ell (r\partial_r)^\alpha \partial_s^\beta \partial_y^\gamma u(r, s, y)\right| < \infty. \tag{1.1.1}$$

For simplicity, assume that $X = [0, \infty)$. We first look at the left and right boundary of X_b^2. Note that they are a pair of disjoint boundary hypersurfaces of X_b^2. Away from both lb and rb, we may use the "global" coordinates

$$(r, s) := (x + x', \ln \frac{x}{x'}). \tag{1.1.2}$$

Then $X_b^2 \backslash lb \cup rb \cong \mathbb{R}^{2,1}$. Note that lb (resp. rb) is correspondent to $\{s = -\infty\}$ (resp. $\{s = +\infty\}$). Near lb, we may use the coordinates

$$(x', t) := (x', \ln \frac{x}{x'}). \tag{1.1.3}$$

We will, with a bit abuse of language, denote a function in various coordinate systems by the same name. For example, given $u(r, s) \in C^\infty(\mathbb{R}^{2,1}) \cong C^\infty(X_b^2 \backslash lb \cup rb)$, the restriction of u near lb can be represented via (1.1.3) by

$$u(x', t) := u(x'(1 + \mathrm{e}^t), t) = u(r, s).$$

Under this recognition, we can express the transition rules of (first order) b-derivatives between (1.1.2) and (1.1.3) as

$$\begin{cases} r\partial_r & = x'\partial_{x'} \\ \partial_s & = \partial_t - \frac{\mathrm{e}^t}{1+\mathrm{e}^t} \cdot x'\partial_{x'} \end{cases} \text{and} \begin{cases} x'\partial_{x'} & = r\partial_r \\ \partial_t & = \partial_s + \frac{\mathrm{e}^s}{1+\mathrm{e}^s} \cdot r\partial_r \end{cases}. \tag{1.1.4}$$

Also, we can study the transition rules for higher order b-derivatives.

Lemma 1.1.1. *Given any index α, $\partial_s^\alpha = p_\alpha(\partial_t, x'\partial_{x'}, \frac{\mathrm{e}^t}{1+\mathrm{e}^t})$, where $p_\alpha(\zeta, \eta, \xi) = \zeta^\alpha + p'_\alpha(\zeta, \eta, \xi)$ is a polynomial with the degree of ζ in $p'_\alpha(\zeta, \eta, \xi)$ less than α. Similarly, $\partial_t^\alpha = q_\alpha(\partial_s, r\partial_r, \frac{\mathrm{e}^s}{1+\mathrm{e}^s})$ with $q_\alpha(\zeta, \eta, \xi) = \zeta^\alpha + q'_\alpha(\zeta, \eta, \xi)$.*

Proof. We use induction on α. The base case $\alpha = 1$ was just (1.1.4). Assume that the claim is true when $\alpha = k$. Then we compute

$$\begin{aligned}
\partial_s^{k+1} &= (\partial_t - \frac{\mathrm{e}^t}{1+\mathrm{e}^t} \cdot x'\partial_{x'})p_k(\partial_t, x'\partial_{x'}, \frac{\mathrm{e}^t}{1+\mathrm{e}^t}) \\
&= (\partial_t - \frac{\mathrm{e}^t}{1+\mathrm{e}^t} \cdot x'\partial_{x'}) \left(\partial_t^k + \sum a_{\alpha_1,\alpha_2,\alpha_3} \left(\frac{\mathrm{e}^t}{1+\mathrm{e}^t} \right)^{\alpha_1} (x'\partial_{x'})^{\alpha_2} \partial_t^{\alpha_3} \right) \\
&= \partial_t^{k+1} - \frac{\mathrm{e}^t}{1+\mathrm{e}^t} \cdot x'\partial_{x'}\partial_t \\
&\quad + \sum a_{\alpha_1,\alpha_2,\alpha_3} \alpha_1 \left(\frac{\mathrm{e}^t}{1+\mathrm{e}^t} \right)^{\alpha_1} (x'\partial_{x'})^{\alpha_2} \partial_t^{\alpha_3} \\
&\quad - \sum a_{\alpha_1,\alpha_2,\alpha_3} \alpha_1 \left(\frac{\mathrm{e}^t}{1+\mathrm{e}^t} \right)^{\alpha_1+1} (x'\partial_{x'})^{\alpha_2} \partial_t^{\alpha_3} \\
&\quad + \sum a_{\alpha_1,\alpha_2,\alpha_3} \left(\frac{\mathrm{e}^t}{1+\mathrm{e}^t} \right)^{\alpha_1} (x'\partial_{x'})^{\alpha_2} \partial_t^{\alpha_3+1} \\
&\quad - \sum a_{\alpha_1,\alpha_2,\alpha_3} \left(\frac{\mathrm{e}^t}{1+\mathrm{e}^t} \right)^{\alpha_1+1} (x'\partial_{x'})^{\alpha_2+1} \partial_t^{\alpha_3},
\end{aligned}$$

hence the first claim follows from induction principle. The second claim is proved in the identical way. □

Similarly, near rb, we may use the coordinate

$$(x, t') := (x, \ln \frac{x'}{x}), \tag{1.1.5}$$

then

$$\begin{cases} r\partial_r &= x\partial_x \\ \partial_s &= \frac{\mathrm{e}^t}{1+\mathrm{e}^t} \cdot x\partial_x - \partial_{t'} \end{cases} \quad \text{and} \quad \begin{cases} x\partial_x &= r\partial_r \\ \partial_{t'} &= \frac{\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}} \cdot r\partial_r - \partial_s \end{cases}, \tag{1.1.6}$$

and results similar to Lemma 1.1.1 also hold.

Proposition 1.1.2. *Let u be a function in $C^\infty(\mathring{X}_b^2)$. The following statements are equivalent:*

(i) Let $u(r,s)$ be the coordinate representation of u under (1.1.2). Then $u(r,s)$ is Schwartz in s within $\{r < a\}$ for any a, that is, given any indexes α, β and

$\ell \in \mathbb{N}$,

$$\sup_{r<a} \left|(1+|s|)^{\ell}(r\partial_r)^{\alpha}\partial_s^{\beta}u(r,s)\right| < \infty. \tag{1.1.7}$$

(ii) Let $u(x',t)$ and $u(x,t')$ be the coordinate representation of u near lb under (1.1.3) and rb under (1.1.5), respectively. $u(x',t)$ is (left-)Schwartz in t within $\{x' < a\}$ for any a, that is, given any $\delta \in \mathbb{R}$, indexes α, β and $\ell \in \mathbb{N}$,

$$\sup_{\substack{x'<a \\ t<\delta}} \left|(1+|t|)^{\ell}(x'\partial_{x'})^{\alpha}\partial_t^{\beta}u(x',t)\right| < \infty, \tag{1.1.8}$$

and $u(x,t')$ is (left-)Schwartz in t' within any $\{x < a\}$ either.

Proof. $(i)\Rightarrow(ii)$. Consider (1.1.8) first. According to (1.1.2), (1.1.3), (1.1.4) and Lemma 1.1.1, we have

$$(1+|t|)^{\ell}(x'\partial_{x'})^{\alpha}\partial_t^{\beta}u(x',t) = (1+|s|)^{\ell}(r\partial_r)^{\alpha}q_{\beta}(\partial_s, r\partial_r, \frac{\mathrm{e}^s}{1+\mathrm{e}^s})u(r,s).$$

Since $|\mathrm{e}^s(1+\mathrm{e}^s)^{-1}| < 1$, by (1.1.7), we have (1.1.8) holds. To see that $u(x,t')$ is Schwartz in t', just observe that

$$(1+|t'|)^{\ell}(x\partial_x)^{\alpha}\partial_{t'}^{\beta}u(x,t') = (1+|s|)^{\ell}(r\partial_r)^{\alpha}\widetilde{q}_{\beta}(\partial_s, r\partial_r, \frac{\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}})u(r,s)$$

for some polynomial $\widetilde{q}_{\beta}(\zeta,\eta,\xi)$, and $|\mathrm{e}^{-s}(1+\mathrm{e}^{-s})^{-1}| < 1$ as well, then apply the same argument as above.

$(ii)\Rightarrow(i)$. Just "reverse" the process in the last paragraph via (the first part of) Lemma 1.1.1. □

Away form the right and left boundary in X_b^2, we could also use the projective coordinates. Near *lb*, they are given by

$$(z,\omega) := (x', \frac{x}{x'}); \tag{1.1.9}$$

near rb, by

$$(w,\gamma) := (x, \frac{x'}{x}). \tag{1.1.10}$$

The transition rules for b-derivatives between (1.1.3) and (1.1.9) are

$$\begin{cases} x'\partial_{x'} &= z\partial_z \\ \partial_t &= \omega\partial_\omega \end{cases};$$

between (1.1.5) and (1.1.10),

$$\begin{cases} x\partial_x &= w\partial_w \\ \partial_{t'} &= \gamma\partial_\gamma \end{cases}.$$

Recall that $u \in C^\infty(\mathring{X}_b^2)$. Let $u(z,\omega)$ be the coordinate representation of u away from rb under (1.1.9). Then (1.1.8) is equivalent to

$$\sup_{\substack{z<a \\ \omega<\mathrm{e}^\delta}} \left|(1+|\ln\omega|)^\ell (z\partial_z)^\alpha (\omega\partial_\omega)^\beta u(z,\omega)\right| < \infty. \tag{1.1.11}$$

Note that (1.1.11) simply means that all b-derivatives of $u(z,\omega)$ decay faster than any negative power of $|\ln\omega|$ when approaching the left boundary of X_b^2. Hence, we introduce the following definition.

Definition 1.1.3. A function $u \in C^\infty(\mathring{X}_b^2)$ is said to be *vanishing to infinite logarithmic order at the left boundary*, if (1.1.8) or (1.1.11) holds.

The condition of vanishing to infinite logarithmic order at the right boundary is defined accordingly. Proposition 1.1.2 implies that u vanishes to infinite logarithmic order at both the left and right boundary if and only if (1.1.7) holds. We denote the collection of functions satisfying (1.1.7) by ${}^{\mathrm{l}}S^0_{lb,rb}(X_b^2)$.

Note that in (1.1.11), ω is just a boundary defining function for lb. In fact, the decaying condition in question can be defined at any boundary hypersurface of a manifold with corners. Roughly speaking, a function is said to be vanishing to infinite logarithmic order at a boundary hypersurface if all b-derivatives decay faster than any negative power of the logarithm of a (hence any) correspondent boundary defining function.

When studying collections of boundary hypersurfaces with non-empty intersection, we will impose *jointly* decaying conditions near the intersections. For example, $u \in {}^{\mathrm{l}}S^{0}_{lb,rb}(X^2) \subset C^{\infty}(\mathring{X}^2)$ if given any $a > 0$,

$$\sup_{x,\, x'<a} \left((1+|\ln x|)^{\ell}(1+|\ln x'|)^{\ell} \left|(x\partial_x)^{\alpha}(x'\partial_{x'})^{\beta}u(x,x')\right|\right) < \infty \tag{1.1.12}$$

for any index α, β, and $\ell \in \mathbb{N}$. However, it is easy to see that one might as well just require decaying at each boundary hypersurface *separately.*

Proposition 1.1.4. *Let $u \in C^{\infty}(\mathring{X}^2)$. $u \in {}^{\mathrm{l}}S^{0}_{lb,rb}(X^2)$ if and only if given any $a > 0$,*

$$\begin{aligned} &\sup_{x,\, x'<a} \left((1+|\ln x|)^{\ell} \left|(x\partial_x)^{\alpha}(x'\partial_{x'})^{\beta}u(x,x')\right|\right) < \infty, \\ &\sup_{x,\, x'<a} \left((1+|\ln x'|)^{\ell} \left|(x\partial_x)^{\alpha}(x'\partial_{x'})^{\beta}u(x,x')\right|\right) < \infty. \end{aligned} \tag{1.1.13}$$

for all indexes α, β and $\ell \in \mathbb{N}$.

Proof. If (1.1.12) holds, then (1.1.13) holds, since $(1+|\ln x|), (1+|\ln x'|) \geqslant 1$.

Assume that (1.1.13) holds. Note that when $x < 1$, $|\ln x|$ is decreasing, thus when both x and x' are less than 1,

$$(1+|\ln x|)(1+|\ln x'|) < \max\{(1+|\ln x|)^2, (1+|\ln x'|)^2\}.$$

Since (1.1.13) is valid for arbitrary $\ell \in \mathbb{N}$, (1.1.12) follows. □

Another example that plays a fundamental role in the rest of this work is the collection of functions that vanish to infinite logarithmic order at the entire boundary of X_b^2.

Proposition 1.1.5. *Assume that u is a function in $C^\infty(\mathring{X}_b^2)$ with $u(r,s)$ the coordinate representation of u under (1.1.2). $u \in {}^1S^0_{lb,\mathit{ff},rb}(X_b^2)$ if and only if given any indexes α, β and $\ell \in \mathbb{N}$,*

$$\sup_{r<a} \left|(1+|\ln r|)^\ell (1+|s|)^\ell (r\partial_r)^\alpha \partial_s^\beta u(r,s)\right| < \infty. \tag{1.1.14}$$

for any $a > 0$.

Proof. Assume that (1.1.14) is true. Observe that, near lb within $\{t < \delta\}$,

$$\begin{aligned}
&(1+|\ln x'|)^\ell (1+|t|)^\ell (x'\partial_{x'})^\alpha \partial_t^\beta u(x',t) \\
=&(1+\left|\ln \frac{r}{1+\mathrm{e}^s}\right|)^\ell (1+|s|)^\ell (r\partial_r)^\alpha q_\beta(\partial_s, r\partial_r, \frac{\mathrm{e}^s}{1+\mathrm{e}^s})u(r,s) \\
=&(1+|\ln r - \ln(1+\mathrm{e}^s)|)^\ell (1+|s|)^\ell (r\partial_r)^\alpha q_\beta(\partial_s, r\partial_r, \frac{\mathrm{e}^s}{1+\mathrm{e}^s})u(r,s),
\end{aligned}$$

hence by Peetre's inequality,

$$\begin{aligned}
&\left|(1+|\ln x'|)^\ell (1+|t|)^\ell (x'\partial_{x'})^\alpha \partial_t^\beta u(x',t)\right| \\
\leqslant&(1+|\ln r|)^\ell (1+\ln(1+\mathrm{e}^s))^\ell (1+|s|)^\ell \left|(r\partial_r)^\alpha q_\beta(\partial_s, r\partial_r, \frac{\mathrm{e}^s}{1+\mathrm{e}^s})u(r,s)\right| \\
\leqslant&\left(1+\ln\left(1+\mathrm{e}^\delta\right)\right)^\ell (1+|\ln r|)^\ell (1+|s|)^\ell \left|(r\partial_r)^\alpha q_\beta(\partial_s, r\partial_r, \frac{\mathrm{e}^s}{1+\mathrm{e}^s})u(r,s)\right| \\
\leqslant&D^\ell_{\alpha\beta}\left(1+\ln\left(1+\mathrm{e}^\delta\right)\right)^\ell = C^\ell_{\alpha\beta}.
\end{aligned}$$

By the equivalence between (1.1.8) and (1.1.11), $u \in {}^1S^0_{lb,\mathit{ff}}(X_b^2)$. Similarly, near rb

within $\{t' < \delta\}$,

$$\begin{aligned}
&(1+|\ln x|)^\ell (1+|t'|)^\ell \left|(x\partial_x)^\alpha \partial_{t'}^\beta u(x,t')\right| \\
=&(1+\left|\ln \frac{r}{1+\mathrm{e}^{-s}}\right|)^\ell (1+|-s|)^\ell \left|(r\partial_r)^\alpha \widetilde{q}_\beta(\partial_s, r\partial_r, \frac{\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}})u(r,s)\right| \\
\leqslant&(1+\ln(1+\mathrm{e}^{-\delta}))^\ell (1+|\ln r|)^\ell (1+|s|)^\ell \left|(r\partial_r)^\alpha \widetilde{q}_\beta(\partial_s, r\partial_r, \frac{\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}})u(r,s)\right| \\
\leqslant&\widetilde{C}^\ell_{\alpha\beta},
\end{aligned}$$

hence, $u \in {}^1S^0_{f\!f,rb}(X_b^2)$. Therefore we have $u \in {}^1S^0_{lb,f\!f}(X_b^2) \cap {}^1S^0_{f\!f,rb}(X_b^2) = {}^1S^0_{lb,f\!f,rb}(X_b^2)$.

That $u \in {}^1S^0_{lb,f\!f,rb}(X_b^2)$ implies (1.1.14) is proved essentially in the same way. $\square$

Recall that $\mathring{X}_b^2 \cong \mathring{X}^2$, hence $C^\infty(\mathring{X}_b^2)$ can be identifies with $C^\infty(\mathring{X}^2)$. The transition rules for b-derivatives between (1.1.2) and the natural coordinates on X^2, i.e., (x, x') are

$$\begin{cases} x\partial_x &= \frac{\mathrm{e}^s}{1+\mathrm{e}^s}\cdot r\partial_r + \partial_s \\ x'\partial_{x'} &= \frac{\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}}\cdot r\partial_r - \partial_s \end{cases} \text{ and } \begin{cases} r\partial_r = x\partial_x + x'\partial_{x'} \\ \partial_s = \frac{x'}{x+x'}\cdot x\partial_x - \frac{x}{x+x'}\cdot x'\partial_{x'} \end{cases}. \tag{1.1.15}$$

Lemma 1.1.6. *Given any indexes α and β,*

$$(x\partial_x)^\alpha (x'\partial_x')^\beta = p_{\alpha\beta}(r\partial_r, \partial_s, \frac{\mathrm{e}^s}{1+\mathrm{e}^s}, \frac{\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}}), \tag{1.1.16}$$

$$(r\partial_r)^\alpha (\partial_s)^\beta = q_{\alpha\beta}(x\partial_x, x'\partial_{x'}, \frac{x}{x+x'}, \frac{x'}{x+x'}) \tag{1.1.17}$$

for some polynomial $p_{\alpha\beta}(\zeta,\eta,\xi,\lambda)$ and $q_{\alpha\beta}(\zeta,\eta,\xi,\lambda)$.

Proof. The idea of the argument is essentially the same as Lemma 1.1.1. We use double induction.

Fix $\alpha = 0$ first, and apply induction on β. The base case $\beta = 1$ is just (1.1.15). Assume that the result holds when $\beta = k$, that is, $(x'\partial_{x'})^k = p_{0,k}(r\partial_r, \partial_s, \frac{\mathrm{e}^s}{1+\mathrm{e}^s}, \frac{\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}})$, where $p_{0,k}(\zeta, \eta, \xi, \lambda)$ is a polynomial. Then

$$
\begin{aligned}
(x'\partial_{x'})^{k+1} &= (\frac{\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}} \cdot r\partial_r - \partial_s) p_{0,k}(r\partial_r, \partial_s, \frac{\mathrm{e}^{s}}{1+\mathrm{e}^{s}}, \frac{\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}}) \\
&= p_{0,k+1}(r\partial_r, \partial_s, \frac{\mathrm{e}^{s}}{1+\mathrm{e}^{s}}, \frac{\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}}),
\end{aligned}
$$

where $p_{0,k+1}(\zeta, \eta, \xi, \lambda)$ is clearly a polynomial. Hence by induction principle, (1.1.16) holds for $\alpha = 0$ and all β.

Now we apply induction on α for an arbitrary fixed β. The base case $\alpha = 0$ was handled in the last paragraph. Assume that the result holds when $\alpha = k$, then

$$
\begin{aligned}
(x\partial_x)^{k+1}(x'\partial_x')^{\beta} &= (\frac{\mathrm{e}^{s}}{1+\mathrm{e}^{s}} \cdot r\partial_r + \partial_s) p_{k\beta}(r\partial_r, \partial_s, \frac{\mathrm{e}^{s}}{1+\mathrm{e}^{s}}, \frac{\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}}) \\
&= p_{k+1,\beta}(r\partial_r, \partial_s, \frac{\mathrm{e}^{s}}{1+\mathrm{e}^{s}}, \frac{\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}})
\end{aligned}
$$

with $p_{k+1,\beta}(\zeta, \eta, \xi, \lambda)$ a polynomial. By induction principle, (1.1.16) holds for any α, and since β is arbitrary, it holds in general.

(1.1.17) is proved in the identical way. $\square$

The transition rules for b-derivatives between the projective coordinates (1.1.9) and the natural coordinates on X^2 are

$$
\begin{cases} x\partial_x &= \omega\partial_\omega \\ x'\partial_{x'} &= z\partial_z - \omega\partial_\omega \end{cases} \quad \text{and} \quad \begin{cases} z\partial_z &= x\partial_x + x'\partial_{x'} \\ \omega\partial_\omega &= x\partial_x \end{cases};
$$

between (1.1.10) and the natural coordinates on X^2,

$$
\begin{cases} x\partial_x &= w\partial_w - \gamma\partial_\gamma \\ x'\partial_{x'} &= \gamma\partial_\gamma \end{cases} \quad \text{and} \quad \begin{cases} w\partial_w &= x\partial_x + x'\partial_{x'} \\ \gamma\partial_\gamma &= x'\partial_{x'} \end{cases}.
$$

Thus, the transitions of higher order b-derivatives between projective coordinates and natural coordinates are very simple to describe:

Lemma 1.1.7. *Given any indexes α and β,*

$$\begin{aligned}(x\partial_x)^\alpha(x'\partial_{x'})^\beta &= p_{\alpha\beta}(z\partial_z, \omega\partial_\omega)\\ &= \widetilde{p}_{\alpha\beta}(w\partial_w, \gamma\partial_\gamma),\\ (z\partial_z)^\alpha(\omega\partial_\omega)^\beta &= q_{\alpha\beta}(x\partial_x, x'\partial_{x'}),\end{aligned}$$

and

$$(w\partial_w)^\alpha(\gamma\partial_\gamma)^\beta = \widetilde{q}_{\alpha\beta}(x\partial_x, x'\partial_{x'}),$$

where $p_{\alpha\beta}, \widetilde{p}_{\alpha\beta}, q_{\alpha\beta}$ and $\widetilde{q}_{\alpha\beta}$ are polynomials in two variables.

Proposition 1.1.8. *With respect to the natural identification,*

$${}^1S^0_{lb,ff,rb}(X^2_b) = {}^1S^0_{lb,rb}(X^2).$$

Proof. Let $u \in {}^1S^0_{lb,ff,rb}(X^2_b)$. Given any indexes α, β and $\ell \in \mathbb{N}$, by (1.1.16),

$$\begin{aligned}&(1+|\ln x|)^\ell(1+|\ln x'|)^\ell \left|(x\partial_x)^\alpha(x'\partial'_x)^\beta u(x,x')\right|\\ =&(1+\left|\ln\frac{r}{1+\mathrm{e}^{-s}}\right|)^\ell(1+\left|\ln\frac{r}{1+\mathrm{e}^{s}}\right|)^\ell\left|p_{\alpha\beta}(r\partial_r,\partial_s,\frac{\mathrm{e}^s}{1+\mathrm{e}^s},\frac{\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}})u(r,s)\right|.\end{aligned}$$

Applying Peetre's inequality, we have

$$\begin{aligned}(1+\left|\ln\frac{r}{1+\mathrm{e}^{\pm s}}\right|)^\ell &\leqslant (1+|\ln r|)^\ell(1+\left|\ln(1+\mathrm{e}^{\pm s})\right|)^\ell\\ &\leqslant C(1+|\ln r|)^\ell(1+|s|)^{2\ell},\end{aligned}$$

thus from Proposition 1.1.5 and that $\left|\frac{\mathrm{e}^s}{1+\mathrm{e}^s}\right|, \left|\frac{\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}}\right| < 1$, it follows that for any $a > 0$,

$$\sup_{x,x'<a}(1+|\ln x|)^\ell(1+|\ln x'|)^\ell\left|(x\partial_x)^\alpha(x'\partial'_x)^\beta u(x,x')\right| < \infty.$$

Now let $v \in {}^1S^0_{lb,rb}(X^2)$. Fix an arbitrary $a > 0$. Note that when $x, x' < 1/2$, $(1 + |\ln x|)^{-1}(1 + |\ln x'|)^{-1} < (1 + |\ln(x + x')|)^{-1}$, thus there exists some constant C_a such that when $x, x' < a$,

$$(1 + |\ln x|)^{-1}(1 + |\ln x'|)^{-1} < C_a(1 + |\ln(x + x')|)^{-1}.$$

Also note that from the Peetre's inequality again,

$$\begin{aligned}(1 + |\ln x|)^{-1}(1 + |\ln x'|)^{-1} &= (1 + \left|\ln \frac{x}{x'} + \ln x'\right|)^{-1}(1 + |\ln x'|)^{-1} \\ &\leqslant (1 + \left|\ln \frac{x}{x'}\right|)^{-1}(1 + |\ln x'|)(1 + |\ln x'|)^{-1} \\ &= (1 + \left|\ln \frac{x}{x'}\right|)^{-1}.\end{aligned}$$

As a consequence, when $x, x' < a$, we have

$$\begin{aligned}\left|(x\partial_x)^\alpha(x'\partial'_x)^\beta v(x, x')\right| &\leqslant D^{2\ell}_{\alpha\beta}(1 + |\ln x|)^{-2\ell}(1 + |\ln x'|)^{-2\ell} \\ &\leqslant D^{2\ell}_{\alpha\beta}C_a(1 + |\ln(x + x')|)^{-\ell}(1 + \left|\ln \frac{x}{x'}\right|)^{-\ell} \\ &\leqslant \widetilde{D}^{\ell}_{\alpha\beta}(1 + |\ln(x + x')|)^{-\ell}(1 + \left|\ln \frac{x}{x'}\right|)^{-\ell}.\end{aligned}$$

Therefore, from (1.1.17), and noting that $\left|\frac{x}{x+x'}\right|, \left|\frac{x'}{x+x'}\right| < 1$, we conclude that

$$\begin{aligned}\left|(r\partial_r)^\alpha(\partial_s)^\beta v(r, s)\right| &= \left|q_{\alpha\beta}(x\partial_x, x'\partial_{x'}, \frac{x}{x + x'}, \frac{x'}{x + x'})v(x, x')\right| \\ &\leqslant C^{\ell}_{\alpha\beta}(1 + |\ln r|)^{-\ell}(1 + |s|)^{-\ell},\end{aligned}$$

for some constant when $C^{\ell}_{\alpha\beta}$ when $r < a$, which implies $v \in {}^1S^0_{lb,ff,rb}(X^2_b)$. □

In the rest of this section, we study ${}^1S^0_{\partial X}(X)$, the subspace of $C^\infty(\mathring{X})$ consisting of functions that vanish to infinite logarithmic order at the boundary, i.e., $u \in {}^1S^0_{\partial X}(X)$

if given $a > 0$,

$$\sup_{x<a} \left((1 + |\ln x|)^{\ell} \, |(x\partial_x)^{\alpha} u(x)| \right) < \infty$$

for any index α, and $\ell \in \mathbb{N}$, and its relation with some previously studied examples.

Lemma 1.1.9. *Given any $k \in \mathbb{R}$, $(1 + \ln^2 x)^k \cdot {}^{\mathrm{l}}S^0_{\partial X}(X) \subset {}^{\mathrm{l}}S^0_{\partial X}(X)$.*

Proof. Let $u \in {}^{\mathrm{l}}S^0_{\partial X}(X)$. Note that

$$x\partial_x \left[(1 + \ln^2 x)^k u(x) \right] = k(1 + \ln^2 x)^{k-1} (\ln x) u(x) + (1 + \ln^2 x)^k (x\partial_x u(x)),$$

and by induction one could show that in general

$$(x\partial_x)^{\alpha} \left[(1 + \ln^2 x)^k u(x) \right] = \sum_{\alpha,\beta \in \mathbb{N}} a_{\beta\gamma\delta} (1 + \ln^2 x)^{k-\beta} (\ln x)^{\gamma} (x\partial_x)^{\delta} u(x)$$

with $a_{\beta\gamma\delta} \neq 0$ for finitely many terms. Now the claim follows immediately from the definition of ${}^{\mathrm{l}}S^0_{\partial X}(X)$. □

Proposition 1.1.10. *Under the natural identification,*

$${}^{\mathrm{l}}S^0_{\partial X}(X) \otimes {}^{\mathrm{l}}S^0_{\partial X}(X) \subset {}^{\mathrm{l}}S^0_{lb,rb}(X^2) = {}^{\mathrm{l}}S^0_{lb,ff,rb}(X^2_b) \subset C_0(X^2_b),$$

where $C_0(X^2_b)$ is the collection of continuous functions on X^2_b that vanish at the boundary.

Proof. The second equality was established in Proposition 1.1.8. The last inclusion follows immediately from the definitions. It is only left to show the first inclusion. Recall that ${}^{\mathrm{l}}S^0_{\partial X}(X) \otimes {}^{\mathrm{l}}S^0_{\partial X}(X)$ is the collection of functions which can be written as

$$u(x, x') = \sum_{j=1}^{N} u_j(x) v_j(x')$$

for some $u_j, v_j \in {}^{\mathrm{l}}S^0_{\partial X}(X)$. By linearity, it suffices to just prove the case with $N = 1$. Fix an $a > 0$. Note that

$$(x\partial_x)^\alpha (x'\partial_{x'})^\beta (u_1(x)v_1(x')) = (x\partial_x)^\alpha u_1(x) \cdot (x'\partial_{x'})^\beta v_1(x'),$$

hence for any $\ell \in \mathbb{N}$, we have

$$\begin{aligned} \left|(x\partial_x)^\alpha (x'\partial_{x'})^\beta (u_1(x)v_1(x'))\right| &= \left|(x\partial_x)^\alpha u_1(x) \cdot (x'\partial_{x'})^\beta v_1(x')\right| \\ &\leqslant C^\ell_\alpha (1 + |\ln x|)^{-\ell} D^\ell_\beta (1 + |\ln x'|)^{-\ell} \\ &\leqslant C^\ell_{\alpha\beta} (1 + |\ln x|)^{-\ell} (1 + |\ln x'|)^{-\ell} \end{aligned}$$

when $x, x' < a$, and the first inclusion follows. □

We conclude this section with the general definition of vanishing to infinite logarithmic order at boundary hypersurfaces in manifolds with corners.

Definition 1.1.11. Let W be a manifold with corners, and $M_1(W)$ the collection of boundary hypersurfaces of W. Let $\mathscr{H} \subset M_1(W)$. u is in ${}^{\mathrm{l}}S^0_{\mathscr{H}}(W)$ if u vanishes to infinite logarithmic order at each $H_j \in \mathscr{H}$, that is, in any product decomposition $D = [0,\infty)_{x_{j_1}} \times \cdots \times [0,\infty)_{x_{j_N}} \times [0,\infty)_{w_1} \times \cdots \times [0,\infty)_{w_L} \times \mathcal{V}_y$ near a boundary face $M \subset H_j$ with $V_y \subset M$ a coordinate patch, where $\{x_{j_k} = 0\} = H_{j_k} \in \mathscr{H}$, where $G_l = \{w_l = 0\}$ is a boundary hypersurface not in $\mathscr{H}$, and $M = \bigcap_k H_{j_k} \cap \bigcap_l G_l$, we have

$$\sup_{\mathcal{K}} \left| \prod_k (1 + |\ln x_{j_k}|)^\ell \prod_{p=1}^N (x_{j_p}\partial_{x_{j_p}})^{\alpha_p} \prod_{q=1}^L (w_q \partial_{w_q})^{\beta_q} \partial_y^\gamma u(x_j, y) \right| < \infty \tag{1.1.18}$$

for any compact set $\mathcal{K} \subset D$, indexes $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_L, \gamma$, and $\ell \in \mathbb{N}$.

Observe that this definition is not too restrictive. By Corollary B.2.1, u is in ${}^{\mathrm{l}}S^0_{\mathscr{H}}(W)$ if and only if (1.1.18) holds for a local product decomposition cover of $\mathscr{H}$.

Moreover, this decay condition can be imposed for functions defined only near $\mathscr{H}$; in particular, for distributions conormal to some p-submanifold disjoint from $\mathscr{H}$. See Section 1.2 and Appendix A.

1.2 Symbols and kernels

We follow the *geometric approach* to define pseudodifferential operators via their Schwartz kernels. The type of operators is determined by the type of their symbols. We begin with a detailed review of our symbols. We follows the approach in [21] closely.

Definition 1.2.1. A function $\varphi \in C^\infty(\mathring{\mathbb{R}}^{n,k}_{(x,y)})$ is in $S^0(\mathbb{R}^{n,k})$ if for any multi-indexes α and β, and compact set $K \subset \mathbb{R}^{n,k}$,

$$\sup_K \left|(x\partial_x)^\alpha \partial_y^\beta \varphi\right| < \infty,$$

where $(x\partial_x)^\alpha = (x_1\partial_{x_1})^{\alpha_1} \dots (x_k\partial_{x_1})^{\alpha_k}$.

It follows immediately from the definition that ${}^1S^0_{lb,rb}(\mathbb{R}^{2,2}) \subset S^0(\mathbb{R}^{2,2})$, for instance.

Recall that $\alpha \leqslant \beta$, where $\alpha = (\alpha_j), \beta = (\beta_j)$ are multi-indexes, if $\alpha_j \leqslant \beta_j$ for every j.

Lemma 1.2.2. *Given any α and β,*

$$(x\partial_x)^\alpha \partial_y^\beta = \sum_\gamma C^\alpha_\gamma x^\gamma \partial_x^\gamma \partial_y^\beta,$$

where C_γ^α is a constant, such that

$$\begin{cases} C_\gamma^\alpha = 0, & \textit{if } \gamma \nleq \alpha; \\ C_\gamma^\alpha = 0, & \textit{if } \gamma_j = 0 < \alpha_j \textit{ for some } j; \\ C_\alpha^\alpha = 1. \end{cases}$$

Proof. Let k be the number of variables of α. We prove the claim by induction on k.

When $k = 0$, the claim is trivially true. In particular, $C_0^0 = 1$. Assume that the claim holds when $k = m$. Let $\alpha = (\alpha', \alpha_{m+1})$ be arbitrary, where α' is an index of m variables. By induction principle, it suffices to show that

$$(x\partial_x)^\alpha \partial_y^\beta = (x_{m+1}\partial_{x_{m+1}})^{\alpha_{m+1}} (x'\partial_{x'})^{\alpha'} \partial_y^\beta = \sum_{\gamma,\ell} C_{(\gamma,\ell)}^\alpha x_{m+1}^\ell \partial_{x_{m+1}}^\ell (x')^\gamma \partial_{x'}^\gamma \partial_y^\beta,$$

where $(x', x_{m+1}) = x$, where $C_{(\gamma,\ell)}^\alpha$ is a constant such that $C_{(\gamma,\ell)}^\alpha = 0$ if $(\gamma, \ell) \nleq \alpha$, $C_{(\gamma,\ell)}^\alpha = 0$ if $\gamma_j = 0 < \alpha'_j$ for some j or $\ell = 0 < \alpha_{m+1}$, and $C_\alpha^\alpha = 1$.

(1.2.1)

which will be proved, in turn, by induction on α_{m+1}. From the inductive hypotheses on k, clearly statement (1.2.1) holds when $\alpha_{m+1} = 0$. Assume that (1.2.1) is true when $\alpha_{m+1} = p$, then

$$\begin{aligned} (x_{m+1}\partial_{x_{m+1}})^{p+1} (x'\partial_{x'})^{\alpha'} \partial_y^\beta &= x_{m+1}\partial_{x_{m+1}} \left[(x_{m+1}\partial_{x_{m+1}})^p (x'\partial_{x'})^{\alpha'} \partial_y^\beta \right] \\ &= x_{m+1}\partial_{x_{m+1}} \left[\sum_{\gamma,\ell} C_{(\gamma,\ell)}^{(\alpha',p)} x_{m+1}^\ell \partial_{x_{m+1}}^\ell (x')^\gamma \partial_{x'}^\gamma \partial_y^\beta \right] \\ &= \sum_{\gamma,\ell} C_{(\gamma,\ell)}^{(\alpha',p)} \left(\ell x_{m+1}^\ell \partial_{x_{m+1}}^\ell + x_{m+1}^{\ell+1} \partial_{x_{m+1}}^{\ell+1} \right) (x')^\gamma \partial_{x'}^\gamma \partial_y^\beta \\ &= \sum_{\gamma,\ell} C_{(\gamma,\ell)}^{(\alpha',p+1)} x_{m+1}^\ell \partial_{x_{m+1}}^\ell (x')^\gamma \partial_{x'}^\gamma \partial_y^\beta, \end{aligned}$$

where

$$C_{(\gamma,\ell)}^{(\alpha',p+1)} = \ell C_{(\gamma,\ell)}^{(\alpha',p)} + C_{(\gamma,\ell-1)}^{(\alpha',p)}, \tag{1.2.2}$$

in which we take the notational convention that $C_\gamma^\alpha = 0$ when $(0) \nleqslant \gamma$. In particular, we see that $C_\alpha^\alpha = C_{(\alpha',p+1)}^{(\alpha',p+1)} = C_{(\alpha',p)}^{(\alpha',p)} = 1$ from the inductive assumptions. The other assumptions can be verified easily via (1.2.2) as well, and the proof is completed by induction principle. $\square$

Lemma 1.2.3. $\varphi \in S^0(\mathbb{R}^{n,1})$ *if and only if given any* α, β *and compact set* $K \subset \mathbb{R}^{n,1}$,

$$\sup_K \left| x^\alpha \partial_x^\alpha \partial_y^\beta \varphi \right| < \infty. \tag{1.2.3}$$

Proof. We argue by induction on α. If $\varphi \in S^0(\mathbb{R}^{n,1})$, then clearly (1.2.3) holds when $\alpha \leqslant 1$. Now assume that (1.2.3) holds when $\alpha \leqslant m$, then by Lemma 1.2.2, we have

$$\begin{aligned} \left| x^{m+1} \partial_x^{m+1} \partial_y^\beta \varphi \right| &= \left| (x\partial_x)^{m+1} \partial_y^\beta \varphi - \sum_{k=1}^m C_k^{m+1} x^k \partial_x^k \partial_y^\beta \varphi \right| \\ &\leqslant \left| (x\partial_x)^{m+1} \partial_y^\beta \varphi \right| + \sum_{k=1}^m C_k^{m+1} \left| x^k \partial_x^k \partial_y^\beta \varphi \right|. \end{aligned}$$

Thus,

$$\sup_K \left| x^{m+1} \partial_x^{m+1} \partial_y^\beta \varphi \right| \leqslant \sup_K \left| (x\partial_x)^{m+1} \partial_y^\beta \varphi \right| + \sum_{k=1}^m C_k^{m+1} \sup_K \left| x^k \partial_x^k \partial_y^\beta \varphi \right| < \infty.$$

Conversely, if (1.2.3) holds for $\varphi \in C^\infty(\mathring{\mathbb{R}}^{n,1})$ with all α, β and K, then by Lemma 1.2.2, we have

$$\sup_K \left| (x\partial_x)^\alpha \partial_y^\beta \varphi \right| \leqslant \sum_{k=0}^\alpha C_k^\alpha \sup_K \left| x^k \partial_x^k \partial_y^\beta \varphi \right| < \infty.$$

Therefore, $\varphi \in S^0(\mathbb{R}^{n,1})$. $\square$

Remark. In fact, this characterization of S^0-type symbols can be generalized to $\mathbb{R}^{n,k}$ for any $k \leqslant n$. The proof is almost identical but with a double induction. The details are left to the interested readers. We do not need this generalization in this work though.

Lemma 1.2.4. *Assume that*

$$
\begin{aligned}
F : \mathbb{R}^{n,1} &\longrightarrow \mathbb{R}^{n,1} \\
(x, y) &\longrightarrow (xf(x,y), g(x,y))
\end{aligned}
$$

with $f > 0$ is smooth on $\mathbb{R}^{n,1}$. Let $\varphi \in S^0(\mathbb{R}^{n,1})$, then $\varphi \circ F \in S^0(\mathbb{R}^{n,1})$.

Proof. Denote the variables of φ by (u, v), and the those of $\varphi \circ F$ by (x, y). For example, we have

$$\varphi \circ F(x, y) = \varphi(xf(x, y), g(x, y)),$$

and

$$
\begin{aligned}
\partial_{y_j} (\varphi \circ F(x, y)) =& x\partial_u \varphi(xf(x, y), g(x, y))\partial_{y_j} f(x, y) \\
& - \sum_k \partial_{v_k} \varphi(xf(x, y), g(x, y))\partial_{y_j} g_k(x, y).
\end{aligned}
$$

We first observe that

$$\partial_y^\beta \varphi \circ F(x, y) = \sum_{\gamma+|\delta|\leqslant|\beta|} x^\gamma \left(\partial_u^\gamma \partial_v^\delta \varphi\right) (xf(x, y), g(x, y))h^\beta_{\gamma\delta}(x, y) \tag{1.2.4}$$

where $h^\beta_{\gamma\delta} \in C^\infty(\mathbb{R}^{n,1})$. To see this, we proceed by induction on $|\beta|$. It was just seen that (1.2.4) holds when $|\beta| \leqslant 1$. Assume that (1.2.4) also holds for $|\beta| = m$. Then

$$
\begin{aligned}
\partial_{y_j}\partial_y^\beta \varphi \circ F &= \partial_{y_j}\left(\sum_{\gamma+|\delta|\leqslant|\beta|} x^\gamma \left(\partial_u^\gamma \partial_v^\delta \varphi\right)(xf,g) h_{\gamma\delta}^\beta\right) \\
&= \sum_{\gamma+|\delta|\leqslant|\beta|} \Bigg(x^{\gamma+1}\left(\partial_u^{\gamma+1}\partial_v^\delta\varphi\right)(xf,g)\partial_{y_j} f h_{\gamma\delta}^\beta \\
&\quad + \sum_k x^\gamma \left(\partial_u^\gamma \partial_{v_k}\partial_v^\delta\varphi\right)(xf,g)\partial_{y_j} g_k h_{\gamma\delta}^\beta + x^\gamma\left(\partial_u^\gamma\partial_v^\delta\varphi\right)(xf,g)\partial_{y_j} h_{\gamma\delta}^\beta \Bigg) \\
&= \sum_{\gamma+|\delta|\leqslant|\beta|+1} x^\gamma \left(\partial_u^\gamma\partial_v^\delta\varphi\right)(xf,g) \\
&\quad \times \left(\sum_{(\tilde\delta,k)=\delta} \left(\partial_{y_j} g_k\right) h_{\gamma\tilde\delta}^\beta + \left(\partial_{y_j} f\right) h_{\gamma-1,\delta}^\beta + \partial_{y_j} h_{\gamma\delta}^\beta \right) \\
&= \sum_{\gamma+|\delta|\leqslant|\beta|+1} x^\gamma \left(\partial_u^\gamma\partial_v^\delta\varphi\right)(xf,g) h_{\gamma\delta}^{(\beta,j)}
\end{aligned}
$$

where $(\tilde\delta, k) = \tilde\delta + (\delta_{lk})$, $(\beta, j) = \beta + (\delta_{kj})$ with δ_{pq} the Kronecker delta, and $h_{\gamma\delta}^\beta = 0$ if $|\gamma| + |\delta| > |\beta|$. Since j is arbitrary, (1.2.4) holds for $|\beta| = m+1$, and the claim follows from induction principle. Arguing similarly, we conclude that

$$
(x\partial_x)^\alpha \partial_y^\beta \varphi \circ F(x,y) = \sum_{\gamma+|\delta|\leqslant\alpha+|\beta|} x^\gamma \left(\partial_u^\gamma\partial_v^\delta\varphi\right)(xf(x,y), g(x,y)) h_{\gamma\delta}^{\alpha\beta}(x,y),
$$

where $h_{\gamma\delta}^{\alpha\beta} \in C^\infty(\mathbb{R}^{n,1})$. Since $f > 0$,

$$
(x\partial_x)^\alpha \partial_y^\beta \varphi \circ F = \sum_{\gamma+|\delta|\leqslant\alpha+|\beta|} (xf)^\gamma \left(\partial_u^\gamma\partial_v^\delta\varphi\right)(xf,g)\frac{h_{\gamma\delta}^{\alpha\beta}}{(f)^\gamma}.
$$

Now let $K \subset \mathbb{R}^{n,1}$ be an arbitrary compact set. Note that every $h_{\gamma\delta}^{\alpha\beta}(f)^{-\gamma}$ is bounded over K, and by Lemma 1.2.3, $(xf)^\gamma \left(\partial_u^\gamma\partial_v^\delta\varphi\right)(xf,g)$ is also bounded over K. Therefore, we observe that

$$
\sup_K \left|(x\partial_x)^\alpha\partial_y^\beta\varphi\circ F\right| \leqslant \sum_{\gamma+|\delta|\leqslant\alpha+|\beta|} \sup_K \left|(xf)^\gamma\left(\partial_u^\gamma\partial_v^\delta\varphi\right)(xf,g)\right| \cdot \sup_K \left|\frac{h_{\gamma\delta}^{\alpha\beta}}{(f)^\gamma}\right| < \infty. \qquad \square
$$

Proposition 1.2.5. *If $F : \mathbb{R}^{n,1} \longrightarrow \mathbb{R}^{n,1}$ is a diffeomorphism, then $F^*(S^0(\mathbb{R}^{n,1})) = S^0(\mathbb{R}^{n,1})$.*

Proof. Just recall that $F(x, y) = (xf(x, y), g(x, y))$ for some $f, g \in C^\infty(\mathbb{R}^{n,1})$ with $f > 0$, then apply Lemma 1.2.4. $\square$

Let W be a manifold with corners. A function $\varphi \in C^\infty(\mathring{W})$ is in $S^0(W)$ if given any coordinate patch $\mathcal{U} \cong \mathbb{R}^{n,k}$ of W, $\varphi|_{\mathcal{U}} \in S^0(\mathbb{R}^{n,k})$. Clearly, ${}^1S^0_{lb,rb}(X^2) \subset S^0(X^2)$ and ${}^1S^0_{lb,ff,rb}(X^2_b) \subset S^0(X^2_b)$. In general, we have ${}^1S^0_{M_1(W)}(W) \subset S^0(W)$. We now introduce the most important function space in this work.

Definition 1.2.6. A function $\kappa \in {}^1S^0_{lb,rb}(X^2_b) \cap C(X^2_b)$ is in $S^0_{bl}(X^2_b)$ if near the front face, κ can be written as

$$\kappa = \kappa_0 + \kappa_1,$$

where $\kappa_0 \in {}^1S^0_{lb,rb}(ff)$ and $\kappa_1 \in {}^1S^0_{lb,ff,rb}(X^2_b)$.

Following the tradition in study of the b-type calculus, pseudodifferential operators are described by their Schwartz kernels. The first category of operators to be defined is the kind of *residual operators*. They are just the operators with S^0_{bl}-kernels.

Definition 1.2.7. K is called a *residual bl-pseudodifferential operator*, denoted by $K \in \Psi^{-\infty}_{bl}(X)$, if K is in $S^0_{bl}(X^2_b, \Omega_{b,R})$, that is, if μ' is a trivialization of $\Omega_{b,R}(X^2_b)$, then there is a $\kappa \in S^0_{bl}(X^2_b)$ such that $K = \kappa \cdot \mu'$.

A bl-pseudodifferential operator of order m with $m \in \mathbb{R}$ is identified with its Schwartz kernel as well, which is presumed to be a distribution on X^2_b conormal to Δ_b. In particular, a residual bl-operator is ought to be an m-th order bl-operator for any $m \in \mathbb{R}$. To understand what local growth/decay behavior of the symbols of the Schwartz kernels of a bl-operator should be expected near the front face of X^2_b, we take the residual operators as models.

Suppose that $\kappa \in S^0_{bl}(X^2_b)$. Let $\mathcal{U} \cong \mathbb{R}^{n,1}_{(x,y)} \times \mathbb{R}^n_z$ be a coordinate patch near $\mathit{ff} \cap \Delta_b$, such that $\Delta_b \cap \mathcal{U} \cong R^{n,1} \times \{0\}$, and $\mathit{ff} \cap \mathcal{U} \cong \{x = 0\} \times \mathbb{R}^{n-1} \times \mathbb{R}^n$. Pick a $\psi \in C^\infty_c(\mathcal{U})$. Continue to write the coordinate representation of $\psi\kappa|_{\mathcal{U}}$ as $\kappa(x, y, z)$. Then by definition of $S^0_{bl}(X^2_b)$, we can write

$$\kappa(x, y, z) = \kappa_0(y, z) + \kappa_1(x, y, z),$$

where $\kappa_0(y, z) \in C^\infty_c(\mathbb{R}^{n-1} \times \mathbb{R}^n)$, and $\kappa_1(x, y, z) \in {}^1S^0_\partial(\mathbb{R}^{n,1} \times \mathbb{R}^n) \cap C_c(\mathbb{R}^{n,1} \times \mathbb{R}^n)$ with $\partial := \partial(\mathbb{R}^{n,1} \times \mathbb{R}^n)$. Since the functions in question are compactly supported in z, they can be expressed in terms of Fourier (inverse) transform:

$$\begin{aligned}\int \mathrm{e}^{iz\cdot\xi}\, \widehat{k}(x, y, \xi)\, đ\xi &= k(x, y, z)\\ &= \kappa_0(y, z) + \kappa_1(x, y, z)\\ &= \int \mathrm{e}^{iz\cdot\xi}\, \widehat{\kappa}_0(y, \xi)\, đ\xi + \int \mathrm{e}^{iz\cdot\xi}\, \widehat{\kappa}_1(x, y, \xi)\, đ\xi\end{aligned}$$

From the uniqueness of Fourier transform, in particular we have

$$\widehat{k}(x, y, \xi) = \widehat{\kappa}_0(y, \xi) + \widehat{\kappa}_1(x, y, \xi).$$

Note that $\widehat{\kappa}_0$ is Schwartz in both variables. Also, observe that

$$\begin{aligned}&\left|[(1 + |\ln x|)(1 + |y|)\xi]^\ell (x\partial_x)^\alpha \partial^\beta_y \partial^\gamma_\xi \widehat{\kappa}_1(x, y, \xi)\right|\\ =&\left|\int \partial^\ell_z\, \mathrm{e}^{-iz\cdot\xi}\, z^\alpha [(1 + |\ln x|)(1 + |y|)]^\ell (x\partial_x)^\alpha \partial^\beta_y \kappa_1(x, y, z)\ \mathrm{d}z\right|\\ =&\left|\sum_{\ell_1+\ell_2=\ell} \int \mathrm{e}^{-iz\cdot\xi} (\partial^{\ell_1}_z z^\alpha) [(1 + |\ln x|)(1 + |y|)]^\ell (x\partial_x)^\alpha \partial^\beta_y \partial^{\ell_2}_z \kappa_1(x, y, z)\ \mathrm{d}z\right| \leqslant C.\end{aligned}$$

The following definition is motivated thereby:

Definition 1.2.8. Let $m \in \mathbb{R}$. Assume that $a(x, y, z, \xi) \in C^\infty(\mathring{\mathbb{R}}^{n,1}_{(x,y)} \times \mathbb{R}^n_z \times \mathbb{R}^n_\xi) \cap$

$C(\mathbb{R}^{n,1}_{(x,y)} \times \mathbb{R}^n_z \times \mathbb{R}^n_\xi)$. a is in $S^m_{bl}(\mathbb{R}^{n,1} \times \mathbb{R}^n; \mathbb{R}^n)$ if a can be written as

$$a(x, y, z, \xi) = a_0(y, z, \xi) + a_1(x, y, z, \xi),$$

such that $a_0 \in S^m(\mathbb{R}^{n-1} \times \mathbb{R}^n; \mathbb{R}^n)$, and a_1 satisfies the following condition: given any multi-indexes α, β, γ and $k \in \mathbb{N}$, there is some constant $C^k_{\alpha\beta\gamma}$, such that

$$\left|(x\partial_x)^\alpha (\partial_y \partial_z)^\beta \partial_\xi^\gamma a_1(x, y, z, \xi)\right| < C^k_{\alpha\beta\gamma} \left[(1 + |\ln x|)\,(1 + |y| + |z|)\right]^{-k} (1 + |\xi|)^{m-|\gamma|}. \tag{1.2.5}$$

Denote the collection of functions satisfying (1.2.5) by ${}^1S^m_\partial(\mathbb{R}^{n,1} \times \mathbb{R}^n; \mathbb{R}^n)$.

Proposition 1.2.9. *Let $\psi \in C^\infty(\mathbb{R}^{n,1} \times \mathbb{R}^n)$ such that $\psi \equiv 0$ near $\mathbb{R}^{n,1} \times \{0\}$. Given any $a \in S^m_{bl}(\mathbb{R}^{n,1} \times \mathbb{R}^n; \mathbb{R}^n)$ with $m \in \mathbb{R}$, define a distribution u by*

$$u := \int e^{iz\xi}\, a(x, y, z, \xi)\, đ\xi,$$

then $\psi u \in S^0_{bl}(\mathbb{R}^{n,1} \times \mathbb{R}^n)$, that is, ψu can be written as

$$\psi u(x, y, z) = u_0(y, z) + u_1(x, y, z),$$

where $u_0 \in S^0(\mathbb{R}^{n-1} \times \mathbb{R}^n) = C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$, and $u_1 \in {}^1S^0_\partial(\mathbb{R}^{n,1} \times \mathbb{R}^n)$.

Proof. Let $\rho \in \mathcal{S}(\mathbb{R}^n)$ with $\rho(0) = 1$. Define

$$u_r := \int e^{iz\xi}\, a_r(x, y, z, \xi)\, đ\xi,$$

where $a_r(x, y, z, \xi) = \rho(r\xi) a(x, y, z, \xi)$ with $0 < r < 1$. Note that by the continuity principle, $u_r \xrightarrow{r \to 0} u$ and $\psi u_r \xrightarrow{r \to 0} \psi u$ as distributions. We will show that ψu_r converges to a function satisfying the desired properties. In what follows, we denote

$\sum_j \partial^2_{\xi_j}$ by $|\partial_\xi|^2$. Given any $z \neq 0$, we compute

$$\begin{aligned}
u_r &= \frac{1}{|z|^{2\delta}} \int (|z|^2)^\delta \, \mathrm{e}^{iz\xi} \, a_r(x,y,z,\xi) \, đ\xi \\
&= \frac{1}{|z|^{2\delta}} \int (-1)^\delta \left(\left(|\partial_\xi|^2\right)^\delta \mathrm{e}^{iz\xi} \right) a_r(x,y,z,\xi) \, đ\xi \\
&= \frac{1}{|z|^{2\delta}} \int (-1)^\delta \, \mathrm{e}^{iz\xi} \left(\left(|\partial_\xi|^2\right)^\delta a_r(x,y,z,\xi) \right) \, đ\xi \\
&= (-1)^\delta \frac{1}{|z|^{2\delta}} \sum_{\substack{|\delta_1|+|\delta_2|=2\delta \\ \delta_1+\delta_2 \in (2\mathbb{N})^n}} \int D_{\delta_1,\delta_2} \, \mathrm{e}^{iz\xi} \, r^{|\delta_1|} (\partial_\xi^{\delta_1} \rho)(r\xi)(\partial_\xi^{\delta_2} a)(x,y,z,\xi) \, đ\xi,
\end{aligned}$$

where D_{δ_1,δ_2} is some constants, hence

$$\begin{aligned}
(x\partial_x)^\alpha \partial_y^\beta \partial_z^\gamma u_r = (-1)^\delta \sum_{\substack{|\delta_1|+|\delta_2|=2\delta \\ \delta_1+\delta_2 \in (2\mathbb{N})^n \\ \gamma_1+\gamma_2+\gamma_3=\gamma}} \int \xi^{\gamma_2} \, \mathrm{e}^{iz\xi} (\partial_\xi^{\delta_1} \rho)(r\xi) \left((x\partial_x)^\alpha \partial_y^\beta \partial_z^{\gamma_3} \partial_\xi^{\delta_2} a \right)(x,y,z,\xi) \, đ\xi \\
\times D_{\delta_1,\delta_2} \partial_z^{\gamma_1} \left(\frac{i^{|\gamma_2|} r^{|\delta_1|}}{|z|^{2\delta}} \right). \qquad (1.2.6)
\end{aligned}$$

Now fix an integer $\delta > \frac{m+2n+|\gamma|}{2}$. Observe that

$$\begin{aligned}
&\left| r^{|\delta_1|} \int \xi^{\gamma_2} \, \mathrm{e}^{iz\xi} (\partial_\xi^{\delta_1} \rho)(r\xi)((x\partial_x)^\alpha \partial_y^\beta \partial_z^{\gamma_3} \partial_\xi^{\delta_2} a)(x,y,z,\xi) \, đ\xi \right| \\
=& \frac{r^{|\delta_1|}}{r} \left| \int \xi^{\gamma_2} \, \mathrm{e}^{iz\xi/r} (\partial_\xi^{\delta_1} \rho)(\xi)((x\partial_x)^\alpha \partial_y^\beta \partial_z^{\gamma_3} \partial_\xi^{\delta_2} a)(x,y,z,\frac{\xi}{r}) \, đ\xi \right| \\
\leqslant& C_{|\alpha|,|\beta|,|\gamma|,\delta} \frac{r^{|\delta_1|}}{r} \int (1+|\xi|)^{|\gamma|} \left| (\partial_\xi^{\delta_1} \rho)(\xi) \right| \left(1 + \frac{|\xi|}{r} \right)^{m-\delta_2} đ\xi \\
=& \begin{cases}
C_{|\alpha|,|\beta|,|\gamma|,\delta} \dfrac{r^{2\delta}}{r^{1+m}} \displaystyle\int \left| (\partial_\xi^{\delta_1} \rho)(\xi) \right| (1+|\xi|)^{m+|\gamma|-\delta_2} \, đ\xi, & \text{if } m-\delta_2 \geqslant 0, \\
C_{|\alpha|,|\beta|,|\gamma|,\delta} \dfrac{r^{|\delta_1|}}{r} \displaystyle\int \left| (\partial_\xi^{\delta_1} \rho)(\xi) \right| (1+|\xi|)^{m+|\gamma|-\delta_2} \, đ\xi, & \text{if } -2n-|\gamma| < m-\delta_2 < 0, \\
C_{|\alpha|,|\beta|,|\gamma|,\delta} \, r^{|\delta_1|} \displaystyle\int \left| (\partial_\xi^{\delta_1} \rho)(r\xi) \right| (1+|\xi|)^{m+|\gamma|-\delta_2} \, đ\xi, & \text{if } m-\delta_2 \leqslant -2n-|\gamma|,
\end{cases}
\end{aligned}$$

thus all terms in (1.2.6) with $|\delta_1| > 0$ vanish as r approaches 0, and consequently,

$$\begin{aligned}\lim_{r\to 0}\left|(x\partial_x)^\alpha \partial_y^\beta \partial_z^\gamma u_r\right| \leqslant & 2^\delta \sum_{\gamma_1+\gamma_2+\gamma_3=\gamma} \left|\partial_z^{\gamma_1} |z|^{-2\delta}\right| \\ & \cdot \int \left|\xi^{\gamma_2}((x\partial_x)^\alpha \partial_y^\beta \partial_z^{\gamma_3} \left(|\partial_\xi|^2\right)^\delta a)(x,y,z,\xi)\right| đ\xi \\ \leqslant & 2^\delta C_{|\alpha|,|\beta|,|\gamma|,\delta} \int (1+|\xi|)^{-2n}\, đ\xi \times \sum_{\gamma_1+\gamma_2+\gamma_3=\gamma} \left|\partial_z^{\gamma_1} |z|^{-2\delta}\right|,\end{aligned}$$

which implies that ψu_r converges to a function in $C^\infty(\mathring{\mathbb{R}}^{n,1}\times\mathbb{R}^n)$ whose b-derivatives are bounded over any compact set, since $\psi \equiv 0$ near $\{z=0\}$, that is, $\psi u = \lim_{r\to 0}\psi u_r \in S^0(\mathbb{R}^{n,1}\times\mathbb{R}^n)$.

Now write $a = a_0 + a_1$ with $a_0 \in S^m(\mathbb{R}^{n-1}\times\mathbb{R}^n;\mathbb{R}^n)$ and $a_1 \in {}^{\mathrm{l}}S_\partial^m(\mathbb{R}^{n,1}\times\mathbb{R}^n;\mathbb{R}^n)$. Then correspondingly $(u_0)_r \in S^0(\mathbb{R}^{n-1}\times\mathbb{R}^n)$ and $(u_1)_r \in {}^{\mathrm{l}}S_\partial^0(\mathbb{R}^{n,1}\times\mathbb{R}^n)$. Note that $\psi(x,y,z) = \psi(0,y,z) + x\psi_1(x,y,z) = \psi_0(y,z) + x\psi_1(x,y,z)$ for some $\psi_1(x,y,z \in) \in C^\infty(\mathbb{R}^{n,1}\times\mathbb{R}^n)$, hence

$$\begin{aligned}\psi u_r(x,y,z) = & \psi_0(y,z)(u_0)_r(y,z) + x\psi_1(x,y,z)(u_0)_R(y,z) \\ & + \psi_0(y,z)(u_1)_r(x,y,z) + x\psi_1(x,y,z)(u_1)_r(x,y,z).\end{aligned}$$

Define $u_0 := \lim_{r\to 0}\psi_0(y,z)(u_0)_r(y,z)$ and $u_1 := \lim_{r\to 0}[x\psi_1(x,y,z)(u_0)_R(y,z) + \psi_0(y,z)\cdot(u_1)_r(x,y,z) + x\psi_1(x,y,z)(u_1)_r(x,y,z)]$. Then similar arguments like above show that u_1 and u_2 satisfy the desired properties. $\square$

Remark. It can further be shown that $\psi u(x,y,z)$ is Schwartz in z. The reader should be advised that this observation is, in fact, the motivation for the notion of vanishing to infinite logarithmic order.

We denote the collection of conormal distributions associated with $S_{bl}^m(\mathbb{R}^{n,1}\times\mathbb{R}^n;\mathbb{R}^n)$ by $I_{bl}^m(\mathbb{R}^{n,1}\times\mathbb{R}^n,\mathbb{R}^n)$. Note that in particular $S_{bl}^0(\mathbb{R}^{n,1}\times\mathbb{R}^n) \cong I_{bl}^{-\infty}(\mathbb{R}^{n,1}\times\mathbb{R}^n) := I_{bl}^{-\infty}(\mathbb{R}^{n,1}\times\mathbb{R}^n,\mathbb{R}^n)$.

Example 1.2.10. Let $m \in \mathbb{N}$ and

$$a(x,y,\xi) = \sum_{|\alpha| \leqslant m} i^{|\alpha|} a_\alpha(x,y) \xi^\alpha,$$

where $a_\alpha \in C_c^\infty(\mathbb{R}^{n,1})$. Then $a \in S_{bl}^m(\mathbb{R}^{n,1}; \mathbb{R}^n)$, since

$$\begin{aligned} a(x,y,\xi) &= \sum_{|\alpha| \leqslant m} i^{|\alpha|} \left(a_\alpha(0,y) + x a'_\alpha(x,y) \right) \xi^\alpha \\ &= \sum_{|\alpha| \leqslant m} i^{|\alpha|} a_\alpha(0,y) \xi^\alpha + \sum_{|\alpha| \leqslant m} i^{|\alpha|} x a'_\alpha(x,y) \xi^\alpha \\ &= a_0(y,\xi) + a_1(x,y,\xi), \end{aligned}$$

where $a'_\alpha \in C_c^\infty(\mathbb{R}^{n,1})$ by the Taylor's theorem. We will see that

$$u := \int \mathrm{e}^{iz\xi}\, a(x,y,\xi)\, đ\xi$$

vanishes outside $\{z = 0\}$. In fact, with the same notations as in Proposition 1.2.9, we compute

$$\begin{aligned} \psi u_r &= \psi \int \mathrm{e}^{iz\xi}\, \rho(r\xi) \sum_{|\alpha| \leqslant m} i^{|\alpha|} a_\alpha(x,y) \xi^\alpha\, đ\xi \\ &= \psi \sum_{|\alpha| \leqslant m} \frac{a_\alpha(x,y)}{r^{|\alpha|+1}} (\partial_z^\alpha \check{p})(\frac{z}{r}) \\ &\leqslant \psi \sum_{|\alpha| \leqslant m} \frac{a_\alpha(x,y)}{r^{|\alpha|+1}} C_\alpha (1 + \left|\frac{z}{r}\right|)^{-|\alpha|-2} \\ &= \psi \sum_{|\alpha| \leqslant m} C_\alpha a_\alpha(x,y) r (r + |z|)^{-|\alpha|-2}. \end{aligned}$$

Hence

$$\psi u = \lim_{r \to 0} \psi u_r = 0.$$

The claim then follows from the arbitrariness of ψ.

The following lemma is proved in the same way as Lemma 1.2.3.

Lemma 1.2.11. *Assume that* $a(x,y,z,\xi) \in C^\infty(\mathring{\mathbb{R}}^{n,1}{}_{(x,y)} \times \mathbb{R}^n_z \times \mathbb{R}^n_\xi) \cap C(\mathbb{R}^{n,1}_{(x,y)} \times \mathbb{R}^n_z \times \mathbb{R}^n_\xi)$ *such that* $a(0,y,z,\xi) \in S^m(\mathbb{R}^{n-1} \times \mathbb{R}^n; \mathbb{R}^n)$*. Then* $a \in S^m_{bl}(\mathbb{R}^{n,1} \times \mathbb{R}^n; \mathbb{R}^n)$ *with* $m \in \mathbb{R}$ *if and only if given any multi-indexes* α, β, γ *and* $\ell \in \mathbb{N}$*, there is some constant* $C^\ell_{\alpha\beta\gamma}$*, such that*

$$\left| x^\alpha \partial_x^\alpha (\partial_y \partial_z)^\beta \partial_\xi^\gamma a_1(x,y,z,\xi) \right| < C^\ell_{\alpha\beta\gamma} \left[(1 + |\ln x|) \left(1 + |y| + |z|\right) \right]^{-\ell} (1 + |\xi|)^{m-|\gamma|},$$

where $a_1(x,y,z,\xi) = a(x,y,z,\xi) - a(0,y,z,\xi)$*.*

The next result is an adaptation of the coordinate invariance in the standard theory of conormal distributions.

Proposition 1.2.12. *Assume that*

$$\begin{aligned} F : \mathbb{R}^{n,1} \times \mathbb{R}^n &\longrightarrow \mathbb{R}^{n,1} \times \mathbb{R}^n \\ (x,y,z) &\longmapsto (xf(x,y,z), g(x,y,z), h(x,y,z)) \end{aligned}$$

is a diffeomorphism with $h(x,y,0) = 0$*. Let* $a \in S^m_{bl}(\mathbb{R}^{n,1} \times \mathbb{R}^n; \mathbb{R}^n)$*,* $\psi \in C^\infty_c(\mathbb{R}^{n,1} \times \mathbb{R}^n)$*, and define* $u \in I^m_{bl}(\mathbb{R}^{n,1} \times \mathbb{R}^n, \mathbb{R}^{n,1} \times \{0\})$ *by*

$$u(u,v,w) = \psi \int \mathrm{e}^{iv\cdot\xi}\, a(u,v,w,\xi)\, đ\xi$$

Then

$$F^* u(x,y,z) = \int \mathrm{e}^{iz\cdot\xi}\, a'(x,y,z,\xi)\, đ\xi.$$

where $a' = \tilde{a} + r$*, with* $\tilde{a}$ *defined in (1.2.7) and* $r \in S^{-\infty}_{bl}(\mathbb{R}^{n,1} \times \mathbb{R}^n; \mathbb{R}^n)$*.*

Proof. Let $\rho(\xi) \in \mathcal{S}(\mathbb{R}^n)$ with $\rho(0) = 1$ and

$$u_r := \psi \int \mathrm{e}^{iv\cdot\xi}\, \rho(r\xi) a(u,v,w,\xi)\, đ\xi,$$

then $u_r \in I_{bl}^{-\infty}(\mathbb{R}^{n,1} \times \mathbb{R}^n)$. Note that $f(x,y,z) > 0$ and $h(x,y,z) = \widetilde{h}(x,y,z)z$, where $\widetilde{h}(x,y,z)$ is a matrix-valued function with $\widetilde{h}(x,y,0) = \partial_z h(x,y,0)$. In addition, $\widetilde{h}$ is invertible in a neighborhood of $\{z = 0\}$. We compute

$$\begin{aligned} F^* u_r(x,y,z) &= \psi \circ F \int \mathrm{e}^{ih(x,y,z)\cdot\xi}\, \rho(r\xi) a(F(x,y,z),\xi)\, đ\xi \\ &= \widetilde{\psi} \int \mathrm{e}^{i\widetilde{h}(x,y,z)z\cdot\xi}\, \rho(r\xi) a(F(x,y,z),\xi)\, đ\xi \\ &= \widetilde{\psi} \int \mathrm{e}^{iz\cdot\widetilde{h}(x,y,z)^T\xi}\, \rho(r\xi) a(F(x,y,z),\xi)\, đ\xi, \end{aligned}$$

where $\widetilde{\psi} = \psi \circ F$. Now let $\phi \in C^\infty(\mathbb{R}^{n,1} \times \mathbb{R}^n)$ such that $\phi \equiv 1$ near $\{z = 0\}$ and supported inside the set where $\widetilde{h}$ is invertible. Write $F^*u_r = \phi F^* u_r + (1-\phi)F^*u_r$. With the change of variables $\xi \to h^T\xi$,

$$\begin{aligned} \phi F^* u_r(x,y,z) &= \int \mathrm{e}^{iz\cdot\widetilde{h}(x,y,z)^T\xi}\, \phi\widetilde{\psi}\rho(r\xi) a(F(x,y,z),\xi)\, đ\xi \\ &= \int \mathrm{e}^{iz\cdot\xi}\, \frac{\rho((\widetilde{h}^T)^{-1} r\xi)\phi\widetilde{\psi}}{\left|\det \widetilde{h}\right|}\, a(F(x,y,z),(\widetilde{h}^T)^{-1}\xi)\, đ\xi \\ &= \int \mathrm{e}^{iz\cdot\xi}\, \widetilde{\rho}(x,y,z,r\xi)\widetilde{a}(x,y,z,\xi)\, đ\xi \end{aligned}$$

where $\widetilde{\rho}(x,y,z,\xi) = \rho((\widetilde{h}^T)^{-1}(x,y,z)\xi)$ (defined over $\operatorname{supp}\phi$), and

$$\widetilde{a}(x,y,z,\xi) = \frac{\phi\widetilde{\psi}}{\left|\det\widetilde{h}\right|}(x,y,z) a(F(x,y,z),(\widetilde{h}^T)^{-1}\xi)\ . \tag{1.2.7}$$

Let $G = \operatorname{supp}\widetilde{\psi} \cap \operatorname{supp}\phi$, then in particular G is compact and $\operatorname{supp}\widetilde{a}(\cdot,\xi) \subset G$. Note that $\widetilde{a}|_{x=0}$ is in $S^m(\mathbb{R}^{n-1}\times\mathbb{R}^n;\mathbb{R}^n)$, since $a|_{u=0} \in S^m(\mathbb{R}^{n-1}\times\mathbb{R}^n;\mathbb{R}^n)$, $\widetilde{\psi}$ is compactly supported and $(\widetilde{h}^T)^{-1}$ is an invertible matrix, in particular $c\,|\xi| \leqslant \left|(\widetilde{h}^T)^{-1}\xi\right| \leqslant C\,|\xi|$ over the support of ψ. Let

$$\widetilde{a}_1(x,y,z,\xi) = \widetilde{a}(x,y,z,\xi) - \widetilde{a}(0,y,z,\xi).$$

Arguing similarly as in Lemma 1.2.4, we have

$$
\begin{aligned}
(x\partial_x)^\alpha(\partial_y\partial_z)^\beta\partial_\xi^\gamma\widetilde{a}_1 &= \sum_{\substack{|\lambda|=|\gamma| \\ \delta+\epsilon+\theta=\alpha+|\beta|}} H^{\alpha\beta\gamma\lambda}_{\delta\epsilon\theta}(Q_\xi)^\theta x^\delta\partial_u^\delta(\partial_v\partial_w)^\epsilon\partial_\xi^\lambda a_1(F,(\widetilde{h}^T)^{-1}\xi) \\
&= \sum_{\substack{|\lambda|=|\gamma| \\ \delta+\epsilon+\theta=\alpha+|\beta|}} P^{\alpha\beta\gamma\lambda}_{\delta\epsilon\theta} a_1(F,(\widetilde{h}^T)^{-1}\xi),
\end{aligned}
$$

where $Q_\xi = \sum_{i,j}\xi_i\partial_{\xi_j}$ and $H^{\alpha\beta\gamma\lambda}_{\delta\epsilon\theta} \in C_c^\infty(\mathbb{R}^{n,1}\times\mathbb{R}^n)$. By Lemma 1.2.11, that $(\widetilde{h}^T)^{-1}$ is an invertible matrix and that $\min_G|f| > 0$, we have

$$
\sup_{(x,y,z)\in G}\left|[(1+|\ln x|)(1+|(y,z)|)]^k(1+|\xi|)^{|\lambda|-m}\frac{f^\delta P^{\alpha\beta\gamma\lambda}_{\delta\epsilon\theta}}{f^\delta}a_1(F,(\widetilde{h}^T)^{-1}\xi)\right| < \infty,
$$

and consequently, $\widetilde{a} \in S^m_{bl}(\mathbb{R}^{n,1}\times\mathbb{R}^n;\mathbb{R}^n)$. In addition, we define

$$
\begin{aligned}
\phi F^*u &:= \lim_{r\to 0}\phi F^*u_r \\
&= \int \mathrm{e}^{iz\cdot\xi}\,\widetilde{a}(x,y,z,\xi)\,đ\xi.
\end{aligned}
$$

The reason for choice of notation above will be clear momentarily. On the other hand, note that $(1-\phi)F^*u = F^*(1-\widetilde{\phi})u$ where $\widetilde{\phi} = \phi\circ F^{-1}$. Since F preserves $\mathbb{R}^{n,1}\times\{0\}$, $1-\widetilde{\phi}$ vanishes near $\mathbb{R}^{n,1}\times\{0\}$ as $1-\phi$ does. Hence by Proposition 1.2.9, $(1-\widetilde{\phi})u \in I^{-\infty}_{bl}(\mathbb{R}^{n,1}\times\mathbb{R}^n)$, then by Proposition 1.2.5, $(1-\phi)F^*u \in I^{-\infty}_{bl}(\mathbb{R}^{n,1}\times\mathbb{R}^n)$. Since $\lim_{r\to 0}(1-\phi)F^*u_r = \lim_{r\to 0}F^*(1-\widetilde{\phi})u_r = F^*(1-\widetilde{\phi})u = (1-\phi)F^*u$, we conclude that

$$
F^*u = \lim_{r\to 0}F^*u_r = \lim_{r\to 0}\phi F^*u_r + \lim_{r\to 0}(1-\phi)F^*u_r = \phi F^*u + (1-\phi)F^*u. \qquad \square
$$

Lemma 1.2.13 (Asymptotic completeness)**.** *Given a sequence* $a_j \in S^{m-j}_{bl}(\mathbb{R}^{k,1};\mathbb{R}^n)$,

$j \in \mathbb{N}$, *there exists an* $a \in S^m_{bl}(\mathbb{R}^{k,1}; \mathbb{R}^n)$, *such that for all* $N \in \mathbb{N}$,

$$a - \sum_{j=0}^{N-1} a_j \in S^{m-N}_{bl}(\mathbb{R}^{k,1}; \mathbb{R}^n).$$

Proof. The argument is standard. We give a formula for a and leave the verification to the reader.

Assume that $a_j = a_0^j + a_1^j$ with $a_0^j \in S^{m-j}(\mathbb{R}^{k-1}_y; \mathbb{R}^n_\xi)$ and $a_1^j \in {}^1S^{m-j}_\partial(\mathbb{R}^{k,1}_{(x,y)}; \mathbb{R}^n_\xi)$ Let $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi \geqslant 0$ and

$$\psi(\xi) = \begin{cases} 0, \text{ if } |\xi| \leqslant 1; \\ 1, \text{ if } |\xi| \geqslant 2. \end{cases}$$

Let

$$\epsilon_j = \frac{1}{2(1+\delta_j)},$$

where

$$\begin{aligned}\delta_j = \max\{ &\sup_{|\alpha|\leqslant j} \left|\partial_\xi^\alpha \psi\right| \cdot \sup_{p,|\beta|,|\gamma|\leqslant j} \left|(1+|y|)^p (1+|\xi|)^{j+|\gamma|-m} \partial_y^\beta \partial_\xi^\gamma a_0^j\right|, \\ &\sup_{|\alpha|\leqslant j} \left|\partial_\xi^\alpha \psi\right| \cdot \sup_{p,q,|\beta|,|\gamma|\leqslant j} \left|[(1+|\ln x| + |y|)]^p (1+|\xi|)^{j+|\gamma|-m} (x\partial_x)^q \partial_y^\beta \partial_\xi^\gamma a_1^j\right|\}.\end{aligned}$$

Then

$$a := \sum_{j=0}^{\infty} \psi(\epsilon_j \xi) a_j$$

satisfies the desired condition. □

We are now prepared to present the precise definition of our operators.

Definition 1.2.14. Let $m \in \mathbb{R}$. $K \in \Psi^m_{bl}(X)$ if K is a distributional right b-density on X_b^2 satisfying the following conditions:

1. Given $\psi \in C_c^\infty(X_b^2 \backslash \Delta_b)$, ψK is in $\Psi^{-\infty}_{bl}(X)$.

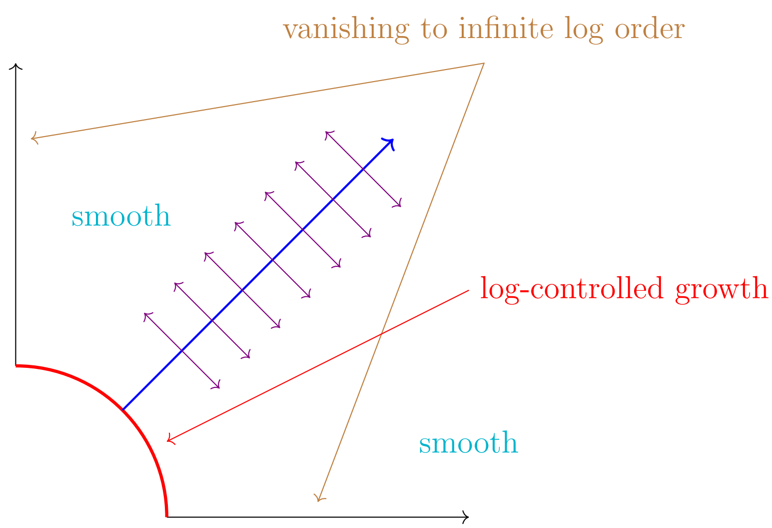


Figure 1.1: Schwartz Kernel of $\Psi^m_{bl}(X)$

2. Given any coordinate patch $\mathcal{U}$ near Δ_b, $\psi \in C^\infty_c(\mathcal{U})$ and a local trivialization μ' of $\Omega_{b,R}$ over $\mathcal{U}$,

   (a) if $\mathcal{U} \cong \mathbb{R}^n_x \times \mathbb{R}^n_z$ away from the front face such that $\Delta_b \cong \mathbb{R}^n_x \times \{0\}_z$, then

   $$\psi K = \int \mathrm{e}^{iz\cdot\xi}\, a(x,\xi)\, đ\xi \cdot \mu'$$

   for some $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$;

   (b) if $\mathcal{U} \cong \mathbb{R}^{n,1}_{(x,y)} \times \mathbb{R}^n_z$ near the front face such that $\Delta_b \cong \mathbb{R}^{n,1}_{(x,y)} \times \{0\}_z$ then

   $$\psi K = \int \mathrm{e}^{iz\cdot\xi}\, a(x,y,\xi)\, đ\xi \cdot \mu'$$

   for some $a \in S^m_{bl}(\mathbb{R}^{n,1};\mathbb{R}^n)$.

Elements in $\Psi^m_{bl}(X)$ are called *bl*-pseudodifferential operators of order m. As a distribution right b-density, $K \in \Psi^m_{bl}(X)$ defines a continuous linear map $K : C^\infty(X^2_b, \Omega_{b,L}) \to \mathbb{C}$. Note that we have $\Psi^m_{bl}(X) \subset I^m_{bl}(X^2_b, \Delta_b, \Omega_{b,R})$, where

$$I^m_{bl}(X^2_b, \Delta_b, \Omega_{b,R}) := I^m_{bl}(X^2_b, \Delta_b) \otimes C^\infty(X^2_b, \Omega_{b,R}).$$

See Figure 1.1 for a diagrammatic description.

Theorem 1.2.15. *For any $m \in \mathbb{R}$, there is a linear map ${}^b\sigma_m : \Psi^m_{bl}(X) \to S^{[m]}_{bl}({}^bT^*X)$ such that the sequence*

$$0 \longrightarrow \Psi^{m-1}_{bl}(X) \longrightarrow \Psi^m_{bl}(X) \xrightarrow{{}^b\sigma_m} S^{[m]}_{bl}({}^bT^*X) \longrightarrow 0$$

is exact.

Proof. Given any $K \in \Psi^m_{bl}(X)$, there exist finitely many $u_j \in I^m_{bl}(X^2_b, \Delta_b)$ and $\mu_j \in C^\infty(X^2_b, \Omega_{b,R})$ such that

$$K = \sum_j u_j \otimes \mu_j.$$

Note that the pullback of $\Omega_{b,R}(X^2_b)\big|_{\Delta_b}$ to $N^*\Delta_b$ is isomorphic to $\Omega_{b,t}(N^*\Delta_b)$ (see (B.2)). Fix a representative of $\sigma_m(u_j)$ for each j and denote it by $\widetilde{\sigma}_m(u_j)$, then by Theorem A.8, (B.3), and Corollary B.3.1, we have

$$\begin{aligned} {}^b\sigma_m(K) &:= \sum_j [\widetilde{\sigma}_m(u_j) \otimes (\mu_j\big|_{\Delta_b})'] \\ &\in S^{[m]}_{bl}(N^*\Delta_b, \Omega_f(N^*\Delta_b) \otimes \Omega_{b,t}(N^*\Delta_b)) \\ &\cong S^{[m]}_{bl}(N^*\Delta_b, \Omega_b(N^*\Delta_b)) \\ &\cong S^{[m]}_{bl}({}^bT^*X), \end{aligned}$$

where $(\mu_j\big|_{\Delta_b})'$ is the lifting of $\mu_j\big|_{\Delta_b}$ to $N^*\Delta_b$. Thus we obtain the map ${}^b\sigma_m : \Psi^m_{bl}(X) \to S^{[m]}_{bl}({}^bT^*X)$. The exactness of the sequence follows from the correspondent property of the (principal) symbol map σ_m of $I^m_{bl}(X^2_b, \Delta_b)$. □

${}^b\sigma_m(K)$ is called the (b-)*principal symbol* of $K \in \Psi^m_{bl}(X)$. K is called *(partially) elliptic* if ${}^b\sigma_m(K)$ is invertible for all $\xi \neq 0$ in ${}^bT^*X$.

For $m \in \mathbb{N}$, an element $P \in I^m(X^2_b, \Delta_b, \Omega_{b,R}) \subset I^m_{bl}(X^2_b, \Delta_b, \Omega_{b,R})$ is said to be in $\mathrm{Diff}^m_b(X)$ if P satisfies condition 2 in Definition 1.2.14 such that the local left symbols

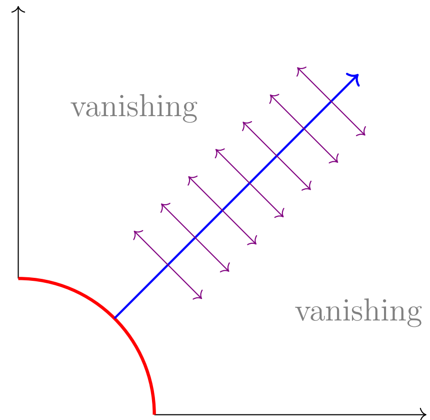


Figure 1.2: Schwartz Kernel of a b-differential operator

are polynomials in ξ of total degree m with compactly supported, smooth coefficients, that is, for example, when $\mathcal{U} \cap \mathit{ff} \cap \Delta_b \neq \varnothing$, then

$$a(x, y, \xi) = \sum_{|\alpha| \leqslant m} i^{|\alpha|} a_\alpha(x, y) \xi^\alpha,$$

where $a_\alpha \in C_c^\infty(\mathcal{U})$. Recall that P is supported in Δ_b (Figure 1.2, see also Example 1.2.10), hence the following result is immediate.

Proposition 1.2.16. *For any $m \in \mathbb{N}$, $\mathrm{Diff}_b^m(X) \subset \Psi_{bl}^m(X)$.*

$\mathrm{Diff}_b^m(X)$ is called the collection of b*-differential operators* of order m by Melrose.

1.3 Mapping properties

In this and the next section, we will review how to interpret elements in $\Psi_{bl}^*(X)$ as linear maps on certain classes of nice (scalar) functions, hence justify the term "operators".

We will go over a model scenario in relatively plain terms first. Let $X = [0, 1)$. Denote a point of X^2 as (x, x') . Let $(s, \rho) \in (-\infty, \infty) \times [0, 1)$ be coordinates of X_b^2

near the front face such that in the interiors of X_b^2 and X^2,

$$s = \ln(\frac{x}{x'}), \qquad \rho = x + x'.$$

Then the blow-down map is given by

$$\beta_b^2 : (s, \rho) \longrightarrow (\frac{\rho}{1+e^{-s}}, \frac{pe^{-s}}{1+e^{-s}}).$$

Hence, the Jacobian is

$$\left|\frac{\partial(x,x')}{\partial(s,\rho)}\right| = \left|\det\begin{pmatrix} \dfrac{1}{1+\mathrm{e}^{-s}} & \dfrac{\rho\,\mathrm{e}^{-s}}{(1+\mathrm{e}^{-s})^2} \\ \dfrac{\mathrm{e}^{-s}}{(1+\mathrm{e}^{-s})} & \dfrac{-\rho\,\mathrm{e}^{-s}}{(1+\mathrm{e}^{-s})^2} \end{pmatrix}\right| = \frac{\rho\,\mathrm{e}^{-s}}{(1+\mathrm{e}^{-s})^2},$$

and consequently the pull back of b-density via β_b^2 is

$$(\beta_b^2)^*(\psi \boxtimes \varphi) = \psi(\frac{\rho}{1+\mathrm{e}^{-s}})\varphi(\frac{\rho\,\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}}) \cdot \left|\frac{\mathrm{d}\rho}{\rho}\,\mathrm{d}s\right|^{\frac{1}{2}}$$

where $\psi, \varphi \in \dot{C}_c^\infty(X, \Omega_b^{\frac{1}{2}})$ and $\psi \boxtimes \varphi = \psi(x)\varphi(x') \cdot \left|\frac{\mathrm{d}x}{x}\frac{\mathrm{d}x'}{x'}\right|^{\frac{1}{2}} \in xx' \cdot C_c^\infty(X^2, \Omega_b^{\frac{1}{2}})$.

Let $\kappa \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$ supported near $\mathit{ff}(X_b^2)$, then

$$< \kappa, (\beta_b^2)^*(\psi \boxtimes \varphi) >= \int_{-\infty}^{\infty}\int_0^1 \kappa(s,\rho)\psi(\frac{\rho}{1+\mathrm{e}^{-s}})\varphi(\frac{p\,\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}})\frac{\mathrm{d}\rho}{\rho}\,\mathrm{d}s.$$

Making the change of variable $\frac{\rho}{1+\mathrm{e}^{-s}} \longleftrightarrow r$ or $\rho \longleftrightarrow r(1+\mathrm{e}^{-s})$, we can write

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_0^1 \kappa(s,\rho)\psi(\frac{\rho}{1+\mathrm{e}^{-s}})\varphi(\frac{p\,\mathrm{e}^{-s}}{1+\mathrm{e}^{-s}})\frac{\mathrm{d}\rho}{\rho}\,\mathrm{d}s \\ &= \int_{-\infty}^{\infty}\int_0^{\frac{1}{1+\mathrm{e}^{-s}}} \kappa(s, r(1+\mathrm{e}^{-s}))\psi(r)\varphi(r\,\mathrm{e}^{-s})\frac{\mathrm{d}r}{r}\,\mathrm{d}s \\ &= \int_0^1 \left\{\int_{-\ln(\frac{1}{r}-1)}^{\infty} \kappa(s, r(1+\mathrm{e}^{-s}))\varphi(r\,\mathrm{e}^{-s})\ \mathrm{d}s\right\}\psi(r)\frac{\mathrm{d}r}{r}. \end{aligned}$$

Thus, we may define the action of K, the operator associated to κ, on φ by

$$K\varphi(r) := \int_{-\ln(\frac{1}{r}-1)}^{\infty} \kappa(s, r(1+\mathrm{e}^{-s}))\varphi(r\,\mathrm{e}^{-s})\ \mathrm{d}s \cdot \left|\frac{\mathrm{d}r}{r}\right|^{\frac{1}{2}}. \tag{1.3.1}$$

Now assume that $\varphi \in S^0(X)$ and $\kappa \in C^\infty(X_b^2, \Omega_b^{\frac{1}{2}})$ such that κ vanishes to infinite logarithmic order at the left and right boundary. Then (1.3.1) still makes sense. More precisely, we have the following computation

$$\begin{aligned}
r\partial_r(K\varphi(r)) =& -\kappa(-\ln(\frac{1}{r}-1), 1)\varphi(1-r)\frac{1}{1-r} \\
&+ \int_{-\ln(\frac{1}{r}-1)}^{\infty} r(1+\mathrm{e}^{-s})\partial_\rho\kappa(s, r(1+\mathrm{e}^{-s}))\varphi(r\,\mathrm{e}^{-s})\ \mathrm{d}s \\
&+ \int_{-\ln(\frac{1}{r}-1)}^{\infty} \kappa(s, r(1+\mathrm{e}^{-s}))r\,\mathrm{e}^{-s}\,\varphi'(r\,\mathrm{e}^{-s})\ \mathrm{d}s \\
=& -\partial_s^i\kappa(-\ln(\frac{1}{r}-1), 1)(r\partial_r)^j\varphi(1-r)p_{ij}(\frac{1}{1-r}, r) \\
&+ \int_{-\ln(\frac{1}{r}-1)}^{\infty} \tilde{\kappa}(s, r(1+\mathrm{e}^{-s}))\varphi(r\,\mathrm{e}^{-s})\ \mathrm{d}s \\
&+ \int_{-\ln(\frac{1}{r}-1)}^{\infty} \kappa(s, r(1+\mathrm{e}^{-s}))\tilde{\varphi}(r\,\mathrm{e}^{-s})\ \mathrm{d}s
\end{aligned}$$

where i and j are both 0, $p_{ij}(x,y)$ a polynomial in (x,y), $\tilde{\kappa}(s,\rho) = \rho\partial_\rho\kappa(s,\rho)$ Schwartz in s, and $\tilde{\varphi} = r\partial_r\varphi(r)$ in $S^0(X)$, hence $r\partial_r(K\varphi)$ is bounded over X. Moreover, by induction, it could be shown that in general $(r\partial_r)^\alpha(K\varphi(r))$ is a summation consisting of the above three types of expressions. In conclusion, we have $K\varphi \in S^0(X)$.

Before move on to the general case, we make a remark on simplification. Suppose in addition that $\varphi, \psi \in C_c^\infty([0, \sqrt{2}/2), \Omega_b^{\frac{1}{2}})$ and recall that κ is supported near the front face, then we could extend every object to $\bar{X} = [0,\infty)$ or $\bar{X}_b^2$ by zero. Still denoted with the same notations, we obtain

$$\begin{aligned} K\varphi(r) &= \int_{-\infty}^{\infty} \kappa(s, r(1+\mathrm{e}^{-s}))\varphi(r\,\mathrm{e}^{-s})\ \mathrm{d}s \cdot \left|\frac{\mathrm{d}r}{r}\right|^{\frac{1}{2}} \\ &= \int \kappa(s, r(1+\mathrm{e}^{-s}))\varphi(r\,\mathrm{e}^{-s})\ \mathrm{d}s \cdot \left|\frac{\mathrm{d}r}{r}\right|^{\frac{1}{2}} \end{aligned} \tag{1.3.2}$$

via the same computation leading to (1.3.1). Since in general we are interested in compact manifolds, assumptions similar to this setting could always be achieved by employing partitions of unity.

In the rest of this section, we will invoke the formulations of the mapping properties and compositions in terms of pullbacks, products and pushforwards of b-densities (see, e.g., [15], [24], [27], [26]). For more details of this (b-geometry) point of view, see Appendix B. First, we look at the mapping properties.

Lemma 1.3.1. *If $K_A \in \Psi_{bl}^{-\infty}(X)$, then it defines linear maps*

(a) $A : S^0(X) \longrightarrow S^0(X)$;

(b) $A : {}^{1}S^0_{\partial X}(X) \longrightarrow {}^{1}S^0_{\partial X}(X)$; *and*

(c) $A : S^0_{bl}(X) \longrightarrow S^0_{bl}(X)$.

Proof. We first derive a local formula for

$$\mu A\varphi := (\pi_{L,b})_* \left(\pi_{L,b}^* \mu \pi_{R,b}^* \varphi \cdot K_A\right)$$

with $\mu \in C^\infty(X, \Omega_b)$, which is essentially the same as (1.3.2). For simplicity, we assume that K_A is supported near the front face of X_b^2. Let $\mathcal{V} \cong \mathbb{R}^{n-1}$ be a coordinate patch on Y. Then $X \cong [0,1)_x \times \mathcal{V}_y$ near Y, $X^2 \cong [0,1)^2_{(x,x')} \times \mathcal{V}^2_{(y,y')}$ near Y^2 and $X_b^2 \cong [0,1)_b^2 \times \mathcal{V}^2$ near *ff*. We may use the coordinates

$$(\rho, \omega, y, z) := (x + x', \ln(\frac{x}{x'}), y, y - y')$$

on X_b^2 near $f\!f$. In particular, we have

$$\begin{aligned}\beta_b^2 : X_b^2 &\longrightarrow X^2\\ (\rho,\omega,y,z) &\longmapsto (\frac{\rho}{1+\mathrm{e}^{-\omega}}, \frac{\rho}{1+\mathrm{e}^{\omega}}, y, y-z),\end{aligned}$$

hence

$$\begin{aligned}\pi_{L,b} : X_b^2 &\longrightarrow X\\ (\rho,\omega,y,z) &\longmapsto (\frac{\rho}{1+\mathrm{e}^{-\omega}}, y)\end{aligned}$$

and

$$\begin{aligned}\pi_{R,b} : X_b^2 &\longrightarrow X\\ (\rho,\omega,y,z) &\longmapsto (\frac{\rho}{1+\mathrm{e}^{\omega}}, y-z).\end{aligned}$$

Now, by employing a partition of unity, we may further assume that $K_A \in C_c^\infty([0,1)_b^2 \times \mathcal{V}^2)$. Them, with a bit of abuse of language, we write

$$K_A = K_A(\rho,\omega,y,z)\cdot\mu',$$

where $K_A(\rho,\omega,y,z) \in S_{bl}^0([0,1)_b^2 \times \mathcal{V}^2)$. As a consequence,

$$\begin{aligned}&\pi_{L,b}^*\mu\pi_{R,b}^*\varphi \cdot K_A\\ &= \mu(\frac{\rho}{1+\mathrm{e}^{-\omega}}, y)\varphi(\frac{\rho}{1+\mathrm{e}^{\omega}}, y-z)K_A(\rho,\omega,y,z)\cdot\left|\frac{\mathrm{d}\rho}{\rho}\mathrm{d}\omega\mathrm{d}y\mathrm{d}z\right|.\end{aligned}$$

Suppose that $\kappa \in C_c^\infty(X_b^2, \Omega_b)$ with $X_b^2 \cong [0,1)_b^2 \times \mathcal{V}^2$. Then for any $\phi \in C_c^\infty(\mathring{X})$,

$$\begin{aligned}\left\langle \kappa, \pi_{L,b}^* \phi \right\rangle &= \int \kappa(\rho, \omega, y, z) \phi(\frac{\rho}{1+\mathrm{e}^{-\omega}}, y) \frac{\mathrm{d}\rho}{\rho} \mathrm{d}\omega \mathrm{d}y \mathrm{d}z \\ &= \int \left\{ \int \kappa((1+\mathrm{e}^{-\omega})r, \omega, y, z)\ \mathrm{d}\omega \mathrm{d}z \right\} \phi(r, y) \frac{\mathrm{d}r}{r} \mathrm{d}y,\end{aligned}$$

thus,

$$(\pi_{L,b})_* \kappa = \int \kappa((1+\mathrm{e}^{-\omega})r, \omega, y, z)\ \mathrm{d}\omega \mathrm{d}z \cdot \left| \frac{\mathrm{d}r}{r} \mathrm{d}y \right|.$$

Now we compute

$$\begin{aligned}&(\pi_{L,b})_* \left(\pi_{L,b}^* \mu \pi_{R,b}^* \varphi \cdot K_A \right) \\ &= \int \mu(r, y) \varphi(\frac{1+\mathrm{e}^{-\omega}}{1+\mathrm{e}^{\omega}} r, y-z) K_A((1+\mathrm{e}^{-\omega})r, \omega, y, z)\ \mathrm{d}\omega \mathrm{d}z \cdot \left| \frac{\mathrm{d}r}{r} \mathrm{d}y \right| \\ &= \int K_A((1+\mathrm{e}^{-\omega})r, \omega, y, z) \varphi(r\,\mathrm{e}^{-\omega}, y-z)\ \mathrm{d}\omega \mathrm{d}z \cdot \mu(r, y) \left| \frac{\mathrm{d}r}{r} \mathrm{d}y \right|.\end{aligned}$$

In summary, to establish the various mapping properties, we need to demonstrate that

$$h := \int K_A((1+\mathrm{e}^{-\omega})r, \omega, y, z) \varphi(r\,\mathrm{e}^{-\omega}, y-z)\ \mathrm{d}\omega \mathrm{d}z$$

satisfies the corresponding characterizing properties of the function spaces in question.

The following notations will be of convenience: we write

$$\begin{aligned}K_A^{ac}((1+\mathrm{e}^{-\omega})r, \omega, y, z) &= \partial_y^a (\rho \partial_\rho)^c K_A(\rho, \omega, y, z)\big|_{\rho=(1+\mathrm{e}^{-\omega})r}, \\ \varphi^{bd}(r\,\mathrm{e}^{-\omega}, y-z) &= \partial_y^b (x\partial_x)^d \varphi(x, y-z)\Big|_{x=r\,\mathrm{e}^{-\omega}}.\end{aligned}$$

In particular, we have

$$\begin{aligned}&\partial_y^\gamma (r\partial_r)^\delta \left(K_A((1+\mathrm{e}^{-\omega})r,\omega,y,z)\varphi(r\,\mathrm{e}^{-\omega},y-z)\right)\\ &=\sum_{\substack{a+b=\gamma\\ c+d=\delta}} C_{abcd}K_A^{ac}((1+\mathrm{e}^{-\omega})r,\omega,y,z)\varphi^{bd}(r\,\mathrm{e}^{-\omega},y-z)\end{aligned} \tag{1.3.3}$$

for some constant C_{abcd}. Let $r_0>0$ be arbitrary.

(a) Let $\varphi\in S^0(X)$. We must show that $h\in S^0(X)$.

$$\mu A\varphi := (\pi_{L,b})_* \left(\pi_{L,b}^*\mu\pi_{R,b}^*\varphi\cdot K_A\right)\in S^0(X,\Omega_b),$$

where $\mu\in C^\infty(X,\Omega_b)$. Since K_A^{ac} is of $O\left((1+|\omega|)^{-\ell}\right)$ for any ℓ, and compactly supported in z, and φ^{bd} is bounded, from (1.3.3), we in fact have

$$\begin{aligned}&\sup_{r<r_0}\left|\partial_y^\gamma(r\partial_r)^\delta h(r,y)\right|\\ &=\sup_{r<r_0}\left|\int \partial_y^\gamma (r\partial_r)^\delta \left(K_A((1+\mathrm{e}^{-\omega})r,\omega,y,z)\varphi(r\,\mathrm{e}^{-\omega},y-z)\right)\,\mathrm{d}\omega\mathrm{d}z\right|\\ &=\sup_{r<r_0}\left|\int\sum_{\substack{a+b=\gamma\\ c+d=\delta}} C_{abcd}K_A^{ac}((1+\mathrm{e}^{-\omega})r,\omega,y,z)\varphi^{bd}(r\,\mathrm{e}^{-\omega},y-z)\,\mathrm{d}\omega\mathrm{d}z\right|\\ &\leqslant\int\sum_{\substack{a+b=\gamma\\ c+d=\delta}} C'_{abcd}(1+|\omega|)^{-2}\,\mathrm{d}\omega<\infty.\end{aligned}$$

(b) Let $\varphi\in {}^{1}S^0_{\partial X}(X)$. We shall show that $h\in {}^{1}S^0_{\partial X}(X)$. Note that, by Peetre's inequality, we have

$$
\begin{aligned}
&\left|\int K_A^{ac}((1+\mathrm{e}^{-\omega})r,\omega,y,z)\varphi^{bd}(r\,\mathrm{e}^{-\omega},y-z)\ \mathrm{d}\omega\mathrm{d}z\right|\\
\leqslant& C\int\frac{(1+|\ln r\,\mathrm{e}^{-\omega}|)^{-\ell}\ \mathrm{d}\omega}{(1+|\omega|)^{\ell+2}}\\
=& C\int\frac{(1+|\ln r+-\omega|)^{-\ell}\ \mathrm{d}\omega}{(1+|\omega|)^{\ell+2}}\\
\leqslant& C\int\frac{(1+|\ln r|)^{-\ell}(1+|\omega|)^{\ell}\ \mathrm{d}\omega}{(1+|\omega|)^{\ell+2}}\\
=& D\,(1+|\ln r|)^{-\ell}.
\end{aligned}
$$

for some constant C and $D = C\int(1+|\omega|)^{-2}\ \mathrm{d}\omega$. Thus, also from (1.3.3),

$$
\sup_{r<r_0}\left|(1+|\ln r|)^{\ell}\partial_y^{\gamma}(r\partial_r)^{\delta}h(r,y)\right|<\infty. \tag{1.3.4}
$$

(c) Let $\varphi\in S^0_{bl}(X)$. We will demonstrate that $h\in S^0_{bl}(X)$. To this end, we write

$$
\begin{aligned}
\varphi(x,y) &= \varphi_0(y)+\varphi_1(x,y),\\
K_A(\rho,\omega,y,z) &= (K_A)_0(\omega,y,z)+(K_A)_1(\rho,\omega,y,z).
\end{aligned}
$$

Let

$$
h_0(y)=\int(K_A)_0(\omega,y,z)\varphi_0(y-z)\ \mathrm{d}\omega\mathrm{d}z,
$$

and $h_1 = h_{11}+h_{12}+h_{13}$, where

$$
\begin{aligned}
h_{11}(r,y) &= \int(K_A)_0(\omega,y,z)\varphi_1(r\,\mathrm{e}^{-\omega},y-z)\ \mathrm{d}\omega\mathrm{d}z,\\
h_{12}(r,y) &= \int(K_A)_1((1+\mathrm{e}^{-\omega})r,\omega,y,z)\varphi_0(y-z)\ \mathrm{d}\omega\mathrm{d}z,\\
h_{13}(r,y) &= \int(K_A)_1((1+\mathrm{e}^{-\omega})r,\omega,y,z)\varphi_1(r\,\mathrm{e}^{-\omega},y-z)\ \mathrm{d}\omega\mathrm{d}z.
\end{aligned}
$$

Since $\partial_y^\beta (K_A)_0(\omega, y, z)$ decays faster than $(1+|\omega|)^{-2}$ for any β, we have $h_0(y) \in S^0(X)$. We study h_{1j}, $j = 1,2,3$, respectively. Nevertheless, the readers should note that the analysis on these three terms is similar. We compute

$$
\begin{aligned}
&\sup_{r<r_0} \left|(1+|\ln r|)^\ell (r\partial_r)^\alpha \partial_y^\beta h_{11}(r,y)\right| \\
=&\sup_{r<r_0} \left|(1+|\ln r|)^\ell \sum_{\beta_1+\beta_2=\beta} \int C_{\beta_1} \partial_y^{\beta_1}(K_A)_0(\omega,y,z)(r\partial_r)^\alpha \partial_y^{\beta_2}\varphi_1(r\,\mathrm{e}^{-\omega}, y-z)\ \mathrm{d}\omega\mathrm{d}z\right| \\
=&\sup_{r<r_0} \left|(1+|\ln r|)^\ell \sum_{\beta_1+\beta_2=\beta} \int C_{\beta_1} \partial_y^{\beta_1}(K_A)_0(\omega,y,z)(x\partial_x)^\alpha \partial_y^{\beta_2}\varphi_1(x, y-z)\big|_{x=r\,\mathrm{e}^{-\omega}}\ \mathrm{d}\omega\mathrm{d}z\right| \\
\leqslant& C\sup_{r<r_0} (1+|\ln r|)^\ell \int (1+\left|\ln r\,\mathrm{e}^{-\omega}\right|)^{-\ell}(1+|\omega|)^{-\ell-2}\ \mathrm{d}\omega \\
\leqslant& C\sup_{r<r_0} (1+|\ln r|)^\ell \int (1+|\ln r|)^{-\ell}(1+\left|\ln \mathrm{e}^{-\omega}\right|)^{\ell}(1+|\omega|)^{-\ell-2}\ \mathrm{d}\omega < \infty,
\end{aligned}
$$

$$
\begin{aligned}
&\sup_{r<r_0} \left|(1+|\ln r|)^\ell (r\partial_r)^\alpha \partial_y^\beta h_{12}(r,y)\right| \\
=&\sup_{r<r_0} \left|\frac{\sum_{\beta_1+\beta_2=\beta} \int C_{\beta_1} (\rho\partial_\rho)^\alpha \partial_y^{\beta_1}(K_A)_1(\rho,\omega,y,z)\big|_{\rho=r(1+\mathrm{e}^{-\omega})} \partial_y^{\beta_2}\varphi_0(y-z)\ \mathrm{d}\omega\mathrm{d}z}{(1+|\ln r|)^{-\ell}}\right| \\
\leqslant& C\sup_{r<r_0} (1+|\ln r|)^\ell \int (1+|\ln r|)^{-\ell}(1+\left|\ln(1+\mathrm{e}^{-\omega})\right|)^{\ell}(1+|\omega|)^{-\ell-2}\ \mathrm{d}\omega,
\end{aligned}
$$

in which we use the fact that

$$
\left(1+\ln(1+\mathrm{e}^{|\omega|})\right) < \left(1+\ln(2\,\mathrm{e}^{|\omega|})\right) = \left(1+\ln 2+\ln \mathrm{e}^{|\omega|}\right) < (1+\ln 2)(1+|\omega|).
$$

Thus, both h_{11} and h_{12} are in ${}^{\mathrm{l}}S^0_{\partial X}(X)$. That $h_{13} \in {}^{\mathrm{l}}S^0_{\partial X}(X)$ follows the same argument leading to (1.3.4). In conclusion, we recognize that $h = h_0 + h_1 \in S^0_{bl}(X)$. □

Example 1.3.2. Let $\psi_1 \in C^\infty(\mathbb{R})$ such that $\psi_1 \equiv 1$ on $(-\infty, -2] \cup [2, \infty)$ and $\psi_1 \equiv 0$ on $[-1, 1]$. Let $\psi_2 \in C_c^\infty([0, 1))$ such that $\psi_2(0) = 1$ and $\psi_2 \equiv 0$ on $[\sqrt{2}/2, 1)$. Let $\nu = \left|\frac{\mathrm{d}x}{x}\right|$ and ν' be its lift to X_b^2 under $\pi_{R,b}$. Then $K_A = \psi_1(\omega)\,\mathrm{e}^{-|\omega|} \cdot \nu' \in \Psi_{bl}^{-\infty}([0,1)_b^2)$ and $\varphi(x) = \psi_2(x)x \in C_c^\infty(0, 1) \subset S^0(0, 1)$. Note that

$$A\varphi(x) = \int_{-\infty}^{\infty} \psi_1(\omega)\,\mathrm{e}^{-|\omega|}\,\psi_2(x\,\mathrm{e}^{-\omega})x\,\mathrm{e}^{-\omega}\;\mathrm{d}\omega \in S^0([0, 1)).$$

However, $A\varphi \notin C^\infty([0, 1))$, since $\lim_{x\to 0}(A\varphi)'(x) = \infty$.

Theorem 1.3.3. *Let $m \in \mathbb{R}$. If $K_A \in \Psi_{bl}^m(X)$, then it define linear maps*

(a) $A : S^0(X) \longrightarrow S^0(X)$;

(b) $A : {}^1S^0_{\partial X}(X) \longrightarrow {}^1S^0_{\partial X}(X)$; and

(c) A: $S^0_{bl}(X) \longrightarrow S^0_{bl}(X)$.

Proof. We only prove (c). For simplicity we assume that K_A is supported near $\mathit{ff}(X_b^2)$. Write $K_A = K_{A_1} + K_{A_2}$ such that K_{A_1} is supported away from lb and rb, and K_{A_2} is supported away from Δ_b. Immediately from the definition of $\Psi_{bl}^m(X)$ and Lemma 1.3.1, K_{A_2} define a linear map $A_2 : S^0_{bl}(X) \to S^0_{bl}(X)$.

Let $\mathcal{V} \cong \mathbb{R}^{n-1}$ be a coordinate patch on $Y = \partial X$. Then $X \cong [0, 1)_x \times \mathcal{V}_y$ near Y, $X^2 \cong [0, 1)^2_{(x,x')} \times \mathcal{V}^2_{(y,y')}$ near Y^2 and $X_b^2 \cong [0, 1)_b^2 \times \mathcal{V}^2$ near $\mathit{ff}(X_b^2)$. We may use the coordinates

$$(x, \omega, y, z) = (x, \ln(\frac{x}{x'}), y, y - y').$$

We further assume that K_{A_1} is compactly supported in this coordinate patch. More explicitly,

$$K_{A_1} = \psi(\omega, z) \int \mathrm{e}^{i\omega\tau + iz\cdot\xi}\, a(x, y, \tau, \xi)\, đ\tau đ\xi,$$

where $\psi \in C_c^\infty(\mathbb{R}_\omega \times \mathbb{R}_z^{n-1})$ and $a \in S_{bl}^m(\mathbb{R}^{n,1}; \mathbb{R}^n)$. Note that

$$\pi_{L,b}(x, \omega, y, z) = (x, y)$$

and

$$\pi_{R,b}(x,\omega,y,z) = (x\,\mathrm{e}^{-\omega}, y-z),$$

thus, given any $\varphi \in S^0_{bl}(X)$ and $\mu \in C^\infty(X,\Omega_b)$,

$$\begin{aligned}
&\pi^*_{L,b}\mu\pi^*_{R,b}\varphi \cdot K_{A_1}\\
&= \int \mathrm{e}^{i\omega\tau+iz\cdot\xi}\, a(x,y,\tau,\xi)\psi(\omega,z)\varphi(x\,\mathrm{e}^{-\omega}, y-z)\, đ\tau đ\xi \cdot \mu(x,y)\left|\frac{\mathrm{d}x}{x}\mathrm{d}\omega\mathrm{d}y\mathrm{d}z\right|\\
&= \int \mathrm{e}^{i\omega\tau+iz\cdot\xi}\, \tilde{a}(x,y,\tau,\xi)\, đ\tau đ\xi \cdot \mu(x,y)\left|\frac{\mathrm{d}x}{x}\mathrm{d}\omega\mathrm{d}y\mathrm{d}z\right|
\end{aligned}$$

where

$$\tilde{a}(x,y,\tau,\xi) = \int a(x,y,\iota,\eta)\tilde{\varphi}(x,\widehat{\tau-\iota},y,\widehat{\xi-\eta})\, đ\iota đ\eta,$$

with

$$\tilde{\varphi}(x,\omega,y,z) = \psi(\omega,z)\varphi(x\,\mathrm{e}^{-\omega}, y-z)$$

and

$$\tilde{\varphi}(x,\hat{\tau},y,\hat{\xi}) = \int \mathrm{e}^{-i\omega\tau-iz\cdot\xi}\,\tilde{\varphi}(x,\omega,y,z)\ \mathrm{d}\omega\mathrm{d}z.$$

See (A.2). Therefore, by (A.7),

$$\begin{aligned}
&(\pi_{L,b})_*\left(\pi^*_{L,b}\mu\pi^*_{R,b}\varphi \cdot K_{A_1}\right)\\
&= \int \mathrm{e}^{i\omega\tau+iz\cdot\xi}\,\tilde{a}(x,y,\tau,\xi)\, đ\tau đ\xi \mathrm{d}\omega\mathrm{d}z \cdot \mu(x,y)\left|\frac{\mathrm{d}x}{x}\mathrm{d}y\right|\\
&= \tilde{a}(x,y,0,0)\cdot\mu(x,y)\left|\frac{\mathrm{d}x}{x}\mathrm{d}y\right|.
\end{aligned}$$

It is left to show that $\tilde{a}(x,y,0,0) \in S^0_{bl}(X)$. Write

$$
\begin{aligned}
\varphi(x,y) &= \varphi_0(y) + \varphi_1(x,y),\\
a(x,y,\tau,\xi) &= a_0(y,\tau,\xi) + a_1(x,y,\tau,\xi),\\
\tilde{\varphi}(x,\omega,y,z) &= \psi(\omega,z)\varphi_0(y-z) + \psi(\omega,z)\varphi_1(x\,\mathrm{e}^{-\omega}, y-z)\\
&= \tilde{\varphi}_0(\omega,y,z) + \tilde{\varphi}_1(x,\omega,y,z).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\left|\tau^\delta \xi^\gamma \partial_y^\beta \tilde{\varphi}_0(\hat{\tau}, y, \hat{\xi})\right| &= \left|\iint \partial_\omega^\delta \partial_z^\gamma \,\mathrm{e}^{-i\omega\tau - iz\cdot\xi}\, \partial_y \tilde{\varphi}_0(\omega,y,z)\ \mathrm{d}\omega\mathrm{d}z\right|\\
&= \left|\iint \mathrm{e}^{-i\omega\tau - iz\cdot\xi}\, \partial_\omega^\delta \partial_z^\gamma \partial_y \tilde{\varphi}_0(\omega,y,z)\ \mathrm{d}\omega\mathrm{d}z\right|\\
&\leqslant C_{\beta\gamma\delta},
\end{aligned}
\tag{1.3.5}
$$

$$
\begin{aligned}
\left|\tau^\delta \xi^\gamma (x\partial_x)^\alpha \partial_y^\beta \tilde{\varphi}_1(x,\omega,y,z)\right| &= \left|\iint \mathrm{e}^{-i\omega\tau - iz\cdot\xi}\, \partial_\omega^\delta \partial_z^\gamma (x\partial_x)^\alpha \partial_y^\beta \tilde{\varphi}_1(x,\omega,y,z)\right|\\
&< C^\ell_{\alpha\beta\gamma\delta}(1+|\ln x|)^{-\ell}.
\end{aligned}
\tag{1.3.6}
$$

Then we compute

$$
\begin{aligned}
&\left|\partial_y^\beta \int a_0(y,\iota,\eta)\tilde{\varphi}_0(\widehat{\tau-\iota}, y, \widehat{\xi-\eta})\, đ\iota đ\eta\right|\\
=&\left|\sum_{\beta_1+\beta_2=\beta} C_{\beta_1} \int \partial_y^{\beta_1} a_0(y,\iota,\eta) \partial_y^{\beta_2} \tilde{\varphi}_0(\widehat{\tau-\iota}, y, \widehat{\xi-\eta})\, đ\iota đ\eta\right|\\
\leqslant& C_\beta (1+|\tau|)^2(1+|\xi|)^{2n}.
\end{aligned}
$$

Therefore, we have

$$
\tilde{a}_0(y,0,0) = \int a_0(y,\iota,\eta)\tilde{\varphi}_0(\widehat{-\iota}, y, \widehat{-\eta})\, đ\iota đ\eta \in S^0(Y).
$$

Similarly, one can verify that

$$\tilde{a}_1(x,y,0,0) = \int a_0(y,\iota,\eta)\tilde{\varphi}_1(x,\widehat{-\iota},y,\widehat{-\eta}) + a_1(x,y,\iota,\eta)\tilde{\varphi}_0(\widehat{-\iota},y,\widehat{-\eta})$$
$$+ a_1(x,y,\iota,\eta)\tilde{\varphi}_1(x,\widehat{-\iota},y,\widehat{-\eta})\, đ\iota đ\eta \in {}^1S^0_{\partial X}(X).$$

via the estimates (1.3.5), ((1.3.6) and the definitions of $S^0_{bl}(X)$, $S^0_{bl}(\mathbb{R}^{n,1};\mathbb{R}^n)$. Consequently, we have $\tilde{a}(x,y,0,0) = \tilde{a}_0(y,0,0) + \tilde{a}_1(x,y,0,0) \in S^0_{bl}(X)$, and the proof is completed. □

Example 1.3.4. Suppose that $K_P \in \mathrm{Diff}^m_b(X)$ is compactly supported in some coordinate patch near $\Delta_b \cap ff$:

$$K_P = \psi(\omega,z)\int \mathrm{e}^{i\omega\tau + iz\cdot\xi}\, a(x,y) i^\alpha \tau^\alpha i^{|\beta|}\xi^\beta\, đ\tau đ\xi$$

with $\alpha + |\beta| = m$ and $\psi \equiv 1$ in a neighborhood of 0. We will derive a local expression of $P : S^0(X) \to S^0(X)$. Given any $\varphi \in S^0(X)$, by Theorem 1.3.3,

$$\begin{aligned}
P\varphi &= \tilde{a}(x,y,0,0) \\
&= \int a(x,y) i^\alpha \iota^\alpha i^{|\beta|}\eta^\beta \tilde{\varphi}(x,\widehat{-\iota},y,\widehat{-\eta})\, đ\iota đ\eta \\
&= a(x,y)\int (\partial^\alpha_\omega \partial^\beta_z \tilde{\varphi})(x,\widehat{-\iota},y,\widehat{-\eta})\, đ\iota đ\eta \\
&= a(x,y)(\partial^\alpha_\omega \partial^\beta_z \tilde{\varphi})(x,0,y,0) \\
&= a(x,y)(x\partial_x)^\alpha \partial^\beta_y \varphi(x,y).
\end{aligned}$$

1.4 Compositions

We now study the compositions of *bl*-pseudodifferential operators. In what follows, we will use A, B, *et cetera*, to denote the operators acting on functions, and K_A, K_B, *et cetera*, to denote the Schwartz kernels as distributional right b-densities. The notations for the collections of both types of objects will not be distinguished, though.

Lemma 1.4.1. *We have* $\Psi_{bl}^{-\infty}(X) \circ \Psi_{bl}^{-\infty}(X) \subset \Psi_{bl}^{-\infty}(X)$.

Proof. Let $A, B \in \Psi_{bl}^{-\infty}(X)$. We must show that

$$\mu K_{AB} = (\pi_C)_* \left(\pi_{C,b}^* \mu \pi_{F,b}^* K_A \pi_{S,b}^* K_B\right) \in S_{bl}^0(X_b^2, \Omega_b).$$

By using a partition of unity, it suffices to assume that K_A, K_B are supported in some coordinate patches of X_b^2.

If $\mathcal{V} \cong \mathbb{R}^{n-1}$ is a coordinate patch on $Y = \partial X$, then near the boundary we can decompose $X \cong [0,1)_x \times \mathcal{V}_y$. Note that $X_b^2 \cong [0,1)_b^2 \times \mathcal{V}^2$ near $ff(X_b^2)$ and $X_b^3 \cong [0,1)_b^3 \times \mathcal{V}^3$ near $ff(X_b^3)$. Let ℓ be an arbitrary natural number. Assume that $\mu = \left|\frac{\mathrm{d}x}{x}\mathrm{d}y\right| \in C^\infty\left([0,1) \times \mathcal{V}, \Omega_b\right)$. We will continue to denote the lift of μ to X_b^2 via $\pi_{L,b}$ by μ, while the lift via $\pi_{R,b}$ is denoted by μ'. We break the analysis into a couple of steps according to various coordinate patches involved. The explicit coordinates used below are collected in Appendix B.

Step 1. We analyze $\mu K_A B$ near the intersection of mb, ff and fs of X_b^3. With a little abuse of notations, we write

$$K_A = K_A(x, \omega, y, y') \cdot \mu'$$

near $rb(X_b^2)$, and

$$K_B = K_B(\gamma, x', y, y') \cdot \mu'$$

near $lb(X_b^2)$. Then

$$\pi_{C,b}^* \mu \pi_{F,b}^* K_A \pi_{S,b}^* K_B = K_A(x'' \mathrm{e}^s, t, y, y') K_B(s+t, x'', y', y'') \left| \mathrm{d}s\mathrm{d}t \frac{\mathrm{d}x''}{x''} \mathrm{d}y\mathrm{d}y'\mathrm{d}y'' \right|.$$

We recall how to derive the local formula for the pushforward via $\pi_{C,b}$. Suppose that $\kappa \in C^\infty(X_b^3, \Omega_b)$ supported near the intersection of mb, ff and fs, then given any

$\varphi \in \dot{C}^\infty(X_b^2)$,

$$
\begin{aligned}
\langle\, (\pi_{C,b})_* \kappa, \varphi \,\rangle &:= \langle\, \kappa, (\pi_{C,b})^* \varphi \,\rangle \\
&= \int \kappa(s,t,x'',y,y',y'')\varphi(s,x'',y,y') \ \mathrm{d}s\mathrm{d}t\frac{\mathrm{d}x''}{x''}\mathrm{d}y\mathrm{d}y'\mathrm{d}y'' \\
&= \int \left\{ \int \kappa(s,t,x'',y,y',y'') \ \mathrm{d}t\mathrm{d}y'' \right\} \varphi(s,x'',y,y') \ \mathrm{d}s\frac{\mathrm{d}x''}{x''}\mathrm{d}y\mathrm{d}y'.
\end{aligned}
$$

Hence we have

$$
(\pi_{C,b})_*\kappa = \int \kappa(s,t,x'',y,y',y'') \ \mathrm{d}t\mathrm{d}y'',
$$

and consequently

$$
\begin{aligned}
\mu K_{AB} &= (\pi_{C,b})_*(\pi_{C,b}^* \mu \pi_{F,b}^* K_A \pi_{S,b}^* K_B) \\
&= \int K_A(x''\,\mathrm{e}^s, t, y, y') K_B(s+t, x'', y', y'') \ \mathrm{d}t\mathrm{d}y'' \cdot \left| \mathrm{d}s\frac{\mathrm{d}x''}{x''}\mathrm{d}y\mathrm{d}y' \right| \\
&= K_C(s,x'',y,y') \cdot \left| \mathrm{d}s\frac{\mathrm{d}x''}{x''}\mathrm{d}y\mathrm{d}y' \right|.
\end{aligned}
$$

Clearly K_C is well defined. In fact, we will show that $K_C(s,x'',y,y') \in S^0_{bl}(X_b^2)$. To see this, recall that we can write

$$
\begin{aligned}
K_A(x''\,\mathrm{e}^s, t, y, y') &= K_A^0(t,y,y') + K_A^1(x''\,\mathrm{e}^s, t, y, y'), \\
K_B(s+t, x'', y', y'') &= K_B^0(s+t, y', y'') + K_B^1(s+t, x'', y', y'').
\end{aligned}
$$

We will use Peetre's inequality (A.5) a couple of times.

$$\begin{aligned}\left|\int K_A^0(t)K_B^0(s+t)\ \mathrm{d}t\right| &\leqslant \int C_\ell\,(1+|t|)^{-\ell-2}\,(1+|s+t|)^{-\ell}\ \mathrm{d}t\\ &= C_\ell \int (1+|t|)^{-\ell-2}\,(1+|s|)^{-\ell}\,(1+|t|)^{\ell}\\ &\leqslant C_\ell \int (1+|t|)^{-2}\ \mathrm{d}t\cdot(1+|s|)^{-\ell}\\ &= D_\ell(1+|s|)^{-\ell}.\end{aligned}$$

for some constant C_ℓ and $D_\ell = C_\ell \int (1+|t|)^{-2}\ \mathrm{d}t$.

$$\begin{aligned}\left|\int K_A^0(t,)K_B^1(s+t,x'')\ \mathrm{d}t\mathrm{d}y''\right| &\leqslant \int C_\ell\,(1+|t|)^{-\ell-2}\,[(1+|s+t|)(1+|\ln x''|)]^{-\ell}\ \mathrm{d}t\\ &\leqslant \int C_\ell\,(1+|t|)^{-2}\ \mathrm{d}t\cdot[(1+|s|)(1+|\ln x''|)]^{-\ell}\\ &\leqslant D_\ell\,[(1+|s|)(1+|\ln x''|)]^{-\ell}.\end{aligned}$$

$$\begin{aligned}\left|\int K_A^1(x''\,\mathrm{e}^s,t)K_B^0(s+t)\ \mathrm{d}t\right| &\leqslant \int C_\ell(1+|\ln x''\,\mathrm{e}^s|)^{-\ell}(1+|t|)^{-2\ell-2}(1+|s+t|)^{-2\ell}\ \mathrm{d}t\\ &\leqslant C_\ell(1+|\ln x''+s|)^{-\ell}(1+|t|)^{-2\ell-2}(1+|s+t|)^{-2\ell}\ \mathrm{d}t\\ &\leqslant C_\ell\int(1+|\ln x''|)^{-\ell}(1+|s|)^{\ell-2\ell}(1+|t|)^{2\ell-2\ell-2}\ \mathrm{d}t\\ &\leqslant D_\ell\,[(1+|\ln x''|)(1+|s|)]^{-\ell}.\end{aligned}$$

With the exact same argument as above, we also have

$$\left|\int K_A^1(x''\,\mathrm{e}^s,t)K_B^1(s+t,x'')\ \mathrm{d}t\right| \leqslant C_\ell\,[(1+|\ln x''|)(1+|s|)]^{-\ell}.$$

Define

$$K_C^0(s,x'',y,y') = \int K_A^0(t,y,y')K_B^0(s+t,y',y'')\ \mathrm{d}t\mathrm{d}y'',$$

and

$$K_C^1(s,x'',y,y') = \int K_A^0(t,y,y')K_B^1(s+t,x'',y',y'')$$
$$+ K_A^1(x''\,\mathrm{e}^s,t,y,y')K_B^0(s+t,y',y'')$$
$$+ K_A^1(x''\,\mathrm{e}^s,t,y,y')K_B^1(s+t,x'',y',y'')\ \mathrm{d}t\mathrm{d}y'',$$

Then $K_C = K_C^0 + K_C^1$. We have seen that

$$\left|K_C^0(s,y,y')\right| = O\left((1+|s|)^\ell\right)$$

and

$$\left|K_C^1(s,x'',y,y')\right| = O\left([(1+|\ln x''|)(1+|s|)]^\ell\right)$$

for an arbitrary $\ell \in \mathbb{N}$.

Observe now that for any α,β,γ, the b-derivative

$$(\partial_y\partial_{y'})^\gamma(x''\partial_{x''})^\alpha(\partial_s)^\beta\left(K_A(x''\,\mathrm{e}^s,t,y,y')K_B(s+t,x'',y',y'')\right)$$

is a sum of terms in the form of

$$(\partial_y\partial_{y'})^{\delta_1}(x\partial_x)^iK_A(x,t,y,y')\cdot(\partial_y\partial_{y'})^{\delta_2}(x''\partial_{x''})^j\partial_\gamma^kK_B(\gamma,x'',y',y'')\Big|_{\substack{x=x''\mathrm{e}^s\\ \gamma=s+t}}$$
$$=(\partial_y\partial_{y'})^{\delta_1}(x\partial_x)^i\left(K_A^0(t,y,y')+K_A^1(x,t,y,y')\right)\Big|_{\substack{x=x''\mathrm{e}^s\\ \gamma=s+t}}$$
$$\cdot(\partial_y\partial_{y'})^{\delta_2}(x''\partial_{x''})^j\partial_\gamma^k\left(K_B^0(\gamma,y',y'')+K_B^1(\gamma,x'',y',y'')\right)\Big|_{\substack{x=x''\mathrm{e}^s\\ \gamma=s+t}}$$
$$=(\partial_y\partial_{y'})^{\delta_1}(x\partial_x)^iK_A^0(t,y,y')\Big|_{\substack{x=x''\mathrm{e}^s\\ \gamma=s+t}}+(\partial_y\partial_{y'})^{\delta_1}(x\partial_x)^iK_A^1(x,t,y,y')\Big|_{\substack{x=x''\mathrm{e}^s\\ \gamma=s+t}}$$
$$+(\partial_y\partial_{y'})^{\delta_2}(x''\partial_{x''})^jK_B^0(\gamma,y',y'')\Big|_{\substack{x=x''\mathrm{e}^s\\ \gamma=s+t}}+(\partial_y\partial_{y'})^{\delta_2}(x''\partial_{x''})^jK_B^1(\gamma,x'',y',y'')\Big|_{\substack{x=x''\mathrm{e}^s,\\ \gamma=s+t}}$$

hence the same argument as to $K_C^0(s,y,y')$ and $K_C^1(s,x'',y,y')$ themselves shows that their b-derivatives also decays in the same way. Since ℓ is arbitrary, $K_C(s,x'',y,y') \in S^0_{bl}(X^2_b)$.

Step 2. Near $fs \cap lb \cap ff \subset X^3_b$, we have

$$\begin{aligned}
&\pi^*_{C,b}\mu\pi^*_{F,b}K_A\pi^*_{S,b}K_B \\
&= K_A(s, x''\,\mathrm{e}^t, y, y')K_B(t, x'', y', y'')\left|\mathrm{d}s\mathrm{d}t\frac{\mathrm{d}x''}{x''}\mathrm{d}y\mathrm{d}y'\mathrm{d}y''\right|.
\end{aligned}$$

Note that

$$\begin{aligned}
&\left\langle \kappa, \pi^*_{C,b}\varphi \right\rangle \\
&= \int \kappa\cdot\varphi(s+t, x'', y, y')\ \mathrm{d}s\mathrm{d}t\frac{\mathrm{d}x''}{x''}\mathrm{d}y\mathrm{d}y'\mathrm{d}y'' \\
&= \int\left\{\int \kappa(u-t, t, x'', y, y', y'')\ \mathrm{d}t\mathrm{d}y''\right\}\varphi(u, x'', y, y')\ \mathrm{d}u\frac{\mathrm{d}x''}{x''}\mathrm{d}y\mathrm{d}y' \\
&= \int\left\{\int \kappa(s, u-s, x'', y, y', y'')\ \mathrm{d}s\mathrm{d}y''\right\}\varphi(u, x'', y, y')\ \mathrm{d}u\frac{\mathrm{d}x''}{x''}\mathrm{d}y\mathrm{d}y',
\end{aligned}$$

hence

$$\begin{aligned}
(\pi_{C,b})_*\kappa &= \int \kappa(s-t, t, x'', y, y', y'')\ \mathrm{d}t\mathrm{d}y'' \\
&= \int \kappa(s, t-s, x'', y, y', y'')\ \mathrm{d}s\mathrm{d}y''.
\end{aligned}$$

Therefore, we compute

$$\begin{aligned}
&\mu K_{AB} = (\pi_{C,b})_*(\pi^*_{C,b}\mu\pi^*_{F,b}K_A\pi^*_{S,b}K_B) \\
&= \int K_A(s-t, x''\,\mathrm{e}^t, y, y')K_B(t, x'', y', y'')\ \mathrm{d}t\mathrm{d}y''\left|\mathrm{d}s\frac{\mathrm{d}x''}{x''}\mathrm{d}y\mathrm{d}y'\right| \\
&= K_C(s, x'', y, y')\cdot\left|\mathrm{d}s\frac{\mathrm{d}x''}{x''}\mathrm{d}y\mathrm{d}y'\right|
\end{aligned}$$

Following the same lines as in Step 1, one could show that $K_C(s,x'',y,y') \in S^0_{bl}(X^2_b)$.

Step 3. Near $cs \cap lb \cap ff \subset X_b^3$, we have

$$\begin{aligned}
&\pi_{C,b}^* \mu \pi_{F,b}^* K_A \pi_{S,b}^* K_B \\
&= K_A(s+t, x', y, y') K_B(x', t, y', y'') \left| \mathrm{d}s \frac{\mathrm{d}x'}{x'} \mathrm{d}t \mathrm{d}y \mathrm{d}y' \mathrm{d}y'' \right| .
\end{aligned}$$

Note that

$$\begin{aligned}
&\left\langle \kappa, \pi_{C,b}^* \varphi \right\rangle \\
&= \int \kappa \cdot \varphi(s, x' \mathrm{e}^t, y, y') \ \mathrm{d}s \frac{\mathrm{d}x'}{x'} \mathrm{d}t \mathrm{d}y \mathrm{d}y' \mathrm{d}y'' \\
&= \int \left\{ \int \kappa(s, u\mathrm{e}^{-t}, t, y, y', y'') \ \mathrm{d}t \mathrm{d}y'' \right\} \varphi(s, u, y, y') \ \mathrm{d}s \frac{\mathrm{d}u}{u} \mathrm{d}y \mathrm{d}y',
\end{aligned}$$

thus,

$$(\pi_{C,b})_* \kappa = \int \kappa(s, x' \mathrm{e}^{-t}, t, y, y', y'') \ \mathrm{d}t \mathrm{d}y''.$$

Consequently,

$$\begin{aligned}
\mu K_{AB} &= (\pi_{C,b})_* (\pi_{C,b}^* \mu \pi_{F,b}^* K_A \pi_{S,b}^* K_B) \\
&= \int K_A(s+t, x' \mathrm{e}^{-t}, y, y') K_B(x' \mathrm{e}^{-t}, t, y', y'') \ \mathrm{d}t \mathrm{d}y'' \left| \mathrm{d}s \frac{\mathrm{d}x'}{x'} \mathrm{d}y \mathrm{d}y' \right| \\
&= K_C(s, x', y, y') \left| \mathrm{d}s \frac{\mathrm{d}x'}{x'} \mathrm{d}y \mathrm{d}y' \right|
\end{aligned}$$

The proof to that $K_C(s, x'', y, y') \in S_{bl}^0(X_b^2)$ is the same as Step 1.

In summary, away from the faces ss and rb, $\pi_{C,b}^* \mu \pi_{F,b}^* K_A \pi_{S,b}^* K_B$ pushes forward under $\pi_{C,b}$ to define the kernel of an element in $S_{bl}^0(X_b^2, \Omega_b)$. Similarly arguments show the same thing away from fs and lb. Note that the interior of X_b^3 is isomorphic to the interior of X^3, so the analysis for the case when $\pi_{C,b}^* \mu \pi_{F,b}^* K_A \pi_{S,b}^* K_B$ is supported away from the front face is identical to that of closed manifolds and consequently the kernel of AB is clearly smooth there. Since μ is a (local) trivialization of $\Omega_{b,L}(X_b^2)$, in conclusion, we have $AB \in \Psi_{bl}^{-\infty}(X)$. $\square$

Theorem 1.4.2. *If $m, m' \in \mathbb{R}$, then $\Psi_{bl}^{m}(X) \circ \Psi_{bl}^{m'}(X) \subseteq \Psi_{bl}^{m+m'}(X)$.*

Proof. Let $K_A \in \Psi_{bl}^{m}(X)$ and $K_B \in \Psi_{bl}^{m'}(X)$. For simplicity, we assume that K_A and K_B are supported near $ff(X_b^2)$. Write $K_A = K_{A_1} + K_{A_2}$ and $K_B = K_{B_1} + K_{B_2}$, where K_{A_1}, K_{B_1} are both supported away from lb and rb, and where K_{A_2}, K_{B_2} are both supported away from Δ_b, thus, by definition, are elements in $\Psi_{bl}^{-\infty}(X)$. Then,

$$AB = A_1B_1 + A_1B_2 + A_2B_1 + A_2B_2.$$

By Lemma 1.4.1, $A_2B_2 \in \Psi_{bl}^{-\infty}(X)$. We will analyze the other three terms. As before, we work with local coordinates. Let $\mathcal{V} \cong \mathbb{R}^{n-1}$ be a coordinate patch on $Y = \partial X$. Then $X \cong [0,1)_x \times \mathcal{V}_y$, $X_b^2 \cong [0,1)_b^2 \times \mathcal{V}^2$ near $ff(X_b^2)$ and $X_b^3 \cong [0,1)_b^3 \times \mathcal{V}^3$ near $ff(X_b^3)$. We break the proof into a couple of steps.

Step 1. We consider first A_1B_1. In X_b^2, near $\Delta_b(X_b^2) \cap ff(X_b^2)$, we may use coordinates $(x, s, y, z) := (x, \ln x'/x, y, y - y')$, then $\Delta_b(X_b^2) = \{(s,z) = (0,0)\}$. Denote $\omega = (s,z)$ and $\eta = (\lambda, \xi)$, we can write

$$\begin{aligned} K_{A_1} &= \varphi(\omega) \int \mathrm{e}^{i\omega\cdot\eta}\, a(x,y,\eta)\, \text{đ}\eta \cdot \mu', \\ K_{B_1} &= \int \mathrm{e}^{i\omega\cdot\eta}\, b(x,y,\eta)\, \text{đ}\eta \cdot \mu', \end{aligned} \tag{1.4.1}$$

where $a \in S_{bl}^{m}(\mathbb{R}^{n,1}; \mathbb{R}^n)$, $b \in S_{bl}^{m'}(\mathbb{R}^{n,1}; \mathbb{R}^n)$, $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi \equiv 1$ on a sufficiently large neighborhood of 0, and where we may assume that $\mu = \left|\frac{\mathrm{d}x}{x}\mathrm{d}y\right| \in C^\infty(X, \Omega_b)$ and $\mu' = \pi_{R,b}^*(\mu)$. Moreover, we can write

$$\begin{aligned} a(x,y,\eta) =& a_0(y,\eta) + a_1(x,y,\eta), \\ b(x,y,\eta) =& b_0(y,\eta) + b_1(x,y,\eta). \end{aligned}$$

Note that $\pi_{C,b}^*\mu\pi_{F,b}^* K_{A_1}\pi_{S,b}^* K_{B_1}$ is supported near $\pi_{F,b}^{-1}(\Delta_b)\cap\pi_{S,b}^{-1}(\Delta_b)\cap ff(X_b^3)$, hence, in X_b^3, we may use coordinates

$$(x,s,t,y,z,w) := (x,\ln\frac{x'}{x},\ln\frac{x''}{x},y,y-y',y-y'').$$

Denote $\gamma=(t,w)$. According to B.8 and B.9, we have

$$\begin{aligned}
\mu K_{A_1B_1} &= (\pi_{C,b})_*(\pi_{C,b}^*\mu\pi_{F,b}^*K_{A_1}\pi_{S,b}^*K_{B_1})\\
&= \int \varphi(\omega)\,\mathrm{e}^{i\omega\cdot\eta_1}\,a(x,y,\eta_1)\,\mathrm{e}^{i(\gamma-\omega)\cdot\eta_2}\,b(x\mathrm{e}^s,y-z,\eta_2)\,đ\eta_1 đ\eta_2 \mathrm{d}\omega\cdot\mu\mu'\\
&= \int \mathrm{e}^{i\gamma\cdot\eta_2}\,a(x,y,\eta_1)\,\mathrm{e}^{i\omega\cdot(\eta_1-\eta_2)}\,\varphi(\omega)b(x\mathrm{e}^s,y-z,\eta_2)\ \mathrm{d}\omega đ\eta_1 đ\eta_2\cdot\mu\mu'\\
&= \int \mathrm{e}^{i\gamma\cdot\eta_2}\,a(x,y,\eta_1+\eta_2)\,\mathrm{e}^{i\omega\cdot\eta_1}\,\varphi(\omega)b(x\mathrm{e}^s,y-z,\eta_2)\ \mathrm{d}\omega đ\eta_1 đ\eta_2\cdot\mu\mu'\\
&= \int \mathrm{e}^{i\omega\cdot\eta}\,c(x,y,\eta)\,đ\eta\cdot\mu\mu'
\end{aligned}$$

where

$$\begin{aligned}
c(x,y,\eta) &= \int \mathrm{e}^{i\omega\cdot\zeta}\,a(x,y,\zeta+\eta)\varphi(\omega)b(x\mathrm{e}^s,y-z,\eta)\ \mathrm{d}\omega đ\zeta \qquad (1.4.2)\\
&= \int \mathrm{e}^{i\omega\cdot\zeta}\,a_0(y,\zeta+\eta)\varphi(\omega)b_0(y-z,\eta)\ \mathrm{d}\omega đ\zeta\\
&\quad+\Big(\int \mathrm{e}^{i\omega\cdot\zeta}\,a_0(y,\zeta+\eta)\varphi(\omega)b_1(x\mathrm{e}^s,y-z,\eta)\ \mathrm{d}\omega đ\zeta\\
&\qquad+\int \mathrm{e}^{i\omega\cdot\zeta}\,a_1(x,y,\zeta+\eta)\varphi(\omega)b_0(y-z,\eta)\ \mathrm{d}\omega đ\zeta\\
&\qquad+\int \mathrm{e}^{i\omega\cdot\zeta}\,a_1(x,y,\zeta+\eta)\varphi(\omega)b_1(x\mathrm{e}^s,y-z,\eta)\ \mathrm{d}\omega đ\zeta\Big)\\
&= c_0(y,\eta)+c_1(x,y,\eta).
\end{aligned}$$

Denote

$$\widetilde{b}_0(y,\eta,\zeta) = \int \mathrm{e}^{i\omega\cdot\zeta}\,\varphi(\omega)b_0(y-z,\eta)\ \mathrm{d}\omega,$$

then by standard approach in Fourier analysis, for any β,δ,ϵ, and $k\in\mathbb{N}$, there is some constant $C^k_{\beta\delta\epsilon}$, such that

$$\left|\partial_y^\beta \partial_\eta^\delta \partial_\zeta^\epsilon \widetilde{b}(y,\eta,\zeta)\right| \leqslant C^k_{\beta\delta\epsilon}[(1+|y|)(1+|\zeta|)]^{-k}(1+|\eta|)^{m'-|\delta|}. \tag{1.4.3}$$

Hence, for any $\ell \in \mathbb{N}$, , by (1.4.3) with $\beta = \delta = \epsilon = 0$, $k = |m| + 2n$, and Peetre's inequality (see (A.5)),

$$\begin{aligned}
|c_0(y,\eta)| &\leqslant \int \left|a_0(y,\zeta+\eta)\widetilde{b}_0(y,\eta,\zeta)\right| \, đ\zeta \\
&\leqslant \int D(1+|y|)^{-\ell}(1+|\zeta+\eta|)^m(1+|\zeta|)^{-(|m|+2n)}(1+|\eta|)^{m'} \, đ\zeta \\
&\leqslant (1+|y|)^{-\ell}(1+|\eta|)^{m+m'} \int D'(1+|\zeta|)^{2n} \, đ\zeta \\
&= D''(1+|y|)^{-\ell}(1+|\eta|)^{m+m'}
\end{aligned}$$

for some constant D' and $D'' = D' \int (1+|\zeta|)^{2n} \, đ\zeta$. Similarly, one can show that

$$\sup \left|(1+|\eta|)^{|\delta|-(m+m')}(1+|y|)^\ell \partial_y^\beta \partial_\eta^\delta c_0(y,\eta)\right| < \infty$$

for any β, δ, and $\ell \in \mathbb{N}$, therefore $c_0(y,\eta) \in S^m(\mathbb{R}^{n-1};\mathbb{R}^n)$.

To analyze $c_1(x,y,\eta)$, we write

$$\widetilde{b}_1(x,y,\eta,\zeta) = \int \mathrm{e}^{i\omega\cdot\zeta} \, \varphi(\omega) b_1(xe^s, y-z, \eta) \, \mathrm{d}\omega.$$

Observe that

$$
\begin{aligned}
&\left|\zeta^\sigma (x\partial_x)^\alpha \partial_y^\beta \partial_\eta^\delta \partial_\zeta^\epsilon \widetilde{b}_1(x,y,\eta,\zeta)\right| \\
&= \left| \int (\partial_\zeta^\epsilon \, \mathrm{e}^{i\omega\cdot\zeta})\zeta^\sigma \varphi(\omega)(x\partial_x)^\alpha \partial_y^\beta \partial_\eta^\delta b_1(xe^s, y-z, \eta) \, \mathrm{d}\omega \right| \\
&= \left| \int \partial_\omega^\sigma \, \mathrm{e}^{i\omega\cdot\zeta} \, \omega^\epsilon \varphi(\omega) \left((x\partial_x)^\alpha \partial_y^\beta \partial_\eta^\delta b_1\right)(x,y,\eta)\Big|_{\substack{x=xe^s\\ y=y-z}} \mathrm{d}\omega \right| \\
&= \left| \sum_{\sigma_1+\sigma_2=\sigma} D_{\sigma 1,\sigma_2} \int \mathrm{e}^{i\omega\cdot\zeta} \, \partial_\omega^{\sigma_1}(\omega^\epsilon \varphi(\omega)) \partial_\omega^{\sigma_2} \left((x\partial_x)^\alpha \partial_y^\beta \partial_\eta^\delta b_1\right)(x,y,\eta)\Big|_{\substack{x=xe^s\\ y=y-z}} \mathrm{d}\omega \right| \\
&= \left| \sum_{\substack{\sigma_1+\sigma_2=\sigma\\ \sigma_2=(\sigma_{21},\sigma_{22})}} D_{\sigma_1,\sigma_2} \int \mathrm{e}^{i\omega\cdot\zeta} \, \partial_\omega^{\sigma_1}(\omega^\epsilon \varphi(\omega)) \left((x\partial_x)^{\alpha+\sigma_{21}} \partial_y^{\beta+\sigma_{22}} \partial_\eta^\delta b_1\right)(x,y,\eta)\Big|_{\substack{x=xe^s\\ y=y-z}} \mathrm{d}\omega \right|,
\end{aligned}
$$

in which we used the fact that $\partial_s f(x\,\mathrm{e}^s) = (x\partial_x f)(x)\big|_{x=x\,\mathrm{e}^s}$. Since

$$
\begin{aligned}
\left| \left((x\partial_x)^\alpha \partial_y^\beta \partial_\eta^\delta b_1\right)(x,y,\eta)\Big|_{\substack{x=xe^s\\ y=y-z}} \right| &\leqslant C(1+|\ln x\,\mathrm{e}^s|)^{-\ell}(1+|y-z|)^{-\ell}(1+|\eta|)^{m'-|\delta|} \\
&\leqslant \frac{C[(1+|s|)(1+|z|)]^\ell (1+|\eta|)^{m'-|\delta|}}{[(1+|\ln x|)(1+|y|)]^\ell},
\end{aligned}
$$

we obtain an estimate similar to (1.4.3), namely,

$$
\left|(x\partial_x)^\alpha \partial_y^\beta \partial_\eta^\delta \partial_\zeta^\epsilon \widetilde{b}_1(x,y,\eta,\zeta)\right| < C_{\alpha\beta\delta\epsilon}^\ell [(1+|\ln x|)(1+|y|)(1+|\eta|)]^{-\ell}(1+|\eta|)^{m'-|\delta|}. \quad (1.4.4)
$$

Consequently, we have the following estimates:

$$\begin{aligned}\left|\int a_0(y,\zeta+\eta)\widetilde{b}_1(x,y,\eta,\zeta)\,đ\zeta\right| &\leqslant \int C\frac{(1+|\zeta+\eta|)^m(1+|\eta|)^{m'}}{[(1+|\ln x|)(1+|y|)]^\ell(1+|\zeta|)^{(|m|+2n)}}\,đ\zeta\\ &\leqslant C\int(1+|\zeta|)^{-2n}\,đ\zeta\cdot\frac{(1+|\eta|)^{m+m'}}{[(1+|\ln x|)(1+|y|)]^\ell}\\ &\leqslant D_1[(1+|\ln x|)(1+|y|)]^{-\ell}(1+|\eta|)^{m+m'},\end{aligned} \tag{1.4.5}$$

$$\left|\int a_1(x,y,\zeta+\eta)\widetilde{b}_0(y,\eta,\zeta)\,đ\zeta\right| \leqslant D_2[(1+|\ln x|)(1+|y|)]^{-\ell}(1+|\eta|)^{m+m'}, \tag{1.4.6}$$

$$\left|\int a_1(x,y,\zeta+\eta)\widetilde{b}_1(x,y,\eta,\zeta)\,đ\zeta\right| \leqslant D_3[(1+|\ln x|)(1+|y|)]^{-\ell}(1+|\eta|)^{m+m'}. \tag{1.4.7}$$

Furthermore, observe, for example, that

$$\begin{aligned}&(x\partial_x)^\alpha\partial_y^\beta\partial_\eta^\delta a_1(x,y,\zeta+\eta)\widetilde{b}_1(x,y,\eta,\zeta)\\ =&\sum_{\substack{\alpha_1+\alpha_2\leqslant\alpha\\ \beta_1+\beta_2=\beta\\ \delta_1+\delta_2=\delta}} C_{\alpha_1,\alpha_2,\beta_1,\delta_1}(x^{\alpha_1}\partial_x^{\alpha_1}\partial_y^{\beta_1}\partial_\eta^{\delta_1}a_1)(x,y,\zeta+\eta)x^{\alpha_2}\partial_x^{\alpha_2}\partial_y^{\beta_2}\partial_\eta^{\delta_2}b_1(x,y,\eta,\zeta),\end{aligned}$$

hence

$$\left|\int(x\partial_x)^\alpha\partial_y^\beta\partial_\eta^\delta a_1(x,y,\zeta+\eta)\widetilde{b}_1(x,y,\eta,\zeta)\,đ\zeta\right| \leqslant \frac{D^\ell_{\alpha\beta\delta}(1+|\eta|)^{m+m'-|\delta|}}{[(1+|\ln x|)(1+|y|)]^\ell},$$

and same symbol-type estimates hold for $\int a_0(y,\zeta+\eta)\widetilde{b}_1(x,y,\eta,\zeta)\,đ\zeta$ and $\int a_1(x,y,\zeta+\eta)\widetilde{b}_0(y,\eta,\zeta)\,đ\zeta$ as well. In summary, we have $c_1(x,y,\eta)\in {}^1S^m_\partial(\mathbb{R}^{n,1};\mathbb{R}^n)$, and $c(x,y,\eta)\in S^m_{bl}(\mathbb{R}^{n,1};\mathbb{R}^n)$.

Step 2. Now we analyze A_1B_2. We first assume that B_2 is supported away from $rb(X^2_b)$. Hence we look at X^3_b near $\pi^{-1}_{F,b}(\Delta_b(X^2_b))\cap fs$ and we may use the coordinates

$$(x'',y,s,w,t,z):=(x'',y,\ln\frac{x}{x''},y-y'',\ln\frac{x}{x'},y-y').$$

Note that $X_b^3 \cong X_b^2 \times \mathbb{R}^n_{(t,z)}$ and $\pi_{F,b}^{-1}(\Delta_b) \cong X_b^2 \times \{0\}$. In X_b^2 near $f\!f(X_b^2) \cap lb$, we use the coordinates

$$(x', y, \gamma, z) := (x', y, \ln \frac{x'}{x}, y - y'),$$

then we can write

$$\begin{aligned}
K_{A_1} &= \psi(\gamma, z) \int \mathrm{e}^{i(\gamma,z)\cdot(\tau,\xi)}\, a(x', y, \tau, \xi)\, đ\tau đ\xi \cdot \mu', \\
K_{B_2} &= b(x', y, \gamma, z) \cdot \mu',
\end{aligned}$$

where $a \in S^m_{bl}(\mathbb{R}^{n,1}; \mathbb{R}^n)$, $b \in S^0_{bl}([0,1)^2_b \times \mathcal{V}^2)$, $\psi \in C_c^\infty(\mathbb{R}^n)$ and $\mu = \left|\frac{\mathrm{d}x'}{x'}\mathrm{d}y\right|$. Write

$$a(x', y, \tau, \xi) = a_0(y, \tau, \xi) + a_1(x', y, \tau, \xi),\ \ b(x', y, \gamma, z) = b_0(y, \gamma, z) + b_1(x', y, \gamma, z).$$

According to (B.10) and (B.11), we have

$$\begin{aligned}
\mu K_{A_1 B_2} &= (\pi_{C,b})_*(\pi^*_{C,b}\mu\pi^*_{F,b}K_{A_1}\pi^*_{S,b}K_{B_2}) \\
&= \int \mathrm{e}^{i\sigma\cdot\eta}\, \psi(\sigma) a'(x'', y, s, \sigma, \eta) b'(x'', y, s, w, \sigma)\ \mathrm{d}\sigma đ\eta \cdot \mu\mu' \qquad (1.4.8) \\
&= c(x'', y, s, w) \cdot \mu\mu',
\end{aligned}$$

where $\sigma = (t, z)$ and $\eta = (\tau, \xi)$, and where

$$\begin{aligned}
a'(x'', y, s, \sigma, \eta) &= a(x''\, \mathrm{e}^{s-t}, y, \eta) \\
&= a_0(y, \eta) + a_1(x''\, \mathrm{e}^{s-t}, y, \eta) \\
&= a_0'(y, \eta) + a_1'(x'', y, s, \sigma, \eta), \\
b'(x'', y, s, w, \sigma) &= b(x'', y - z, s - t, w - z) \\
&= b_0(y - z, s - t, w - z) + b_1(x'', y - z, s - t, w - z) \\
&= b_0'(y, s, w, \sigma) + b_1'(x'', y, s, w, \sigma).
\end{aligned}$$

Given any indexes $\alpha, \beta, \delta, \epsilon, \theta, \kappa$ and $\ell \in \mathbb{N}$, we have the following estimates:

$$\sup \left|(1+|y|)^{\ell}(1+|\eta|)^{|\delta|-m}\partial_y^{\beta}\partial_\eta^{\delta}a_0'(y,\eta)\right| < \infty,$$
$$\sup \left|(1+|s-t|)^{\ell}\partial_y^{\beta}\partial_s^{\epsilon}\partial_w^{\theta}\partial_\sigma^{\kappa}b_0'(y,s,w,\sigma)\right| < \infty,$$
$$\sup \left|[(1+\left|\ln x'' e^{s-t}\right|)(1+|y|)]^{\ell}(1+|\eta|)^{|\delta|-m}(x''\partial_{x''})^{\alpha}\partial_y^{\beta}\partial_s^{\epsilon}\partial_\sigma^{\kappa}\partial_\eta^{\delta}a_1'(x'',y,s,\sigma,\eta)\right| < \infty,$$
$$\sup \left|[(1+|\ln x''|)(1+|s-t|)]^{\ell}(x''\partial_{x''})^{\alpha}\partial_y^{\beta}\partial_s^{\epsilon}\partial_w^{\theta}\partial_\sigma^{\kappa}b_1'(x'',y,s,w,\sigma)\right| < \infty.$$

Hence we have, for instance,

$$\begin{aligned}
&\left|\eta^{\kappa}(x''\partial_{x''})^{\alpha}\partial_y^{\beta}\partial_s^{\epsilon}\partial_w^{\theta}\int \mathrm{e}^{i\sigma\cdot\eta}\,\varphi(\sigma)a_1'(x'',y,s,\sigma,\eta)b_0'(y,s,w,\sigma)\ \mathrm{d}\sigma\right| \\
&= \left|\int \left(\partial_\sigma^{\kappa}\,\mathrm{e}^{i\sigma\cdot\eta}\right)\varphi(\sigma)\partial_y^{\beta}\partial_s^{\epsilon}\partial_w^{\theta}\left((x''\partial_{x''})^{\alpha}a_1'(x'',y,s,\sigma,\eta)\right)b_0'(y,s,w,\sigma)\ \mathrm{d}\sigma\right| \\
&= \left|\int \mathrm{e}^{i\sigma\cdot\eta}\,\partial_\sigma^{\kappa}\varphi(\sigma)\partial_y^{\beta}\partial_s^{\epsilon}\partial_w^{\theta}\left((x''\partial_{x''})^{\alpha}a_1'(x'',y,s,\sigma,\eta)\right)b_0'(y,s,w,\sigma)\ \mathrm{d}\sigma\right| \\
&\leqslant \left(\sum_{\kappa_1\leqslant\kappa}\int C[(1+\left|\ln x'' e^{s-t}\right|)(1+|s-t|)^2]^{-\ell}\partial_\sigma^{\kappa_1}\varphi(\sigma)\ \mathrm{d}\sigma\right)\cdot(1+|y|)^{-\ell}(1+|\eta|)^{m} \\
&\leqslant \left(\sum_{\kappa_1\leqslant\kappa}\int C[(1+|s-t|)(1+|t|)^2]^{\ell}\partial_\sigma^{\kappa_1}\varphi(\sigma)\ \mathrm{d}\sigma\right)\cdot\frac{(1+|\eta|)^{m}}{(1+|s|)^2(1+|\ln x''|)(1+|y|)^{\ell}} \\
&\leqslant \left(\sum_{\kappa_1\leqslant\kappa}\int C(1+|t|)^{3\ell}\partial_\sigma^{\kappa_1}\varphi(\sigma)\ \mathrm{d}\sigma\right)\cdot[(1+|s|)(1+|\ln x''|)(1+|y|)]^{-\ell}(1+|\eta|)^{m} \\
&\leqslant D[(1+|s|)(1+|\ln x''|)(1+|y|)]^{-\ell}(1+|\eta|)^{m},
\end{aligned}$$

and consequently,

$$\sup\left|\frac{(x''\partial_{x''})^{\alpha}\partial_y^{\beta}\partial_s^{\epsilon}\partial_w^{\theta}\int \mathrm{e}^{i\sigma\cdot\eta}\,\varphi(\sigma)a_1'(x'',y,s,\sigma,\eta)b_0'(y,s,w,\sigma)\ \mathrm{d}\sigma}{[(1+|s|)(1+|\ln x''|)(1+|y|)(1+|\eta|)]^{-\ell}}\right| < \infty.$$

Therefore,

$$\int \mathrm{e}^{i\sigma\cdot\eta}\,\varphi(\sigma)a_1'(x'',y,s,\sigma,\eta)b_0'(y,s,w,\sigma)\ \mathrm{d}\sigma\mathrm{đ}\eta \in {}^{1}S^{0}_{lb,ff,rb}(X_b^2).$$

Proceed similarly, we see that $c \in S^0_{bl}(X^2_b)$.

Secondly we assume that B_2 is supported away from $lb(X^2_b)$, and work near $\pi^{-1}_{F,b}(\Delta_b) \cap rb \in X^3_b$. We use the coordinates

$$(x, y, s, w, t, z) := (x, y, \ln \frac{x''}{x}, y - y'', \ln \frac{x'}{x}, y - y'). \tag{1.4.9}$$

in X^3_b and the coordinates

$$(x, y, \omega, z) := (x, y, \ln \frac{x'}{x}, y - y'),$$

in X^2_b near $ff \cap rb$. We can write

$$\begin{aligned} K_{A_1} &= \psi(\omega, z) \int e^{i(\omega,z)\cdot(\tau,\xi)} \, a_0(x, y, \tau, \xi) \, đ\tau đ\xi \cdot \mu', \\ K_{B_2} &= b_0(x, y, \omega, z) \cdot \mu' \end{aligned} \tag{1.4.10}$$

where $a_0 \in S^m_{bl}(\mathbb{R}^{n,1}; \mathbb{R}^n)$, $b_0 \in S^0_{bl}([0,1)^2_b \times \mathcal{V}^2)$, $\psi \in C^\infty_c(\mathbb{R}^n)$ and $\mu = \left|\frac{dx'}{x'} dy\right|$. Then according to (B.12) and (B.13), we have

$$\begin{aligned} \mu K_{A_1 B_2} &= (\pi_{C,b})_* (\pi^*_{C,b} \mu \pi^*_{F,b} K_{A_1} \pi^*_{S,b} K_{B_2}) \\ &= c_0(x, y, s, w) \cdot \mu\mu'. \end{aligned}$$

where $\sigma = (t, z)$, $\eta = (\tau, \xi)$ and

$$c_0(x, y, s, w) = \int e^{i\sigma\cdot\eta} \, \psi(t, z) a_0(x, y, \eta) b_0'(x'', y, s, w, t, z) \, d\sigma đ\eta,$$

with $b_0'(x, y, s, w, t, z) = b_0(x\, e^t, y - z, s - t, w - z)$. Observe then that also $c_0 \in S^0_{bl}(X^2_b)$. In summary, we have

$$A_1 B_2 \in \Psi^{-\infty}_{bl}(X).$$

Step 3. Lastly we study A_2B_1. We first assume that A_2 is supported away from $lb(X_b^2)$, then

$$\begin{aligned}
K_{A_2} &= a_1(x,y,\omega,z)\cdot\mu', \\
K_{B_1} &= \int e^{i(\omega,z)\cdot(\tau,\xi)}\, b_1(x,y,\tau,\xi)\, đ\tau đ\xi\cdot\mu'.
\end{aligned}$$

Making use of (B.14) and (B.15), near $\pi_{S,b}^{-1}(\Delta_b)\cap ss$ we have

$$\begin{aligned}
\mu K_{A_2B_1} &= (\pi_{C,b})_*(\pi_{C,b}^*\mu\pi_{F,b}^*K_{A_2}\pi_{S,b}^*K_{B_1}) \\
&= \int e^{i(t,w-z)\cdot(\tau,\xi)}\, a_1(x,y,t-s,z)b_1(x\,e^{t-s},y-z,\tau,\xi)\ dtdzđ\tau đ\xi\cdot\mu\mu' \\
&= c_1(x,y,s,w)\cdot\mu\mu'.
\end{aligned}$$

Secondly, assume that A_2 is supported away from $rb(X_b^2)$, and

$$\begin{aligned}
K_{A_2} &= a_2(x',y,\gamma,z)\cdot\mu' \\
K_{B_1} &= \int e^{i(\gamma,z)\cdot(\tau,\xi)}\, b_2(x',y,\tau,\xi)\, đ\tau đ\xi\cdot\mu'.
\end{aligned}$$

By (B.16) and (B.17), we have

$$\begin{aligned}
\mu K_{A_2B_1} &= (\pi_{C,b})_*(\pi_{C,b}^*\mu\pi_{F,b}^*K_{A_2}\pi_{S,b}^*K_{B_1}) \\
&= \int e^{i(t,w-z)\cdot(\tau,\xi))}\, a_2(x''\,e^{t},y,s-t,z)b_2(x'',y-z,\tau,\xi)\ dtdzđ\tau đ\xi\cdot\mu\mu' \\
&= c_2(x'',y,s,w)\cdot\mu\mu'.
\end{aligned}$$

Note that both c_1 and c_2 have very similar structure to c in (1.4.8), thus with arguments comparable to those in Step 2, one can show that $c_1,c_2\in S_{bl}^0(X_b^2)$, hence $A_2B_1\in\Psi_{bl}^{-\infty}(X)$ also.

In conclusion, we have obtained that

$$AB \in \Psi_{bl}^{m+m'}(X). \qquad \square$$

Corollary 1.4.2.1. ${}^b\sigma_{m+m'}(AB) = {}^b\sigma_m(A){}^b\sigma_{m'}(B)$.

Proof. Recall that the m-th principal symbols are obtained by first gluing up the local total symbols, then projecting onto $S_{bl}^{[m]}({}^bT^*X)$. Thus it suffices to work in local coordinate patches. Assume that both K_A and K_B are supported in a coordinate patch over the b-diagonal. Assume further that they are supported near the front face of X_b^2, for simplicity. Then by the discussion leading to (1.4.2), we have

$$\begin{aligned} K_A &= \varphi(\omega) \int \mathrm{e}^{i\omega\cdot\eta}\, a(x, y, \eta)\, đ\eta \cdot \mu', \\ K_B &= \int \mathrm{e}^{i\omega\cdot\eta}\, b(x, y, \eta)\, đ\eta \cdot \mu', \end{aligned}$$

and

$$\begin{aligned} K_{AB} &= \int \mathrm{e}^{i\gamma\cdot\eta}\, c(x, y, \eta)\, đ\eta \\ &= \int \mathrm{e}^{i\gamma\cdot\eta} \left(\int \mathrm{e}^{i\omega\cdot\zeta}\, a(x, y, \zeta + \eta)\varphi(\omega) b(x\,\mathrm{e}^s, y - z, \eta)\ \mathrm{d}\omega đ\zeta \right)\, đ\eta, \end{aligned}$$

where $\varphi \equiv 1$ on a neighborhood of 0. We must show that $a \cdot b$ and c determine the same equivalent class in $S_{bl}^{[m+m']}(\mathbb{R}^{n,1}; \mathbb{R}^n)$. Applying Taylor expansion to $a(x, y, \zeta + \eta)$ at the last variable, we have

$$a(x, y, \zeta + \eta) = \sum_{|\alpha| \leqslant k} \frac{(\partial_\eta^\alpha a)(x, y, \eta)\zeta^\alpha}{\alpha!} + \sum_{|\beta|=k+1} \frac{(-1)^{k+1}(k+1)R_\beta(x, y, \eta, \zeta)}{\beta!},$$

where $R_\beta = \zeta^\beta \int_0^1 (1-t)^k (\partial_\eta^\beta a)(x,y,\eta+t\zeta)\ \mathrm{d}t$. Observe that

$$
\begin{aligned}
&\int \mathrm{e}^{i\omega\cdot\zeta} \sum_{|\alpha|\leqslant k} \frac{(\partial_\eta^\alpha a)(x,y,\eta)\zeta^\alpha}{\alpha!} \varphi(\omega) b(x\,\mathrm{e}^s, y-z, \eta)\ \mathrm{d}\omega đ\zeta \\
&= \sum_{|\alpha|\leqslant k} \frac{1}{\alpha!} (\partial_\eta^\alpha a)(x,y,\eta) \int \left[(-D_\omega)^\alpha \mathrm{e}^{i\omega\cdot\zeta}\right] \varphi(\omega) b(x\,\mathrm{e}^s, y-z,\eta)\ \mathrm{d}\omega đ\zeta \\
&= \sum_{|\alpha|\leqslant k} \frac{1}{\alpha!} (D_\eta^\alpha a)(x,y,\eta) \int \mathrm{e}^{i\omega\cdot\zeta}\, \partial_\omega^\alpha \left[\varphi(\omega) b(x\,\mathrm{e}^s, y-z,\eta)\right]\ \mathrm{d}\omega đ\zeta \\
&= \sum_{|\alpha|\leqslant k} \frac{1}{\alpha!} (D_\eta^\alpha a)(x,y,\eta) \int \mathrm{e}^{i\omega\cdot\zeta} \left[\sum_{\alpha_1+\alpha_2=\alpha} C_{\alpha_1} (\partial_\omega^{\alpha_1}\varphi(\omega))(\partial_\omega^{\alpha_2} b(x\,\mathrm{e}^s, y-z,\eta))\right] \mathrm{d}\omega đ\zeta \\
&= \sum_{\substack{|\alpha|\leqslant k \\ (\beta,\gamma)=\alpha}} \frac{1}{\alpha!} (D_\eta^\alpha a)(x,y,\eta)((x\partial_x)^\beta \partial_y^\gamma b)(x,y,\eta),
\end{aligned}
$$

since $\varphi(0) = 1$ and $\partial_\omega^\alpha \varphi(0) = 0$ for any $\alpha \neq (0)$. Note that one can also check that

$$
\int \mathrm{e}^{i\omega\cdot\zeta} \sum_{|\beta|=k+1} \frac{(-1)^{k+1}(k+1) R_\beta(x,y,\eta,\zeta)}{\beta!} \varphi(\omega) b(x\,\mathrm{e}^s, y-z,\eta)\ \mathrm{d}\omega đ\zeta
$$

is in $S_{bl}^{m+m'-k-1}(\mathbb{R}^{n,1};\mathbb{R}^n)$ (see also Corollay A.4.1). In particular, we have

$$
c(x,y,\eta) - a(x,y,\eta)b(x,y,\eta) \in S_{bl}^{m+m'-1}(\mathbb{R}^{n,1};\mathbb{R}^n). \qquad \square
$$

The next result now follows from the well-known symbolic calculus of pseudodifferential operators.

Theorem 1.4.3. *If $A \in \Psi_{bl}^m(X)$ is elliptic, then there exists $B_s \in \Psi_{bl}^{-m}(X)$ such that*

$$
A \circ B_s = \mathrm{Id} - R_s,
$$

$$
B_s \circ A = \mathrm{Id} - S_s,
$$

where $R_s, S_s \in \Psi_{bl}^{-\infty}(X)$. Moreover, B_s is unique modulo $\Psi_{bl}^{-\infty}(X)$.

1.5 Normal operators

In this section, we will study the *normal operator* associated to a *bl*-pseudodifferential operator. The normal operator captures the behavior of the pseudodifferential operator near the boundary, or, in the geometric point of view, at the infinity. It will play an essential role in the study of the Fredholm property and, moreover, the index formula.

Let $R \in \Psi_{bl}^{-\infty}(X)$. There is a family of linear operators depending on a parameter $\tau \in \mathbb{R}$, $\mathcal{N}(R)(\tau)$, at the boundary $Y = \partial X$, induced by R via applying Fourier transform along the *boundary fiber* to the restriction of K_R at the front face of X_b^2. That is, the Schwartz kernel of $\mathcal{N}(R)(\tau)$ is obtained by

$$\mathcal{N}(R)(\tau) = \int \mathrm{e}^{-i\omega\tau} \, K_R\big|_{\mathit{ff}} \, \mathrm{d}\omega, \tag{1.5.1}$$

where ω denotes the boundary fiber variable. Note that by definition K_R is Schwartz in ω, hence (1.5.1) makes sense. $\mathcal{N}(R)(\tau)$ is called the normal operator of R. It follows immediately from the definition that $\mathcal{N}(R)(\tau)$, as a function on $\mathbb{R} \times Y$, is Schwartz in τ.

The normal operators share an important feature with the principal symbols, namely, they induce algebra homomorphisms. Here is a preliminary version.

Lemma 1.5.1. *Let $R, S \in \Psi_{bl}^{-\infty}(X)$. Then*

$$\mathcal{N}(R \circ S)(\tau) = \mathcal{N}(R)(\tau) \circ \mathcal{N}(S)(\tau).$$

Proof. Recall that near $\mathit{ff}(X_b^2)$, $X_b^2 \cong [0,1)_b^2 \times Y^2$. We use the coordinates

$$(\rho, \omega, y, y') = (x + x', \ln(\frac{x'}{x}), y, y').$$

Let ν be a trivialization of Ω_Y, and μ' be the lift of $\left|\frac{\mathrm{d}x}{x}\right| \cdot \nu$ to X_b^2 via $\pi_{R,b}$. Assume that near the front face,

$$K_R = a(\rho, \omega, y, y') \cdot \mu',$$
$$K_S = b(\rho, \omega, y, y') \cdot \mu'.$$

We first compute

$$\begin{aligned}\mathcal{N}(R)(\tau) \circ \mathcal{N}(S)(\tau) &= \int \mathrm{e}^{-i\omega\tau}\, a(0, \omega, y, y'')\, \mathrm{e}^{-i\gamma\tau}\, b(0, \gamma, y'', y')\ \mathrm{d}\omega\mathrm{d}\gamma\nu(y'') \cdot \nu(y') \\ &= \int \mathrm{e}^{-i(\omega+\gamma)\tau}\, a(0, \omega, y, y'')b(0, \gamma, y'', y')\ \mathrm{d}\omega\mathrm{d}\gamma\nu(y'') \cdot \nu(y').\end{aligned}$$

To compute $\mathcal{N}(R \circ S)(\tau)$, we first recall that the Schwartz kernels of R and S as distributional right-density on X^2 are given by $\widetilde{a}(x, x', y, y') \cdot \left|\frac{\mathrm{d}x'}{x'}\right| \nu(y')$ and $\widetilde{b}(x, x', y, y') \cdot \left|\frac{\mathrm{d}x'}{x'}\right| \nu(y')$ respectively, where

$$\begin{aligned}\widetilde{a}(x, x', y, y') &= a(x + x', \ln(\frac{x'}{x}), y, y') \\ \widetilde{b}(x, x', y, y') &= b(x + x', \ln(\frac{x'}{x}), y, y')\end{aligned}$$

Moreover, the kernel of $R \circ S$ is given by $\widetilde{c}(x, x', y, y') \cdot \left|\frac{\mathrm{d}x'}{x'}\right| \nu(y')$, where

$$\widetilde{c}(x, x', y, y') = \int \widetilde{a}(x, x'', y, y'')\widetilde{b}(x'', x', y'', y') \frac{\mathrm{d}x''}{x''} \nu(y'').$$

Lifting $\widetilde{c}$ to X_b^2, we have

$$\begin{aligned}
c(\rho,\omega,y,y') &= \widetilde{c}(\frac{\rho}{1+\mathrm{e}^{\omega}},\frac{\rho\,\mathrm{e}^{\omega}}{1+\mathrm{e}^{\omega}},y,y') \\
&= \int \widetilde{a}(\frac{\rho}{1+\mathrm{e}^{\omega}},x'',y,y'')\widetilde{b}(x'',\frac{\rho\,\mathrm{e}^{\omega}}{1+\mathrm{e}^{\omega}},y'',y')\frac{\mathrm{d}x''}{x''}\nu(y'') \\
&= \int a(\frac{\rho}{1+\mathrm{e}^{\omega}}+x'',\ln(\frac{(1+\mathrm{e}^{\omega})x''}{\rho}),y,y'') \\
&\quad \times b(x''+\frac{\rho\,\mathrm{e}^{\omega}}{(1+\mathrm{e}^{\omega})x''},\ln(\frac{\rho\,\mathrm{e}^{\omega}}{(1+\mathrm{e}^{\omega})x''}),y'',y')\frac{\mathrm{d}x''}{x''}\nu(y'').
\end{aligned}$$

With the change of variable $\gamma=\ln(\frac{(1+\mathrm{e}^{\omega})x''}{\rho})$,

$$c(\rho,\omega,y,y') = \int a(\frac{(1+\mathrm{e}^{\gamma})\rho}{1+\mathrm{e}^{\omega}},\gamma,y,y'')b(\frac{(\mathrm{e}^{\omega}+\mathrm{e}^{\gamma})\rho}{1+\mathrm{e}^{\omega}},\omega-\gamma,y'',y')\ \mathrm{d}\gamma\nu(y''). \tag{1.5.2}$$

Thus,

$$\begin{aligned}
\mathcal{N}(R\circ S)(\tau) &= \int \mathrm{e}^{-i\omega\tau}\,c(0,\omega,y,y')\ \mathrm{d}\omega\cdot\nu(y') \\
&= \int \mathrm{e}^{-i\omega\tau}\,a(0,\gamma,y,y'')b(0,\omega-\gamma,y'',y')\ \mathrm{d}\gamma\nu(y'')\mathrm{d}\omega\cdot\nu(y') \\
&= \int \mathrm{e}^{-i(\omega+\gamma)\tau}\,a(0,\omega,y,y'')b(0,\gamma,y'',y')\ \mathrm{d}\omega\mathrm{d}\gamma\nu(y'')\cdot\nu(y') \\
&= \mathcal{N}(R)(\tau)\circ\mathcal{N}(S)(\tau).
\end{aligned}$$

□

Now consider $A\in\Psi_{bl}^m(X)$. Write $K_A=K_{A_1}+K_{A_2}$ such that K_{A_1} is supported away from $lb\cup rb\in X_b^2$ and K_{A_2} is supported away from $\Delta_b\in X_b^2$. Recall that $K_{A_2}\in\Psi_{bl}^{-\infty}(X)$, and locally near *ff*,

$$K_{A_1}=\int \mathrm{e}^{i(s,z)\cdot(\tau,\xi)}\,a(r,y,\tau,\xi)\,đ\tau đ\xi,$$

where $(r,s,y,z)=(x+x',\ln\frac{x}{x'},y,y-y')$. The normal operator of A is the family of operators $\mathcal{N}(A)(\tau)\in\Psi^m(Y)$, with $\tau\in\mathbb{R}$, defined by $\mathcal{N}(A)(\tau):=\mathcal{N}(A_1)(\tau)+$

$\mathcal{N}(A_2)(\tau)$ where, locally,

$$\mathcal{N}(A_1)(\tau) := \int e^{i(y-y')\xi}\, a(0,y,\tau,\xi)\, đ\xi.$$

Note that $\mathcal{N}(A_1)(\tau)$ is also obtained by applying Fourier transform along the boundary fiber to the kernel of A_1 restricted to the front face of X_b^2, that is,

$$\begin{aligned}&\int e^{-i\tau s}\left(\int e^{is\lambda+i(y-y')\cdot\xi}\, a(0,y,\lambda,\xi)\, đ\lambda\, đ\xi\right) ds\cdot |\,dy'| \\ =&\int e^{i(y-y')\cdot\xi}\, a(0,y,\tau,\xi)\, đ\xi\cdot |\,dy'|\,.\end{aligned}$$

Note that the continuity principle endows us with a good notion of restriction of conormal distributions to certain submanifolds. In combination with above mentioned point of view, we see that the normal operators are well defined, in the sense that they are independent of the ways how conormal distributions are decomposed into sums of compactly supported ones and smooth ones.

Remark. We also use the notation $\widehat{A}(\tau)$ interchangeably with $\mathcal{N}(A)(\tau)$.

Theorem 1.5.2. *Let $A \in \Psi_{bl}^m(X)$, $B \in \Psi_{bl}^{m'}(X)$. Then*

$$\mathcal{N}(B\circ A)(\tau) = \mathcal{N}(B)(\tau)\circ\mathcal{N}(A)(\tau).$$

Proof. Due to the close relation, we will use the same set of notations and settings as in Theorem 1.4.2. In particular, we write $A = A_1 + A_2$ and $B = B_1 + B_2$. Note that

$$\mathcal{N}(A\circ B)(\tau) = \sum_{j,k=1}^{2}\mathcal{N}(A_k\circ B_j)(\tau)$$

and

$$\mathcal{N}(A)(\tau)\circ\mathcal{N}(B)(\tau) = \sum_{j,k=1}^{2}\mathcal{N}(A_k)(\tau)\circ\mathcal{N}(B_j)(\tau),$$

It suffices to show that $\mathcal{N}(A_k\circ B_j)(\tau) = \mathcal{N}(A_k)(\tau)\circ\mathcal{N}(B_j)(\tau)$ for $j,k = 1,2$. Immediately, $\mathcal{N}(A_2 \circ B_2)(\tau) = \mathcal{N}(A_2)(\tau) \circ \mathcal{N}(B_2)(\tau)$ follows from Lemma 1.5.1.

Step 1. We begin with $j, k = 1$. Under the same assumption as in (1.4.1), we compute

$$\begin{aligned}
&\mu\mathcal{N}(A_1)(\tau)\mathcal{N}(B_1)(\tau) = (\pi_C)_* \left(\pi_C^* \mu \pi_F^* \mathcal{N}(A_1)(\tau)\pi_S^* \mathcal{N}(B_1)(\tau)\right) \\
&= \int \mathrm{e}^{-is\tau}\, \varphi(s, y - y')\, \mathrm{e}^{is\lambda + i(y-y')\cdot\xi}\, a(0, y, \lambda, \xi)\, \mathrm{e}^{i(y'-y'')\cdot\zeta}\, b(0, y', \tau, \zeta)\, đ\lambda đ\xi \mathrm{d}s đ\zeta \mathrm{d}y' \cdot |\,\mathrm{d}y''| \\
&= \int \mathrm{e}^{is(\kappa-\tau)} \varphi(s, y - y') \mathrm{e}^{i(y-y')\cdot\xi}\, a(0, y, \kappa, \xi) \mathrm{e}^{i(y'-y'')\cdot\zeta}\, b(0, y', \tau, \zeta)\, \mathrm{d}s đ\xi đ\kappa đ\zeta \mathrm{d}y' \cdot |\,\mathrm{d}y''| \\
&= \int \mathrm{e}^{-iy''\cdot\zeta}\, a(0, y, \kappa, \xi)\, \mathrm{e}^{i[s(\kappa-\tau) + (y-y')\cdot\xi + y'\cdot\zeta]}\, \varphi(s, y - y') b(0, y', \tau, \zeta)\, \mathrm{d}s \mathrm{d}y' đ\kappa đ\xi đ\zeta \cdot |\,\mathrm{d}y''| \\
&= \int \mathrm{e}^{i(y-y'')\cdot\zeta}\, \mathrm{e}^{is\kappa + iz\cdot\xi} \varphi(s, z) a(0, y, \kappa + \tau, \xi + \zeta) b(0, y - z, \tau, \zeta)\, \mathrm{d}s \mathrm{d}z đ\kappa đ\xi đ\zeta \cdot |\,\mathrm{d}y''| \\
&= \int \mathrm{e}^{i(y-y')\cdot\xi}\, c(0, y, \tau, \xi)\, đ\xi \cdot |\,\mathrm{d}y'|\,,
\end{aligned}$$

where

$$c(0, y, \tau, \xi) = \int \mathrm{e}^{i(s,z)\cdot(\kappa,\xi)}\, \varphi(s, z) a(0, y, \kappa + \tau, \xi + \zeta) b(0, y - z, \tau, \zeta)\ \mathrm{d}s \mathrm{d}z đ\kappa đ\xi. \tag{1.5.3}$$

Comparing (1.4.2) and (1.5.3), we have $\mathcal{N}(A_1 \circ B_1)(\tau) = \mathcal{N}(A_1)(\tau) \circ \mathcal{N}(B_1)(\tau)$.

Step 2. Next we consider $k = 1, j = 2$. Recall that, near $\mathcal{V} \times \mathcal{V} \subset \partial X \times \partial X$, $X^2 \cong [0, 1)^2 \times \mathcal{V}^2$ with the coordinates (x, x', y, y'). In X_b^2, we may use the coordinates

$$(r, y, s, z) := (x + x', y, \ln \frac{x}{x'}, y - y').$$

Assume that the kernels of A_1 and B_2 in X_b^2 are

$$\begin{aligned} K_{A_1} &= \psi(s,z)\int \mathrm{e}^{i(s,z)\cdot(\tau,\xi)}\, a(r,y,\tau,\xi)\, đ\tau đ\xi \cdot \mu' \\ &= \int \mathrm{e}^{i(s,z)\cdot(\tau,\xi)}\, \widetilde{a}(r,y,\tau,\xi)\, đ\tau đ\xi \cdot \mu', \qquad (1.5.4) \\ K_{B_2} &= b(r,y,s,z)\cdot \mu', \end{aligned}$$

where $\widetilde{a}(r,y,\tau,\xi) = \int \psi(\widehat{\tau-\iota},\widehat{\xi-\eta})a(r,y,\iota,\eta)\, đ\iota đ\eta$, then their kernels as living in X^2 are

$$\begin{aligned} \bar{K}_{A_1} &= \psi(\ln\frac{x}{x'}, y-y')\int \mathrm{e}^{i(\ln(x/x'),y-y')\cdot(\tau,\xi)}\, a(x+x',y,\tau,\xi)\, đ\tau đ\xi \cdot \left|\frac{\mathrm{d}x'}{x'}\,\mathrm{d}y'\right|, \\ \bar{K}_{B_2} &= b(x+x',y,\ln\frac{x}{x'},y-y')\cdot\left|\frac{\mathrm{d}x'}{x'}\,\mathrm{d}y'\right|. \end{aligned}$$

Consequently, the compositional kernel is

$$\begin{aligned} \widetilde{c}(x,x',y,y') = \int \mathrm{e}^{i(\ln\frac{x}{x''},y-y'')\cdot(\tau,\xi)}\, &\psi(\ln\frac{x}{x''},y-y'')a(x+x'',y,\tau,\xi) \\ &\times b(x''+x',y'',\ln\frac{x''}{x'},y''-y')\frac{\mathrm{d}x''}{x''}\mathrm{d}y'' đ\tau đ\xi. \end{aligned}$$

Lifting back to X_b^2 and restricted to $r=0$ (the front face), we have

$$c(0,s,y,y') = \int \mathrm{e}^{i(t,y-y'')\cdot(\tau,\xi)}\, \psi(t,y-y'')a(0,y,\tau,\xi)b(0,y'',s-t,y''-y')\ \mathrm{d}t\mathrm{d}y'' đ\tau đ\xi,$$

and moreover,

$$\begin{aligned} \mathcal{N}(A_1B_2)(\tau) &= \int \mathrm{e}^{-is\tau}\, c(0,s,y,y')\ \mathrm{d}s\cdot|\,\mathrm{d}y'| \\ &= \int \mathrm{e}^{i(t,y-y'')\cdot(\iota,\xi)-is\tau}\psi(t,y-y'')a(0,y,\iota,\xi) \\ &\qquad \cdot b(0,y'',s-t,y''-y')\,\mathrm{d}t\mathrm{d}y'' đ\iota đ\xi \mathrm{d}s\cdot|\,\mathrm{d}y'|\,. \end{aligned}$$

On the other hand, also by (1.5.4), we have

$$\mathcal{N}(A_1)(\tau) = \int \mathrm{e}^{i(y-y')\cdot\xi}\, \widetilde{a}(0,y,\tau,\xi)\, đ\xi \cdot |\, \mathrm{d}y'|\,,$$
$$\mathcal{N}(B_2)(\tau) = \int \mathrm{e}^{-is\tau}\, b(0,y,s,y-y')\ \mathrm{d}s \cdot |\, \mathrm{d}y'|\,,$$

and

$$\begin{aligned}
&\mathcal{N}(A_1)(\tau)\mathcal{N}(B_2)(\tau)\\
&= \int \mathrm{e}^{i(y-y'')\xi - is\tau}\, \psi(\widehat{\tau-\iota}, \widehat{\xi-\eta}) a(0,y,\iota,\eta) b(0,y'',s,y''-y')\, đ\iota đ\eta đ\xi \mathrm{d}s \mathrm{d}y'' \cdot |\, \mathrm{d}y'|\\
&= \int \mathrm{e}^{i(y-y'')\eta - is\tau - it(\tau-\iota)}\, \psi(t, y-y'') a(0,y,\iota,\eta) b(0,y'',s,y''-y')\ \mathrm{d}t đ\iota đ\eta \mathrm{d}s \mathrm{d}y'' \cdot |\, \mathrm{d}y'|\\
&= \int \mathrm{e}^{it\iota + i(y-y'')\eta - is\tau}\, \psi(t, y-y'') a(0,y,\iota,\eta) b(0,y'',s-t,y''-y')\ \mathrm{d}t đ\iota đ\eta \mathrm{d}s \mathrm{d}y'' \cdot |\, \mathrm{d}y'|\,,
\end{aligned}$$

hence $\mathcal{N}(A_1)(\tau)\mathcal{N}(B_2)(\tau) = \mathcal{N}(A_1B_2)(\tau)$.

Step 3. Lastly, we look at $k = 2, j = 1$. We use a same set of coordinates as in *Step 2*, and assume that

$$K_{A_2} = a(r,y,s,z) \cdot \mu',$$
$$K_{B_1} = \int \mathrm{e}^{i(s,z)\cdot(\tau,\xi)}\, b(r,y,\tau,\xi)\, đ\tau đ\xi \cdot \mu'.$$

Therefore,

$$\bar{K}_{A_2} = a(x+x', y, \ln\frac{x}{x'}, y-y') \cdot \left| \frac{\mathrm{d}x'}{x'}\mathrm{d}y' \right|,$$
$$\bar{K}_{B_1} = \int \mathrm{e}^{i(\ln(x/x'), y-y')\cdot(\tau,\xi)}\, b(x+x', y,\tau,\xi)\, đ\tau đ\xi \cdot \left| \frac{\mathrm{d}x'}{x'}\mathrm{d}y' \right|,$$

and

$$\widetilde{c}(x,x',y,y') = \int \mathrm{e}^{i(\ln(w/x'), v-y')\cdot(\tau,\xi)}\, a(x+w, y, \ln(x/w), y-v) b(w+x', v,\tau,\xi) \frac{\mathrm{d}w}{w} \mathrm{d}v đ\tau đ\xi$$

is the compositional kernel of A_2B_1 as living in X^2. Lifting to X_b^2, we have

$$c(r,y,s,z) = \int \mathrm{e}^{i(s-u,v-y+z)\cdot(\tau,\xi)}\, a(\frac{r(1+\mathrm{e}^{-u})}{1+\mathrm{e}^{-s}}, y, u, y-v)$$
$$\times\, b(\frac{r(\mathrm{e}^{-u}+\mathrm{e}^{-s})}{1+\mathrm{e}^{-s}}, v, \tau, \xi)\ \mathrm{d}u\mathrm{d}v\text{đ}\tau\text{đ}\xi.$$

Thus,

$$\begin{aligned}
\mathcal{N}(A_2B_1)(\tau) &= \int \mathrm{e}^{-is\tau}\, c(0,y,s,y-y')\ \mathrm{d}s\cdot|\,\mathrm{d}y'| \\
&= \int \mathrm{e}^{-is\tau}\, \mathrm{e}^{i(s-u,v-y')\cdot(\iota,\xi)}\, a(0,y,u,y-v)b(0,v,\iota,\xi)\ \mathrm{d}u\mathrm{d}v\text{đ}\iota\text{đ}\xi\mathrm{d}s\cdot|\,\mathrm{d}y'| \\
&= \int \mathrm{e}^{-is\tau+is\iota}\, \mathrm{e}^{-iu\iota}\, a(0,y,u,y-v)\, \mathrm{e}^{i(v-y')\cdot\xi}\, b(0,v,\iota,\xi)\ \mathrm{d}u\text{đ}\iota\mathrm{d}s\mathrm{d}v\text{đ}\xi\cdot|\,\mathrm{d}y'| \\
&= \int \mathrm{e}^{-iu\iota}\, a(0,y,u,y-v)\, \mathrm{e}^{i(v-y')\cdot\xi}\, b(0,v,\iota,\xi)\ \mathrm{d}u\mathrm{d}v\text{đ}\xi\cdot|\,\mathrm{d}y'|\,.
\end{aligned}$$

Recall that

$$\begin{aligned}
\mathcal{N}(A_2)(\tau) &= \int \mathrm{e}^{-is\tau}\, a(0,y,s,y-y')\ \mathrm{d}s\cdot|\,\mathrm{d}y'|\,, \\
\mathcal{N}(B_1)(\tau) &= \int \mathrm{e}^{i(y-y')\cdot\xi}\, b(0,y,\tau,\xi)\,\text{đ}\xi\cdot|\,\mathrm{d}y'|\,,
\end{aligned}$$

thus,

$$\begin{aligned}
\mathcal{N}(A_2)(\tau)\circ\mathcal{N}(B_1)(\tau) &= \int \mathrm{e}^{-is\tau}\, a(0,y,s,y-v)\, \mathrm{e}^{i(v-y')\cdot\xi}\, b(0,v,\tau,\xi)\ \mathrm{d}s\mathrm{d}v\text{đ}\xi\cdot|\,\mathrm{d}y'| \\
&= \mathcal{N}(A_2B_1)(\tau),
\end{aligned}$$

and the proof is completed. □

Example 1.5.3. If $P \in \mathrm{Diff}_b^m(X)$, then $\widehat{P}(\tau) \in \mathrm{Diff}^m(Y)$. In particular, if over

$\mathcal{U} = [0,1)_x \times \mathcal{V}_y$, $P\big|_{\mathcal{U}} = \sum_{|(\alpha,\beta)|\leqslant m} a_{\alpha\beta}(x,y)(x\partial_x)^\alpha \partial_y^\beta$, then

$$\widehat{P}(\tau)\big|_{\mathcal{V}} = \sum_{|(\alpha,\beta)|\leqslant m} a_{\alpha\beta}(0,y)(i\tau)^\alpha \partial_y^\beta.$$

Another feature shared by both principal symbol (maps) and normal operator (maps) is surjectivity. We first classify the range of the normal operator maps.

Definition 1.5.4. A function $a(y,\tau,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ is in $\widehat{S}^m(\mathbb{R}^n;\mathbb{R}^n)$ if given any multi-indexes α,β and $k \in \mathbb{N}$, a satisfies the following symbolic estimate

$$\sup \left|(1+|\tau|+|\eta|)^{|\beta|-m}(1+|y|)^k \partial_y^\alpha (\partial_\tau \partial_\eta)^\beta a(y,\tau,\eta)\right| < \infty.$$

Define $\widehat{S}^{-\infty}(\mathbb{R}^n;\mathbb{R}^n) := \bigcap_{m\in\mathbb{R}} \widehat{S}^m(\mathbb{R}^n;\mathbb{R}^n)$.

In particular, if $a \in \widehat{S}^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$, then a is Schwartz in τ. (In fact, a is Schwartz in all variables.)

Recall that symbols in $S^m(\mathbb{R}^n;\mathbb{R}^n)$ induce smooth family of pseudodifferential operators on $\mathcal{S}(\mathbb{R}^n)$, the space of Schwartz functions on $\mathbb{R}^n$, via the formula

$$A(\tau)\varphi = \int \mathrm{e}^{i(y-y')\xi}\, a(y,\tau,\xi)\varphi(y')\ \mathrm{d}y'đ\xi, \tag{1.5.5}$$

where $a \in \widehat{S}^m(\mathbb{R}^n;\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. The collection of families of operators with symbols in $S^m(\mathbb{R}^n;\mathbb{R}^n)$ is denoted by $\widehat{\Psi}^m(\mathbb{R}^n)$. We review the derivation of the composition formula of pseudodifferential operators. Let $A(\tau) \in \widehat{\Psi}^m(\mathbb{R}^n)$ and

$B(\tau) \in \widehat{\Psi}^{m'}(\mathbb{R}^n)$, then

$$\begin{aligned}(A(\tau)B(\tau)\varphi)(y) &= \int \mathrm{e}^{i(y-z)\cdot\eta}\, a(y,\tau,\eta)\, \mathrm{e}^{i(z-y')\cdot\xi}\, b(z,\tau,\xi)\varphi(y')\ \mathrm{d}y'\text{đ}\xi\mathrm{d}z\text{đ}\eta \\ &= \int \mathrm{e}^{iy\cdot\eta - iy'\cdot\xi}\, \mathrm{e}^{-iz\cdot(\eta-\xi)}\, a(y,\tau,\eta)b(z,\tau,\xi)\varphi(y')\ \mathrm{d}z\text{đ}\eta\mathrm{d}y'\text{đ}\xi \\ &= \int \mathrm{e}^{iy\cdot(\eta+\xi) - iy'\cdot\xi}\, \mathrm{e}^{-iz\cdot\eta}\, a(y,\tau,\eta+\xi)b(z,\tau,\xi)\varphi(y')\ \mathrm{d}z\text{đ}\eta\mathrm{d}y'\text{đ}\xi \\ &= \int \mathrm{e}^{i(y-y')\xi}\left(\int \mathrm{e}^{iy\eta}\, a(y,\tau,\eta+\xi)b(\widehat{\eta},\tau,\xi)\,\text{đ}\eta\right)\varphi(y')\ \mathrm{d}y'\text{đ}\xi,\end{aligned} \tag{1.5.6}$$

where $b(\widehat{\eta},\tau,\xi) = \int \mathrm{e}^{-iz\cdot\eta}\, b(z,\tau,\xi)\ \mathrm{d}z$. Note that $b(\widehat{\eta},\tau,\xi) \in \widehat{S}^{m'}(\mathbb{R}^n;\mathbb{R}^n)$

Lemma 1.5.5. *If $a \in \widehat{S}^m(\mathbb{R}^n;\mathbb{R}^n)$, $b \in \widehat{S}^{m'}(\mathbb{R}^n;\mathbb{R}^n)$, then*

$$c(y,\tau,\xi) := \int \mathrm{e}^{iy\eta}\, a(y,\tau,\eta+\xi)b(\widehat{\eta},\tau,\xi)\,\text{đ}\eta$$

is in $\widehat{S}^{m+m'}(\mathbb{R}^n;\mathbb{R}^n)$.

Proof. Just observe that

$$\begin{aligned}&\left|\partial_y^\alpha \partial_\tau^\beta \partial_\xi^\gamma c(y,\tau,\xi)\right| \\ =&\left|\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta\\ \gamma_1+\gamma_2=\gamma}} C_{\alpha_1,\beta_1,\gamma_1} \int \mathrm{e}^{iy\eta}\, \eta^{\alpha_1} \partial_y^{\alpha_2}\partial_\tau^{\beta_1}\partial_\xi^{\gamma_1} a(y,\tau,\eta+\xi)\partial_\tau^{\beta_2}\partial_\xi^{\gamma_2} b(\widehat{\eta},\tau,\xi)\,\text{đ}\eta\right| \\ \leqslant& C \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2=\beta\\ \gamma_1+\gamma_2=\gamma}} \int \frac{(1+|\tau|)^{m-|\beta_1|+m'-|\beta_2|}(1+|\eta+\xi|)^{m-|\gamma_1|}(1+|\xi|)^{m'-|\gamma_2|}\text{đ}\eta}{(1+|\eta|)^{-|\alpha|}(1+|y|)^\ell(1+|\eta|)^{2n+|\alpha|+|(m-|\gamma_1|)|}} \\ \leqslant& D \int \frac{(1+|\tau|)^{m+m'-|\beta|}(1+|\xi|)^{m+m'-|\gamma|}(1+|\eta|)^{|(m-|\gamma_1|)|}\,\text{đ}\eta}{(1+|y|)^\ell(1+|\eta|)^{2n+|(m-|\gamma_1|)|}} \\ \leqslant& D \int (1+|\eta|)^{2n}\,\text{đ}\eta \cdot \frac{(1+|\tau|)^{m+m'-|\beta|}(1+|\xi|)^{m+m'-|\gamma|}}{(1+|y|)^\ell},\end{aligned}$$

for any indexes α,β,γ and $\ell \in \mathbb{N}$. □

A parameter-dependent pseudodifferential operator $R(\tau)$ is said to be in $\widehat{\Psi}^{-\infty}(Y)$, where Y is a closed manifold, if $R(\tau)$ is locally defined by (left) symbols in $\widehat{S}^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$. Observe that the (kernels of) operators in $\widehat{\Psi}^{-\infty}(Y)$ can be identified with density sections in $C^\infty(Y^2 \times \mathbb{R}_\tau, \Omega_R)$ that are Schwartz in τ.

Definition 1.5.6. A parameter-dependent operator $\mathcal{A}(\tau)$ is said to be in $\widehat{\Psi}^m(Y)$ if the Schwartz kernel $K_{\mathcal{A}(\tau)}$ satisfies

1. for any $\psi \in C_c^\infty(Y^2 \backslash \Delta)$, $\psi K_{\mathcal{A}(\tau)} \in \widehat{\Psi}^{-\infty}(Y)$;
2. if $\mathcal{U} \cong \mathbb{R}_y^n \times \mathbb{R}_z^n$ is a coordinate patch of Y^2 such that $\mathcal{U} \cap \Delta \cong \mathbb{R}^n \times \{0\}$, $\psi \in C_c^\infty(\mathcal{U})$ and ν' a local trivialization of Ω_R, then

   $$\psi K_{\mathcal{A}(\tau)} = \int \mathrm{e}^{iz\xi}\, a(y,\tau,\xi)\, đ\xi \cdot \nu'$$

   for some $a \in \widehat{S}^m(\mathbb{R}^n;\mathbb{R}^n)$.

The following results are extensions of (1.5.5) and (1.5.6), whose proofs are adaptations of the standard approaches in the theory of pseudodifferential operators on closed manifolds.

Proposition 1.5.7. *1. If $A(\tau) \in \widehat{\Psi}^m(Y)$, then*

$$A(\tau) : C^\infty(Y) \longrightarrow C^\infty(Y).$$

2. $\widehat{\Psi}^m(Y) \circ \widehat{\Psi}^{m'}(Y) \subset \widehat{\Psi}^{m+m'}(Y)$.

Clearly if $A \in \Psi_{bl}^m(X)$, then $\widehat{A}(\tau) \in \widehat{\Psi}^m(Y)$. We now demonstrate the passage from parameter-dependent pseudodifferential operators on $Y = \partial X$ to *bl*-pseudodifferential operators on X. Let $\mathcal{A}(\tau) \in \widehat{\Psi}^{-\infty}(Y)$. Recall that $\mathcal{A}(\tau)$ is just a smooth family of right densities on Y^2 that is Schwartz in τ. In particular, the Fourier (inverse) transform with respect to τ is well defined for $\mathcal{A}(\tau)$. Choose a cut-off function $\psi \in C_c^\infty([0,1)_r \times \mathbb{R}_s)$ such that $\psi \equiv 1$ in a neighborhood of $\{r = 0\}$. Define

$$K_A(r,s,y,y') := \psi(r,s) \int \mathrm{e}^{is\tau}\, \mathcal{A}(\tau)(y,y')\, đ\tau = \psi(r,s) a(s,y,y').$$

By Taylor's theorem, ψ can be expanded near $r = 0$ as

$$\psi(r,s) = \psi(0,s) + r\widetilde{\psi}(r,s) = 1 + r\widetilde{\psi}(r,s),$$

where $\widetilde{\psi} \in C_c^\infty([0,1) \times \mathbb{R})$. Consequently,

$$\begin{aligned} K_A(r,s,y,y') &= \left(1 + r\widetilde{\psi}(r,s)\right) a(s,y,y') \\ &= a(s,y,y') + r\widetilde{\psi}(r,s) a(s,y,y') \\ &= (K_A)_0(s,y,y') + (K_A)_1(r,s,y,y'). \end{aligned}$$

Clearly $(K_A)_0(s,y,y')$ is Schwartz in s. Observe that, e.g., by L'Hospital's rule,

$$\lim_{r \to 0} r \cdot (\ln r)^\ell = 0$$

for any $\ell \in \mathbb{N}$, hence $(K_A)_1 \in {}^1S^0_{lb,ff,rb}(X_b^2)$. In a word, we showed that $K_A \in S^0_{bl}(X_b^2) = \Psi_{bl}^{-\infty}(X)$. Note that the restriction of K_A to $ff(X_b^2)$ is just the Fourier inverse transform of $\mathcal{A}(\tau)$ along the parameter. Immediately from the definition of A, we have

$$\widehat{A}(\tau) = \int \mathrm{e}^{-is\tau}\, K_A\big|_{ff}\, \mathrm{d}s = \int \mathrm{e}^{-is\tau} \left(\int \mathrm{e}^{is\kappa}\, \mathcal{A}(\kappa)\, đ\kappa \right) \mathrm{d}s = \mathcal{A}(\tau).$$

In general, we have the following result:

Theorem 1.5.8. *The normal operator map $\mathcal{N}(\cdot)(\tau) : \Psi^m_{bl}(X) \to \widehat{\Psi}^m(Y)$ is surjective. Moreover, $R \in \Psi^{-\infty}_{bl}(X) \cap \mathrm{null}(\mathcal{N}(\cdot)(\tau))$ if and only if $K_R\big|_{ff} \equiv 0$.*

1.6 Full ellipticity and Fredholm property

We will first construct a full parametrix of a fully elliptic operator in $\Psi^m_{bl}(X)$ with the help of normal operators. Fredholm property will be studied afterwards. Recall that $Y = \partial X$. Let ν be a global trivialization of $\Omega(Y)$, and ν' the lift of ν to $\Omega_R(Y^2)$. Fix a global trivialization μ of $\Omega_b(X)$ such that in a collar $\mathcal{C} \cong [0,1)_x \times Y$ near the boundary, $\mu\big|_{\mathcal{C}} = \left|\frac{\mathrm{d}x}{x}\right| \otimes \nu$. Denote the lift of μ to $\Omega_{b,R}(X_b^2)$ by μ'.

1.6.1 Construction of full parametrices

Definition 1.6.1. $A \in \Psi^m_{bl}(X)$ is called *fully elliptic* if A is elliptic and $\widehat{A}(\tau)^{-1}$ exists for all $\tau \in \mathbb{R}$.

We will rely heavily on a special technique developed by Paul Loya, called *finite-rank-operator method*, to construct most of the arguments in this section. See [19], or Appendix C for a fairly detailed user guide. To familiarize the readers with the finite-rank-operator technique, we retrieve a well-known fact first.

Proposition 1.6.2. *Let Y be a closed manifold, and ν a global trivialization of $\Omega(Y)$. If $A \in \Psi^m(Y)$ is elliptic and invertible, then $A^{-1} \in \Psi^{-m}(Y)$.*

Proof. Recall that there exists a parametrix $B \in \Psi^{-m}(Y)$ of A, such that

$$AB = \mathrm{Id} + F,$$

where $F \in \Psi^{-\infty}(Y)$ is of $C^\infty(Y)$-finite rank. Note that

$$A^{-1} = B - A^{-1}F.$$

Assume that $F = \sum_j f_j \otimes g_j \cdot \nu'$, where ν' is the lift of ν to $\Omega_R(Y^2)$, then given any

$\varphi \in C^\infty(Y)$, we have

$$(A^{-1}F\varphi)(y) = \sum_j (A^{-1}f_j)(y) \int g_j(y')\varphi(y')\nu(y').$$

Since A^{-1} maps $C^\infty(Y)$ to $C^\infty(Y)$, $A^{-1}F$ is in fact a $C^\infty(Y)$-finite rank operator, hence $A^{-1}F \in \Psi^{-\infty}(Y)$. Consequently, $A^{-1} = B - A^{-1}F \in \Psi^{-m}(Y)$. □

Note that the result above is valid no matter whether Y is the boundary of a manifold or not. An immediate consequence is that if A is fully elliptic, then $\widehat{A}(\tau)^{-1} \in \Psi^{-m}(Y)$ for every τ. In fact, we will demonstrate that a much stronger result holds. We now introduce our replacement of the analytic Fredholm theory.

Lemma 1.6.3. *Let $I \subset \mathbb{R}$ be an open interval and $F(\tau) \in \Psi^{-\infty}(Y)$ depending on $\tau \in I$ smoothly. Assume that*

$$F(\tau) = \sum_{j=1}^{N} \varphi_j(y)\psi_j(y', \tau) \cdot \nu' \tag{1.6.1}$$

for some smooth functions φ_j, ψ_j, and that $(\mathrm{Id} - F(\tau))^{-1}$ exists for every $\tau \in I$, then $(\mathrm{Id} - F(\tau))^{-1} = \mathrm{Id} + S(\tau)$ with some $S(\tau) \in \Psi^{-\infty}(Y)$ depending smoothly on $\tau \in I$.

Proof. Applying Gram-Schmidt process to rewrite (1.6.1) if necessary, we henceforth assume that $\{\varphi_j\}$ is orthonormal with respect to the L^2-inner product against ν. Let $V = \mathrm{span}(\{\bar{\varphi}_j\})$, where $\bar{\varphi}_j$ denotes the complex conjugate of φ_j. Let π_V stand for the orthogonal projection onto V. Then we have

$$\pi_V \psi_j(y, \tau) = \sum_k \langle\, \psi_j(\tau), \bar{\varphi}_k \,\rangle\, \bar{\varphi}_k = \sum_k a_{jk}(\tau)\bar{\varphi}_k. \tag{1.6.2}$$

$a_{jk}(\tau)$ is smooth in τ for all i, j, hence so is $\pi_V \psi_i^S(\tau)$. Thus we can write

$$F(\tau) = \sum_{j,k} a_{jk}(\tau)\varphi_j \otimes \bar{\varphi}_k + \sum_j \varphi_j \otimes (\mathrm{Id} - \pi_V)\psi_j(\tau) = A(\tau) + B(\tau).$$

Now let $W = \operatorname{span}(\{\varphi_i\})$. Note that the images of $F(\tau)$, $A(\tau)$ and $B(\tau)$ are all contained in W for any τ. Given any $u \in C^\infty(Y)$, it can be uniquely written as $u = \pi_W u + (\mathrm{Id} - \pi_W)u = u' + u''$, where $u' \in W$ and $u'' \in W^\perp$, then consequently we have

$$F(\tau) = A(\tau)u' + B(\tau)u''.$$

In other words, with respect to the decomposition $C^\infty(Y) = W \oplus W^\perp$, we can write

$$F(\tau) = \begin{bmatrix} A(\tau) & B(\tau) \\ O & O \end{bmatrix}$$

and thus

$$(\mathrm{Id} - F(\tau)) = \begin{bmatrix} \mathrm{Id}_W - A(\tau) & B(\tau) \\ O & \mathrm{Id}_{W^\perp} \end{bmatrix}.$$

Henceforth we will continue to denote the restriction of $A(\tau)$, $B(\tau)$ and Id on W by the same names. Therefore, $(\mathrm{Id} - F(\tau))^{-1}$ exists if and only if $(\mathrm{Id} - A(\tau))^{-1}$ exists, in which case,

$$(\mathrm{Id} - F(\tau))^{-1} = \begin{bmatrix} (\mathrm{Id} - A(\tau))^{-1} & -(\mathrm{Id} - A(\tau))^{-1}B(\tau) \\ O & \mathrm{Id} \end{bmatrix}. \tag{1.6.3}$$

Since $\mathrm{Id} - A(\tau)$ is a linear operator on a finite dimensional vector space, the inverse of $\mathrm{Id} - A(\tau)$, if exists, is just an algebraic expression of its entries, hence also smooth in τ. Therefore, the claimed form of $(\mathrm{Id} - F(\tau))^{-1}$ and the smoothness in τ follows from (1.6.3). $\square$

Lemma 1.6.4. *If $A \in \Psi^m_{bl}(X)$ is fully elliptic, then $\widehat{A}(\tau)^{-1} \in \Psi^{-m}(Y)$ depends smoothly on τ.*

Proof. Assume that A is supported near $\mathit{ff}(X^2_b)$. By the ellipticity, there exist $B \in$

$\Psi_{bl}^{-m}(X)$ and $R \in \Psi_{bl}^{-\infty}(X)$, both supported near $\mathit{ff}(X_b^2)$, such that

$$AB = \mathrm{Id} - R.$$

Near the front face of X_b^2, with the coordinates $(\rho, \omega, y, y') = (x + x', \ln(\frac{x'}{x}), y, y')$, we have $X_b^2 \backslash (rb \cup lb) \cong [0,1)_\rho \times \mathbb{R}_\omega \times Y^2$, and $R = R(\rho, \omega, y, y') \cdot \mu'$ Schwartz in ω and vanishing identically in $\{\rho > \epsilon\}$ for some $\epsilon > 0$. By Stone-Weierstrass theorem on locally compact Hausdorff space, there exists

$$G(\rho, \omega, y, y') = \sum_{j=1}^{N} \varphi_j(y) \psi_j(\rho, \omega, y')$$

with $\varphi_j \in C^\infty(Y)$, $\psi_j \in C^\infty([0,1) \times \mathbb{R} \times Y)$ Schwartz in ω and vanishing in $\{\rho > \epsilon\}$, such that $S(\rho, \omega, y, y') := (R - G)(\rho, \omega, y, y')$ satisfies

$$\int \left| S(\frac{(1 + \mathrm{e}^\gamma)\rho}{1 + \mathrm{e}^\omega}, \gamma, y, y') \right| \, \mathrm{d}\gamma \nu(y') < \delta < 1.$$

Observe that

$$\begin{aligned} \left| S^2(\rho, \omega, y, y') \right| &:= \left| \int S(\frac{(1 + \mathrm{e}^\gamma)\rho}{1 + \mathrm{e}^\omega}, \gamma, y, y'') S(\frac{(\mathrm{e}^\omega + \mathrm{e}^\gamma)\rho}{1 + \mathrm{e}^\omega}, \omega - \gamma, y'', y') \, \mathrm{d}\gamma \nu(y'') \right| \\ &\leqslant \|S\|_\infty \int \left| S(\frac{(1 + \mathrm{e}^\gamma)\rho}{1 + \mathrm{e}^\omega}, \gamma, y, y'') \right| \, \mathrm{d}\gamma \nu(y'') \\ &< \delta \, \|S\|_\infty \, . \end{aligned}$$

In general, we have

$$\begin{aligned} \left| S^{k+1}(\rho, \omega, y, y') \right| &:= \int S(\frac{(1 + \mathrm{e}^\gamma)\rho}{1 + \mathrm{e}^\omega}, \gamma, y, y'') S^k(\frac{(\mathrm{e}^\omega + \mathrm{e}^\gamma)\rho}{1 + \mathrm{e}^\omega}, \omega - \gamma, y'', y') \, \mathrm{d}\gamma \nu(y'') \\ &\leqslant \left\| S^k \right\|_\infty \int \left| S(\frac{(1 + \mathrm{e}^\gamma)\rho}{1 + \mathrm{e}^\omega}, \gamma, y, y'') \right| \, \mathrm{d}\gamma \nu(y'') \\ &< \delta^k \, \|S\|_\infty \, . \end{aligned}$$

Hence, according to (1.5.2), we have

$$(\mathrm{Id} - S)^{-1} = \mathrm{Id} + \sum_{k \geqslant 1} S^k = 1 + Q,$$

and $Q(\rho, \omega, y, y')$ is continuous in all variables. Note that

$$\begin{aligned}
&Q(\rho, \omega, y, y') \\
=&S(\rho, \omega, y, y') + S^2(\rho, \omega, y, y') + (SQS)(\rho, \omega, y, y') \\
=&S(\rho, \omega, y, y') + \int S(\frac{(1 + \mathrm{e}^{\gamma})\rho}{1 + \mathrm{e}^{\omega}}, \gamma, y, y'')S(\frac{(\mathrm{e}^{\omega} + \mathrm{e}^{\gamma})\rho}{1 + \mathrm{e}^{\omega}}, \omega - \gamma, y'', y') \, \mathrm{d}\gamma\nu(y'') \\
&\quad + \int S(\frac{(1 + \mathrm{e}^{\gamma})\rho}{1 + \mathrm{e}^{\omega}}, \gamma, y, y'')Q(\frac{\mathrm{e}^{\gamma}(1 + \mathrm{e}^{\iota})\rho}{1 + \mathrm{e}^{\omega}}, \iota, y'', y''') \\
&\qquad \times S(\frac{\mathrm{e}^{\gamma}(\mathrm{e}^{\omega - \gamma} + \mathrm{e}^{\iota})\rho}{1 + \mathrm{e}^{\omega}}, \omega - \gamma - \iota, y''', y') \, \mathrm{d}\iota\nu(y''')\mathrm{d}\gamma\nu(y'').
\end{aligned}$$

Thus, $Q(0, \omega, y, y')$ is smooth in all variables and Schwartz in ω. Denote the Fourier transform of $Q\big|_{\mathit{ff}}$ along ω by $\widehat{Q}(\tau)$, then $\widehat{Q}(\tau)$ is a smooth family in $\Psi^{-\infty}(Y)$. Let $B_0 = B(1 + Q)$, $H = G + GQ$, then $AB_0 = \mathrm{Id} - H$, and

$$\widehat{A}(\tau)\widehat{B_0}(\tau) = \mathrm{Id} - \widehat{H}(\tau).$$

Note that $\widehat{H}(\tau)$ is in the form of (1.6.1). Fix an arbitrary point $\tau_0 \in \mathbb{R}$, and let $\widehat{B_{\tau_0}}(\tau) = \widehat{B_0}(\tau) + \widehat{A}(\tau_0)^{-1}\widehat{H}(\tau)$, then

$$\mathrm{Id} - F(\tau) := \mathrm{Id} - (\widehat{H}(\tau) - \widehat{A}(\tau)\widehat{A}(\tau_0)^{-1}\widehat{H}(\tau)) = \widehat{A}(\tau)\widehat{B_{\tau_0}}(\tau),$$

hence $\mathrm{Id} - F(\tau_0) = \mathrm{Id}$ is trivially invertible. Since $F(\tau)$ is smooth in τ and satisfies (1.6.1), by Lemma 1.6.3, there is an open interval $I \subset \mathbb{R}$ over which $\mathrm{Id} - F(\tau)$ is invertible, and $\mathrm{Id} + T(\tau) := (\mathrm{Id} - F(\tau))^{-1}$ is smooth in $\tau \in I$, hence so is $\widehat{A}(\tau)^{-1} = \widehat{B_{\tau_0}}(\tau)(\mathrm{Id} + T(\tau))$. Since τ_0 is arbitrary, $\widehat{A}(\tau)^{-1}$ is smooth in $\tau \in \mathbb{R}$. □

Lemma 1.6.5. *If $A \in \Psi^m_{bl}(X)$ is fully elliptic, then $\widehat{A}(\tau)^{-1} \in \widehat{\Psi}^{-m}(Y)$.*

Proof. Recall that there exist $\widehat{B}(\tau) \in \widehat{\Psi}^{-m}(Y)$ and $\widehat{R}(\tau) \in \widehat{\Psi}^{-\infty}(Y)$ such that

$$\widehat{A}(\tau)\widehat{B}(\tau) = \mathrm{Id} - \widehat{R}(\tau).$$

Since $\widehat{R}(\tau)$ is Schwartz in τ, for sufficiently large τ, $(\mathrm{Id} - \widehat{R}(\tau))^{-1}$ exists and

$$(\mathrm{Id} - \widehat{R}(\tau))^{-1} = \mathrm{Id} + T(\tau),$$

where

$$T(\tau) = \sum_{k \geqslant 1} \widehat{R}(\tau)^k.$$

Consequently, we have

$$\widehat{A}(\tau)^{-1} = \widehat{B}(\tau) + \widehat{B}(\tau)T(\tau)$$

for sufficiently large τ. Since $\widehat{A}(\tau)^{-1}$ is smooth in τ, $\widehat{B}(\tau)T(\tau) \in \Psi^{-\infty}(Y)$ is smooth in τ wherever it is defined. Even though $T(\tau)$ is not necessarily smooth in τ, we will show that $T(\tau)$ maintains some fairly rapid decay at infinity.

Denote $\int_Y \nu$ by $\mathrm{vol}_\nu(Y)$. Given any $q \in \mathbb{N}$, let $C_q > 0$ be a constant such that

$$\left|\partial_\tau^j \widehat{R}(\tau)\right| < C_q(1 + |\tau|)^{-1}, \ j \leqslant q.$$

Note that

$$\left|\partial_\tau^q \widehat{R}(\tau)^k\right| \leqslant k^q \,\mathrm{vol}_\nu(Y)^{k-1} C_q^k (1 + |\tau|)^{-k}.$$

Hence, when $|\tau| > 2\,\mathrm{vol}_\nu(Y) C_q$, we have

$$\sum_{k \geqslant 1} \left|\partial_\tau^q \widehat{R}(\tau)^k\right| \leqslant \mathrm{vol}_\nu(Y)^{-1} \sum_{k \geqslant 1} \frac{k^q}{2^k} < \infty.$$

Therefore, when τ is sufficiently large, $\partial_\tau^q T(\tau) = \sum_{k \geqslant 1} \partial_\tau^q \widehat{R}(\tau)^k$ exists and uniformly bounded. Now recall that

$$\begin{aligned} T(\tau)(y,y') = {} & \widehat{R}(\tau)(y,y') + \int \widehat{R}(\tau)(y,y_1)\widehat{R}(\tau)(y_1,y')\nu(y_1) \\ & + \int \widehat{R}(\tau)(y,y_1)T(\tau)(y_1,y_2)\widehat{R}(\tau)(y_2,y')\nu(y_1)\nu(y_2), \end{aligned}$$

hence for any index α,

$$\begin{aligned} \partial_y^\delta \partial_{y'}^\epsilon \partial_\tau^\alpha T(\tau)(y,y') = {} & \partial_y^\delta \partial_{y'}^\epsilon \partial_\tau^\alpha \widehat{R}(\tau)(y,y') \\ & + \sum_{\beta_1+\beta_2=\alpha} C_{\beta_1} \int \partial_y^\delta \partial_\tau^{\beta_1} \widehat{R}(\tau)(y,y_1) \partial_\tau^{\beta_2} \partial_{y'}^\epsilon \widehat{R}(\tau)(y_1,y')\nu(y_1) \\ & + \sum_{\gamma_1+\gamma_2+\gamma_3=\alpha} C_{\gamma_1,\gamma_2} \int \partial_y^\delta \partial_\tau^{\gamma_1} \widehat{R}(\tau)(y,y_1) \partial_\tau^{\gamma_2} T(\tau)(y_1,y_2) \\ & \qquad \cdot \partial_\tau^{\gamma_3} \partial_{y'}^\epsilon \widehat{R}(\tau)(y_2,y')\nu(y_1)\nu(y_2). \end{aligned}$$

Consequently,

$$\left| \partial_y^\delta \partial_{y'}^\epsilon \partial_\tau^\alpha T(\tau)(y,y') \right| \leqslant C_{\alpha\delta\epsilon}^\ell (1+|\tau|)^{-\ell}$$

for some constant $C_{\alpha\delta\epsilon}^\ell$, and thus,

$$\left| \partial_y^\delta \partial_{y'}^\epsilon \partial_\tau^\alpha \left(\widehat{B}(\tau) T(\tau)(y,y') \right) \right| \leqslant \widetilde{C}_{\alpha\delta\epsilon}^\ell (1+|\tau|)^{-\ell} \tag{1.6.4}$$

with some constant $\widetilde{C}_{\alpha\delta\epsilon}^\ell$. Therefore, near the diagonal of Y^2, we have

$$\widehat{A}(\tau)^{-1} = \int e^{iz\xi}\, a(y,\tau,\xi)\, đ\xi,$$

such that when τ is sufficiently large, $a = b + r$ where $b \in \widehat{S}^{-m}(\mathbb{R}^{n-1};\mathbb{R}^{n-1})$, and $r(y,\tau,\xi)$ satisfies the symbolic estimate at where it is defined, hence a is also in $\widehat{S}^{-m}(\mathbb{R}^{n-1};\mathbb{R}^{n-1})$. Similarly, away from the diagonal, $\widehat{A}(\tau)^{-1} \in \widehat{\Psi}^{-\infty}(Y)$. In conclusion, we have $\widehat{A}(\tau)^{-1} \in \widehat{\Psi}^{-m}(Y)$. $\square$

Proposition 1.6.6. *If $A \in \Psi_{bl}^{m}(X)$ is fully elliptic, then there exists a $B \in \Psi_{bl}^{-m}(X)$ such that $AB = \mathrm{Id} - R$, where $R \in \Psi_{bl}^{-\infty}(X)$ with $\widehat{R}(\tau) = 0$. In particular, $R \in {}^{1}S^{0}_{lb,ff,rb}(X_b^2)$.*

Proof. Since A is elliptic, there exists a B' such that

$$AB' = \mathrm{Id} - R',$$

where $R' \in \Psi_{bl}^{-\infty}(X)$. On the other hand, since $\widehat{A}(\tau)^{-1} \in \widehat{\Psi}^{-m}(Y)$, by Theorem 1.5.8, there exists a $C \in \Psi_{bl}^{-m}(X)$ such that $\widehat{C}(\tau) = \widehat{A}(\tau)^{-1}$. Now define

$$B := B' + CR'.$$

Note that $B \in \Psi_{bl}^{-m}(X)$. Then

$$\begin{aligned}\widehat{A}(\tau)\widehat{B}(\tau) &= \widehat{A}(\tau)(\widehat{B'}(\tau) + \widehat{C}(\tau)\widehat{R}'(\tau)) \\ &= \mathrm{Id} - \widehat{R}'(\tau) + \widehat{A}(\tau)\widehat{A}(\tau)^{-1}\widehat{R}'(\tau) \\ &= \mathrm{Id},\end{aligned}$$

which implies that if $R = R'(\mathrm{Id} - AC)$, then $\widehat{R}(\tau) = 0$.

Write $R(r,s,y,y') = R_0(s,y,y') + R_1(r,s,y,y')$, then $R_0(s,y,y') = R\big|_{ff}(s,y,y') = 0$. Therefore, we conclude that $R = R_1 \in {}^{1}S^{0}_{lb,ff,rb}(X_b^2)$. □

The operator B in Proposition 1.6.6 is called a *full parametrix* of A.

1.6.2 Fredholm

For an arbitrary manifold X with boundary, denote the collection of continuous functions that vanish at ∂X by $C_0(X)$. Clearly ${}^{1}S^{0}_{\partial X}(X) \subset C_0(X)$. Note that if $\widehat{R}(\tau) \equiv 0$, then $R \in C_0(X_b^2)$. In this case, R can be identified with a function, still denoted by R, in $C_0(X^2)$. Let $\mathscr{F} = {}^1S^0_{\partial X}(X) \otimes {}^1S^0_{\partial X}(X)$. Recall that by Proposition 1.1.10, $\mathscr{F} \subset {}^1S^0_{lb,rb}(X^2) \subset {}^1S^0_{lb,rb}(X^2_b) \cap C_0(X^2_b)$.

Proposition 1.6.7. *If $\mathscr{A} \subset C_0(X)$ is a subalgebra containing $C_c^\infty(X\backslash\partial X)$ and closed under complex conjugation, then $\mathscr{A} \otimes \mathscr{A}$ is dense in $C_0(X^2)$ in the uniform convergence topology.*

Proof. The idea is to apply Stone-Weierstrass theorem on the quotient space $\widetilde{X^2} = X^2/\partial X^2$.

Given any $p = (p_1, p_2), q = (q_1, q_2) \in X^2\backslash\partial X^2$, there exist bump functions $u_j \in C_c^\infty(X\backslash(\partial X \cup q_j))$, $j = 1, 2$, with $u_j(p_j) = 1$, and consequently

$$u_1 \otimes u_2(p) = u_1(p_1)u_2(p_2) = 1 \neq 0 = u_1(q_1)u_2(q_2) = u_1 \otimes u_2(q). \tag{1.6.5}$$

Also, given any $q' = (q'_1, q'_2) \in \partial X^2$, since either q'_1 or q'_2 is in ∂X,

$$u_1 \otimes u_2(p) = 1 \neq 0 = u_1 \otimes u_2(q'). \tag{1.6.6}$$

Now pass onto the quotient space $\widetilde{X^2}$. Note that $C_0(X^2)$ determines a subset $\widetilde{C}_0$ of $C(\widetilde{X^2})$, since elements in $C_0(X^2)$ have identical values at ∂X^2. In particular,

$$\widetilde{C}_0 = \{f \in C(\widetilde{X^2}) \mid f([\partial X^2]) = 0\}.$$

Similarly, $\mathscr{A} \otimes \mathscr{A}$ determines $\widetilde{\mathscr{A} \otimes \mathscr{A}} \subset \widetilde{C}_0$. Clearly $\widetilde{\mathscr{A} \otimes \mathscr{A}}$ is a subalgebra that is closed under complex conjugation, since $\mathscr{A} \otimes \mathscr{A}$ satisfies the same property. Moreover, (1.6.5) and (1.6.6) together implies that $\widetilde{\mathscr{A} \otimes \mathscr{A}}$ separates points in $\widetilde{X^2}$, since $\mathscr{A}$ contains $C_c^\infty(X^2\backslash\partial X^2)$. Therefore, by Stone-Weierstrass theorem, $\widetilde{\mathscr{A} \otimes \mathscr{A}}$ is dense in $\widetilde{C}_0$ in the uniform convergence topology.

Pulling back to the original space X^2, we conclude that $\mathscr{A} \otimes \mathscr{A}$ is dense in $C_0(X^2)$. $\square$

Corollary 1.6.7.1. *$\mathscr{F}$ is dense in $C_0(X^2)$ in the uniform convergence topology.*

Proof. Clearly ${}^1S^0_{\partial X}(X)$ is closed under complex conjugation and contains $C_c^\infty(X\backslash\partial X)$. To see that ${}^1S^0_{\partial X}(X)$ is a subalgebra, one can employ an obvious adaptation of the argument to Lemma 1.2.3. □

Lemma 1.6.8. *If $R \in {}^1S^0_{lb,rb}(X^2)$, then there exists an $F \in \mathscr{F}$ such that*

$$\sup \int |R - F|\, \mu' < 1.$$

Proof. Let $R' = (1 + |\ln x'|^2)R$. Then R' is also in $C_0(X^2)$. Thus there is an $F' \in \mathscr{F}$ such that

$$\|R' - F'\|_\infty < \left(\int \frac{1}{1 + |\ln x|^2}\mu\right)^{-1}.$$

Note that $F := (1 + |\ln x'|)^{-2}F' \in \mathscr{F}$. Hence

$$\begin{aligned}
\int |R - F|\, \mu' &= \int \frac{|(1 + |\ln x'|)^2 R - (1 + |\ln x'|)^2 F|}{1 + |\ln x'|^2}\mu' \\
&\leqslant \|R' - F'\|_\infty \int \frac{1}{1 + |\ln x'|^2}\mu' \\
&\leqslant \delta < 1.
\end{aligned}$$

□

Lemma 1.6.9. *1. ${}^1S^0_{lb,rb}(X^2) \circ C(X^2) \circ {}^1S^0_{lb,rb}(X^2) \subset {}^1S^0_{lb,rb}(X^2)$.*

2. If $Q \in {}^1S^0_{lb,rb}(X^2)$ and $f \in {}^1S^0_{\partial X}(X)$, then

$$\int f(x)Q(x,x')\mu(x), \int Q(x,x')f(x')\mu(x') \in {}^1S^0_{\partial X}(X).$$

Proof. 1. Let $u, v \in {}^1S^0_{lb,rb}(X^2)$ and $f \in C(X^2)$. Given any indexes α, β and $\ell \in \mathbb{N}$, we have

$$\begin{aligned}
&\left|\int (x\partial_x)^\alpha u(x,x_1)f(x_1,x_2)(x'\partial_{x'})^\beta v(x_2,x')\mu(x_1)\mu(x_2)\right| \\
\leqslant &\int \left|(x\partial_x)^\alpha u(x,x_1)f(x_1,x_2)(x'\partial_{x'})^\beta v(x_2,x')\right| \mu(x_1)\mu(x_2) \\
\leqslant &D^\ell_{\alpha\beta} \left\|f\right\|_\infty [(1+|\ln x|)(1+|\ln x'|)]^{-\ell} \int [(1+|\ln x_1|)(1+|\ln x_2|)]^{-2}\mu(x_1)\mu(x_2) \\
\leqslant &C^\ell_{\alpha\beta}[(1+|\ln x|)(1+|\ln x'|)]^{-\ell}
\end{aligned}$$

for some constant $C^\ell_{\alpha\beta}$, hence $(x\partial_x)^\alpha(x'\partial_{x'})^\beta u\circ f\circ v(x,x')$ exists and

$$\left|(x\partial_x)^\alpha(x'\partial_{x'})^\beta u\circ f\circ v(x,x')\right| \leqslant C^\ell_{\alpha\beta}\left[(1+|\ln x|)(1+|\ln x'|)\right]^{-\ell}.$$

2. Given any index α and $\ell \in \mathbb{N}$, we have

$$\begin{aligned}
\left|\int f(x)(x'\partial'_x)^\alpha Q(x,x')\mu(x)\right| &\leqslant \int \left|f(x)(x'\partial'_x)^\alpha Q(x,x')\right| \mu(x) \\
&\leqslant \int D^\ell_\alpha(1+|\ln x|)^{-2}(1+|\ln x'|)^{-\ell}\mu(x) \\
&\leqslant C^\ell_\alpha(1+|\ln x'|)^{-\ell},
\end{aligned}$$

hence $\left|(x'\partial'_x)^\alpha \int f(x)Q(x,x')\mu(x)\right| \leqslant C^\ell_\alpha(1+|\ln x'|)^{-\ell}$. The other claim is proved identically. $\square$

Theorem 1.6.10. *If $A \in \Psi^m_{bl}(X)$ is fully elliptic, then $A : S^0(X) \to S^0(X)$ is Fredholm.*

Proof. For simplicity, we assume that A is self-adjoint. Let $\Phi = S^0(X)$, $\Psi(X) = \Psi^*_{bl}(X)$, $\Psi^{-\infty}_0(X) = {}^1S^0_{lb,ff,rb}(X^2_b)$ and $\mathcal{B} = {}^1S^0_{\partial X}(X)$. Then the hypotheses in Lemma C.1 are satisfied. Hence, by Corollary C.3.1, A is Fredholm. $\square$

Remark. The general case can be proved by applying Theorem C.4.

Chapter 2

Dirac operators and product-type structures

2.1 Infinite cylindrical end

The non-compact manifolds with a cylindrical end was considered in [1] to give an alternative interpretation of the non-local boundary condition introduced in the same paper. We review the geometric setting in this section.

Let $\widetilde{M} = M_c \coprod_Y M$ be a manifold with a cylindrical end (Figure 2.1), where $M_c = (-\infty, 0]_s \times Y = \mathbb{R}_0^- \times Y$, and M a closed manifold with boundary and $\partial M = Y$. The subscript c stands for "cylinder". We will review a few notions that behave nicely

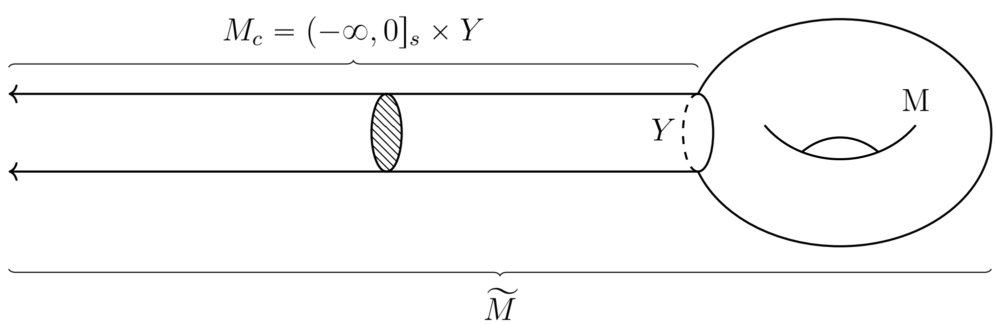


Figure 2.1: Manifold with a Cylindrical End

with the product structure of M_c, and they will be labeled as *product type*.

- Bundle. Denote $\pi_Y : M_c \to Y$ as the projection onto the second factor. Let $E^0 \to Y$ be a $\mathbb{Z}_2$-graded Hermitian bundle. Define $E := \pi_Y^* E^0$, that is, the pullback bundle of E^0 via π_Y. Then E is said to be of product type.

- Metric. Let g_0 be a Riemannian metric on Y, then the metric $g = \mathrm{d}s^2 + g_0$ on M_c is of product type. Note that $\mathrm{d}s^2$ is the natural (Euclidean) metric on $(-\infty, 0]$, and this notion of product metric is classical.

  We discuss the $\hat{A}$-genus of M_c. Let (s, y) be a local coordinate chart at an arbitrary point of M_c. Let $\mathcal{R}$ be the Riemannian curvature operator associated with the product metric g. Under the local frame $\{\partial_s, \partial_y\}$ of TM_c, $\mathcal{R}$ can be represented by an anti-symmetric, matrix-value 2-form, $[\mathcal{R}_{ij}]$. Observe that $\mathcal{R}_{ij} = 0$ if either $i = 1$ or $j = 1$. Consequently, the $\dim(M_c)$-th degree component of

$$\hat{A}(TM_c) = \det{}^{1/2}\left(\frac{\mathcal{R}/4\pi i}{\sinh \mathcal{R}/4\pi i}\right)$$

  is 0.

- Clifford multiplication. Assume that

$$\sigma_0 : \mathbb{C} \oplus \mathbb{C}T_y^*Y \longrightarrow \hom(E_y^0)$$

  such that $\sigma_0(\xi) : (E_y^0)^{\pm} \to (E_y^0)^{\mp}$, $\sigma_0(\xi)$ is self-adjoint and $\sigma_0(\xi)^2 = |\xi|^2$. Here $\xi = z \oplus \eta \in \mathbb{C} \oplus \mathbb{C}T_y^*Y$ and $|\xi|^2 = |z|^2 + |\eta|^2$ with the metric on $\mathbb{C}T^*Y$ induced by g. Let $p = (s, y) \in M_c$. Since

$$T_p^*M_c \cong T_s\mathbb{R}_0^- \oplus T_y^*Y \cong \mathbb{R} \oplus T_y^*Y,$$

  we can induce a Clifford multiplication on E from σ_0 as follow. Given any $\xi =$

$z\mathrm{d}s + \eta \in \mathbb{C}T^*_pM_c$ with $\eta \in \mathbb{C}T^*_yY$ and $v \in E_p$, we define

$$\sigma(\xi)v := \sigma_0(z \oplus \eta)v$$

in which we identify the elements in E_p and E^0_y in the canonical way. σ is called a Clifford multiplication of product type.

- Connection. Let

$$\nabla_0 : C^\infty(Y, E^0) \longrightarrow C^\infty(Y, T^*M \otimes E^0)$$

be a connection such that it is $\mathbb{Z}_2$-graded, unitary and compatible with the Clifford multiplication σ_0, by which we mean

$$\nabla_0(\sigma_0(\xi)\boldsymbol{v}) = \sigma_0(\nabla_0^{\mathcal{LC}}\xi)\boldsymbol{v} + \sigma_0(\xi)\nabla_0\boldsymbol{v}$$

where $\nabla_0^{\mathcal{LC}}\xi = \mathrm{d}z \otimes 1 + \nabla_0^{\mathcal{LC}}\eta$ with $\xi = z \oplus \eta \in \mathbb{C} \oplus \mathbb{C}T^*Y$ and $\nabla_0^{\mathcal{LC}}$ the Levi-Civita connection on T^*Y. Define

$$\nabla := \mathrm{d}s \otimes \partial_s + \nabla_0.$$

We recall how ∇ acts on section of E precisely. Let $\{\boldsymbol{e}_k(y)\}$ be a local frame of E^0 and $\{\boldsymbol{e}_k(s, y)\}$ be its pullback frame of E over M_c, namely,

$$\boldsymbol{e}_k(s, y) = \pi_Y^*\boldsymbol{e}_k(y).$$

For any fixed s, $\nabla_0\boldsymbol{e}_k(s, y)$ is identified with/defined as $\nabla_0\boldsymbol{e}_k(y)$, viewing T^*Y as a subset of $T^*M_c \cong \mathbb{R} \oplus T^*Y$. For any section $\boldsymbol{f}$ of E, locally we have

$$\boldsymbol{f} = \sum_k f_k(s, y)\boldsymbol{e}_k$$

and hence

$$\begin{aligned}\nabla \boldsymbol{f} &= \sum_k \partial_s f_k \,\mathrm{d}s \otimes \boldsymbol{e}_k + \sum_k (\mathrm{d}_y f_k \otimes \boldsymbol{e}_k + f_k \nabla_0 \boldsymbol{e}_k) \\ &= \sum_k \mathrm{d} f_k \otimes \boldsymbol{e}_k + f_k \nabla_0 \boldsymbol{e}_k.\end{aligned}$$

In addition, we will use the notation

$$\partial_s \boldsymbol{f} := \sum_k \partial_s f_k \boldsymbol{e}_k$$

freely. Since the bundle E is of product type, $\partial_s \boldsymbol{f}$ has an unambiguous meaning, as long as the frame $\{\boldsymbol{e}_k\}$ is a pullback frame. We will show that the following properties hold for ∇.

(i) ∇ is $\mathbb{Z}_2$-graded. Assume that $\boldsymbol{v} \in C^\infty(M_c, E^+)$, then locally

$$\boldsymbol{v} = \sum_j f_j \boldsymbol{e}_j^+$$

where $\{\boldsymbol{e}_j^+\}$ is a local frame of E^+. Since

$$\nabla \boldsymbol{v} = \sum_j \mathrm{d} f_j \otimes \boldsymbol{e}_j^+ + f_j \nabla_0 \boldsymbol{e}_j^+$$

and ∇_0 is $\mathbb{Z}_2$-graded, we have $\nabla \boldsymbol{v} \in C^\infty(M_c, E^+)$ as well. Same argument works for sections of E^-.

(ii) ∇ is unitary. Pick a local orthonormal frame $\{\boldsymbol{e}_k\}$ of E. Since the original ∇_0 is unitary on E^0, given any $\boldsymbol{v} \in C^\infty(M_c, TM_c)$, we have

$$\langle (\nabla_0)_{\boldsymbol{v}} \boldsymbol{e}_i, \boldsymbol{e}_j \rangle + \langle \boldsymbol{e}_i, (\nabla_0)_{\boldsymbol{v}} \boldsymbol{e}_j \rangle = 0,$$

hence, for any $\boldsymbol{f}, \boldsymbol{g} \in C^\infty(M_c, E)$,

$$\begin{aligned}
\boldsymbol{v}\langle \boldsymbol{f}, \boldsymbol{g} \rangle &= \boldsymbol{v}\left\langle \sum_i f_i \boldsymbol{e}_i, \sum_j g_j \boldsymbol{e}_j \right\rangle \\
&= \boldsymbol{v}\left(\sum_i f_i g_i\right) = \sum_i (\boldsymbol{v} f_i) g_i + f_i(\boldsymbol{v} g_i) \\
&= \left(\sum_i (\boldsymbol{v} f_i) g_i + f_i(\boldsymbol{v} g_i)\right) + \sum_{i,j} f_i g_j \left(\langle (\nabla_0)_{\boldsymbol{v}} \boldsymbol{e}_i, \boldsymbol{e}_j \rangle + \langle \boldsymbol{e}_i, (\nabla_0)_{\boldsymbol{v}} \boldsymbol{e}_j \rangle\right) \\
&= \langle \nabla_{\boldsymbol{v}} \boldsymbol{f}, \boldsymbol{g} \rangle + (\boldsymbol{f}, \nabla_{\boldsymbol{v}} \boldsymbol{g}).
\end{aligned}$$

(iii) ∇ is compatible with the Clifford multiplication σ. That is, we want to show that

$$\nabla(\sigma(\xi)\boldsymbol{v}) = \sigma(\nabla^{\mathcal{LC}}\xi)\boldsymbol{v} + \sigma(\xi)\nabla\boldsymbol{v}.$$

Note that

$$\nabla(\sigma(\xi)\boldsymbol{v}) = \mathrm{d}s \otimes \partial_s(\sigma(\xi)\boldsymbol{v}) + \nabla_0(\sigma(\xi)\boldsymbol{v})$$

So we compute

$$\mathrm{d}s \otimes \partial_s(\sigma(\xi)\boldsymbol{v}) = \mathrm{d}s \otimes (\sigma(\partial_s\xi)\boldsymbol{v} + \sigma(\xi)\partial_s\boldsymbol{v})$$

and

$$\nabla_0(\sigma(\xi)\boldsymbol{v}) = \sigma(\nabla_0^{\mathcal{LC}}\xi)\boldsymbol{v} + \sigma(\xi)\nabla_0\boldsymbol{v}.$$

Since

$$\mathrm{d}s \otimes (\sigma(\partial_s\xi)\boldsymbol{v}) + \sigma(\nabla_0^{\mathcal{LC}}\xi)\boldsymbol{v} = \sigma(\nabla^{\mathcal{LC}}\xi)\boldsymbol{v}$$

and

$$\mathrm{d}s \otimes \sigma(\xi)\partial_s\boldsymbol{v} + \nabla_0(\sigma(\xi)\boldsymbol{v}) = \sigma(\xi)\nabla\boldsymbol{v},$$

the claim follows.

- Dirac operator. Define

$$\eth := \frac{1}{i}\sigma \circ \nabla$$

with σ and ∇ given above. Then $\eth$ is called a Dirac operator of product type.

Lemma 2.1.1. *Over M_c, the Dirac operator $\eth$ has the following product type structure*

$$\eth = \frac{1}{i}\sigma(\mathrm{d}s)\left(\partial_s + \mathcal{D}_0\right)$$

where $\mathcal{D}_0 : C^\infty(Y, E^0) \to C^\infty(Y, E^0)$ is

a) *self-adjoint;*

b) *such that $-\sigma(\mathrm{d}s)\mathcal{D}_0 = \mathcal{D}_0\sigma(\mathrm{d}s)$; and*

c) *even with respect to the $\mathbb{Z}_2$-grading of E^0.*

Proof. Computing directly, we have

$$\begin{aligned}
\eth &= \frac{1}{i}\sigma \circ \nabla \\
&= \frac{1}{i}\sigma \circ \left(\mathrm{d}s \otimes \partial_s + \nabla_0\right) \\
&= \frac{1}{i}\left(\sigma(\mathrm{d}s)\partial_s + \sigma \circ \nabla_0\right) \\
&= \frac{1}{i}\sigma(\mathrm{d}s)\left(\partial_s + \sigma(\mathrm{d}s)^{-1}\sigma \circ \nabla_0\right).
\end{aligned}$$

Hence, we define $\mathcal{D}_0 := \sigma(\mathrm{d}s)^{-1}\sigma \circ \nabla_0$. To complete the proof, we will show that $\mathcal{D}_0$ satisfies conditions (a) - (b).

a) Note that $\mathcal{D}_0 = \left(i\sigma(\mathrm{d}s)^{-1}\right)\left(i^{-1}\sigma \circ \nabla_0\right)$. Since $i^{-1}\sigma \circ \nabla_0$ is a Dirac operator on

E^0, it is self-adjoint. We go onto compute

$$\begin{aligned}
\mathcal{D}_0^* &= \left(i^{-1}\sigma \circ \nabla_0\right) \circ \left(-i\sigma(\mathrm{d}s)^{-1}\right) \\
&= -\sigma \circ \nabla_0 \circ \sigma(\mathrm{d}s)^{-1} && (2.1.1) \\
&= -\sigma \circ \sigma(\mathrm{d}s)^{-1} \circ \nabla_0 && (2.1.2) \\
&= \sigma(\mathrm{d}s)^{-1}\sigma \circ \nabla_0 && (2.1.3) \\
&= \mathcal{D}_0.
\end{aligned}$$

Note that from (2.1.1) to (2.1.2), we used the compatibility of ∇_0 with the Clifford multiplication, and from (2.1.2) to (2.1.3) we used the anti-commutativity of Clifford multiplications.

b) It is the exact same argument as the one from (2.1.1) to (2.1.2) with $\sigma(\mathrm{d}s)^{-1}$ replcaed by $\sigma(\mathrm{d}s)$.

c) Recall that

$$\sigma \circ \nabla_0 = \sum_j \sigma(\boldsymbol{\varphi}_j) \circ (\nabla_0)_{\boldsymbol{v}_j}$$

with $\{\boldsymbol{\varphi}_j\}$ a local frame of T^*Y and $\{\boldsymbol{v}_j\}$ its dual frame, then the claim follows from that both $(\nabla_0)_{\boldsymbol{v}_j}$ and $\sigma(\mathrm{d}s)^{-1}\sigma(\boldsymbol{\varphi}_j)$ are even in the grading. □

2.2 Melrose's compactification

In 1993, the underdeveloped idea of Atiyah, Patodi and Singer was picked up by R. Melrose, and extended into a full-fledged theory and framework for analysts to work with singular spaces of various sorts. The first step is to pull the attention back to the compact universe.

We consider the change of variable $x = \mathrm{e}^s$. The non-compact manifold $\widetilde{M}$ is compactified to a manifold with boundary, $X = [0,1]_x \times Y \coprod_Y M$, under this process.

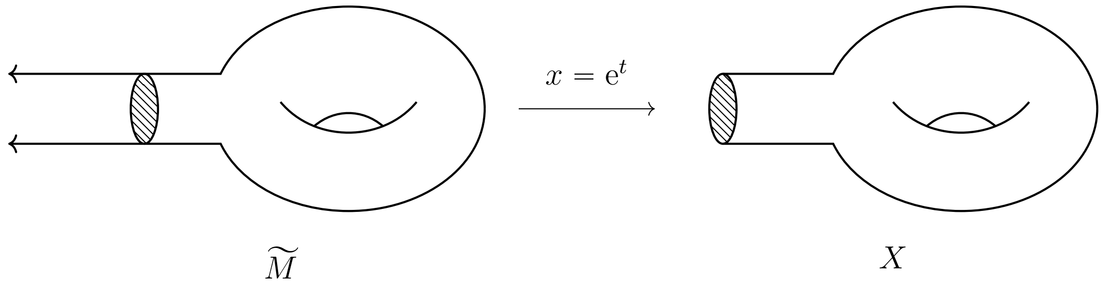


Figure 2.2: Melrose's Compactification

See Figure 2.2.

We will work out how ∂_s transforms. Note that $f(x) \leftrightarrow \tilde{f}(s) := f(\mathrm{e}^s)$. By chain rule,

$$\begin{aligned} \frac{\partial \tilde{f}}{\partial s}(s) &= \frac{\partial}{\partial s} f(\mathrm{e}^s) \\ &= \mathrm{e}^s (\frac{\partial f}{\partial x})(\mathrm{e}^s) \\ &= x \frac{\partial f}{\partial x}(x). \end{aligned}$$

In other words, $\partial_s \leftrightarrow x\partial_x$. Therefore, instead of studying $\eth = \frac{1}{i}\sigma\left(\partial_s + \mathcal{D}_0\right)$, it is equivalent to study the b-differential operator $\frac{1}{i}\sigma\left(x\partial_x + \mathcal{D}_0\right)$, still denoted by $\eth$, and the machinery developed in Chapter 1 can be applied. In particular, as have seen in Example 1.5.3, we have

$$\mathcal{N}(\eth)(\tau) = \frac{1}{i}\sigma(i\tau + \mathcal{D}_0).$$

We discuss the invertibility of $\mathcal{N}(\eth)(\tau)$. Since σ is an isomorphism, it suffices to study $i\tau + \mathcal{D}_0$. Recall that $\mathcal{D}_0$ is self-adjoint, hence $i\tau + \mathcal{D}_0$ is invertible when $\tau \neq 0$. Consequently, $\mathcal{N}(\eth)(\tau)$ is invertible for all $\tau \in \mathbb{R}$ if and only if $\mathcal{D}_0$ is invertible.

Now recall that $\eth$ is $\mathbb{Z}_2$-graded, we can write $\eth = \eth^+ \oplus \eth^-$, where $\eth^\pm : S^0(X, E^\pm) \to S^0(X, E^\mp)$. Note that $\boldsymbol{e}^+ \oplus \boldsymbol{e}^- \in \ker \eth$ if and only if $\boldsymbol{e}^+ \in \ker \eth^+$ and $\boldsymbol{e}^- \in \ker \eth^-$. Hence $\eth^+$ is Fredholm if and only if $\eth$ is Fredholm. We summarize the discussion as follow.

Proposition 2.2.1. $\eth$ *is an elliptic, self-adjoint, $\mathbb{Z}_2$-graded, first order b-differential operator of product type. If* $\eth = i^{-1}\sigma(x\partial_x + \mathcal{D}_0)$ *along* $[0,1] \times Y$ *with* $\mathcal{D}_0 \in \mathrm{Diff}^1(Y, E_0)$ *and* $\mathcal{D}_0$ *is invertible, then both* $\eth$ *and* $\eth$ *are Fredholm over the* S^0*-sections.*

Chapter 3

The heat kernel

3.1 Essentials of heat calculus on closed manifolds

Heat calculus is an approach to construct the heat kernel of a generalized Laplacian on a manifold modeling the Fourier transform technique in Euclidean spaces. We follow [19]. The standard reference is [25], in which the blow-up technique is applied and formulations are in higher generality.

Definition 3.1.1. A function $q(s,x,\omega) \in C^\infty(\mathbb{R}_s \times \mathbb{R}^n_x \times \mathbb{R}^n_\omega)$ is in the *partial heat symbol space* $\mathscr{S}(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n)$ if q is Schwartz in ω, that is, given any $\ell \in \mathbb{N}$, multi-index α, and compact set $K \subset \mathbb{R} \times \mathbb{R}^n$,

$$\sup_{(s,x,\omega)\in K\times\mathbb{R}^n} \left|(1+|\omega|)^\ell (\partial_s \partial_x \partial_\omega)^\alpha q(s,x,\omega)\right| < \infty.$$

Remark. Readers are advised that the choice of terminology here is not standard. On the other hand, in the rest of this work, we will just call the function q a heat symbol or simply a symbol, when no ambiguity arises.

Definition 3.1.2. A linear operator $Q : C^\infty_c(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$ is in $\Psi^p_{\mathcal{H}}(\mathbb{R}^n), p \in$

$\mathbb{Z}$, if there exists a function $q \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n)$ with

$$q(-t, x, -\omega) = (-1)^p q(t, x, \omega),$$

such that, given any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$Q\varphi(t, x) = t^{-n/2-p/2-1} \int q(t^{1/2}, x, \frac{x-y}{t^{1/2}})\varphi(y)\nu(y).$$

Remark. One could think of the function $s^p q(s, x, \omega)$ as the "*full heat symbol*" of Q, analogous to the standard theory of pseudodifferential operators, with p serving as the delimiter of the orders. Nevertheless, as already mentioned, the word "heat symbol" was reserved for the partial heat symbol $q(s, x, \omega)$.

Observe that for $t > 0$, the integral kernel of $Q \in \Psi\!-\!\mathcal{H}^p(\mathbb{R}^n)$, denoted by $Q(t, x, y)$, is smooth in all variables. In particular, the partial derivatives with respect to t of $Q(t, x, y)$ are sums of terms in the form of

$$t^\gamma \partial_s^\alpha ((x-y) \cdot \partial_\omega)^\beta q(t^{1/2}, x, \frac{x-y}{t^{1/2}}).$$

Hence, given any compact set $K \subset \mathbb{R}^n \times \mathbb{R}^n$ and index δ, there exists some constant k, such that

$$\sup_{\mathbb{R}^+ \times K} \left| t^k \partial_t^\delta Q(t, x, y) \right| < \infty. \tag{3.1.1}$$

Proposition 3.1.3. *Let $Q \in \Psi_{\mathcal{H}}^p(\mathbb{R}^n)$.*

1. *If p is even, then*

$$Q : C_c^\infty(\mathbb{R}^n) \longrightarrow t^{-p/2-1} C^\infty \left([0, \infty)_t \times \mathbb{R}^n \right).$$

2. *If p is odd, then*

$$Q : C_c^\infty(\mathbb{R}^n) \longrightarrow t^{-p/2-1/2} C^\infty\left([0,\infty)_t \times \mathbb{R}^n\right).$$

Proposition 3.1.4. *Let $p \in \mathbb{Z}$, $m \in \mathbb{N}$ and $P \in \mathrm{Diff}^m(\mathbb{R}^n)$.*

1. $\Psi_{\mathcal{H}}^p(\mathbb{R}^n) \subset \Psi_{\mathcal{H}}^{p+m}(\mathbb{R}^n)$;

2. $\partial_t : \Psi_{\mathcal{H}}^p(\mathbb{R}^n) \longrightarrow \Psi_{\mathcal{H}}^{p+2}(\mathbb{R}^n)$;

3. $P : \Psi_{\mathcal{H}}^p(\mathbb{R}^n) \longrightarrow \Psi_{\mathcal{H}}^{p+m}(\mathbb{R}^n)$.

Let $\dot{\Psi}_{\mathcal{H}}^{-\infty}(\mathbb{R}^n) := \bigcap_{p\in\mathbb{Z}} \Psi_{\mathcal{H}}^p(\mathbb{R}^n)$. The following result gives a characterization of $\dot{\Psi}_{\mathcal{H}}^{-\infty}(\mathbb{R}^n)$.

Proposition 3.1.5. *If $Q \in \dot{\Psi}_{\mathcal{H}}^{-\infty}(\mathbb{R}^n)$, then there exists a (unique) function $\kappa(t,x,y) \in C^\infty(\overline{\mathbb{R}^+} \times \mathbb{R}^n \times \mathbb{R}^n)$ with*

$$\partial_t^m \kappa(0,x,y) = 0$$

for all $m \in \mathbb{N}$, such that

$$Q\varphi(t,x) = \int \kappa(t,x,y)\varphi(y)\ \mathrm{d}y.$$

Hence, we define

$$\Psi_{\mathcal{H}}^{-\infty}(\mathbb{R}^n) := \left\{\kappa(t,x,y) \in C^\infty(\overline{\mathbb{R}^+} \times \mathbb{R}^n \times \mathbb{R}^n)\middle|\forall m \in \mathbb{N}, \partial_t^m \kappa(0,x,y) = 0\right\}, \tag{3.1.2}$$

the collection of *residual/negligible operators* in heat calculus.

Definition 3.1.6. Let $Q \in \Psi_{\mathcal{H}}^p(\mathbb{R}^n)$ with

$$Q\varphi(t,x) = t^{-n/2-p/2-1} \int q(t^{1/2}, x, \frac{x-y}{t^{1/2}})\varphi(y)\ \mathrm{d}y,$$

then $q(0,x,\omega)$ is called the *normal heat symbol* of Q. The normal symbol map is thus defined as

$$\begin{aligned}\sigma_p^{\mathcal{H}} : \Psi_{\mathcal{H}}^p(\mathbb{R}^n) &\longrightarrow \mathscr{S}(\mathbb{R}^n;\mathbb{R}^n)\\ Q &\longmapsto q(0,x,\omega).\end{aligned}$$

Proposition 3.1.7. *The following sequence is exact:*

$$0 \longrightarrow \Psi_{\mathcal{H}}^{p-1}(\mathbb{R}^n) \longrightarrow \Psi_{\mathcal{H}}^{p}(\mathbb{R}^n) \xrightarrow{\sigma_p^{\mathcal{H}}} \mathscr{S}(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow 0,$$

where $\mathscr{S}(\mathbb{R}^n;\mathbb{R}^n) := \left\{q(x,\omega) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)\middle| q \text{ is Schwartz in } \omega.\right\}$.

Theorem 3.1.8. *Let L be a generalized Laplacian on $\mathbb{R}^n$ equipped with a Riemannian metric. Then there exist $Q \in \Psi_{\mathcal{H}}^{-2}(\mathbb{R}^n)$ and $R \in \Psi_{\mathcal{H}}^{-\infty}(\mathbb{R}^n)$ such that*

$$\begin{cases}(\partial_t + L)Q = R\\ Q\big|_{t=0} = \mathrm{Id}\end{cases}.$$

We now move on the manifolds. The residual space is defined similarly as in (3.1.2):

$$\Psi_{\mathcal{H}}^{-\infty}(Y) := \left\{\kappa(t,x,y) \in C^\infty(\overline{\mathbb{R}^+} \times Y^2)\middle|\forall m \in \mathbb{N}, \partial_t^m\kappa(0,x,y) = 0\right\}.$$

Definition 3.1.9. A linear operator $Q : C_c^\infty(Y) \to C^\infty(\mathbb{R}^+ \times Y)$ is in $\Psi_{\mathcal{H}}^p(Y), p \in \mathbb{Z}$, if given $\{\mathcal{U}_\alpha\}$ a cover of coordinate patches of Y, $\{\phi_\alpha\}$ a partition of unity subordinate to $\{\mathcal{U}_\alpha\}$ and $\{\psi_\alpha \in C_c^\infty(\mathcal{U}_\alpha)\}$ with $\psi_\alpha \equiv 1$ on $\operatorname{supp}\phi_\alpha$, there exist $\{\widetilde{Q}_\alpha \in \Psi_{\mathcal{H}}^p(\mathbb{R}^n)\}$ and $R \in \Psi_{\mathcal{H}}^{-\infty}(Y)$, such that

$$Q = \sum_\alpha \psi_\alpha \widetilde{Q}_\alpha \phi_\alpha + R.$$

We have a variant of Theorem 3.1.8.

Theorem 3.1.10. *Let L be a generalized Laplacian on Y. Then there exist $Q \in \Psi_{\mathcal{H}}^{-2}(Y)$ and $R \in \Psi_{\mathcal{H}}^{-\infty}(Y)$ such that*

$$\begin{cases} (\partial_t + L)Q = R \\ Q\big|_{t=0} = \mathrm{Id} \end{cases}.$$

Proof. Assume that the collection of triple $\{(\mathcal{U}_\alpha, \phi_\alpha, \psi_\alpha)\}$ satisfies the hypotheses in Definition 3.1.9. Write $L_\alpha = L|_{\mathcal{U}_\alpha}$. Then by Theorem 3.1.8, for each α, there exists some $\widetilde{Q}_\alpha \in \Psi_{\mathcal{H}}^{-2}(\mathbb{R}^n)$ and $R_\alpha \in \Psi_{\mathcal{H}}^{-\infty}(\mathbb{R}^n)$, such that $(\partial_t + L_\alpha)\widetilde{Q}_\alpha = R_\alpha$. Define

$$Q := \sum_\alpha \psi_\alpha \widetilde{Q}_\alpha \phi_\alpha = \sum_\alpha Q_\alpha,$$

then $Q \in \Psi_{\mathcal{H}}^{-2}(Y)$. We will show that $R := (\partial_t + L)Q \in \Psi_{\mathcal{H}}^{-\infty}(Y)$.

Recall that $R(t,x,y) \in C^\infty(\mathbb{R}^+ \times Y^2)$, since every Q_α is smooth in all variables when $t > 0$. When $x \neq y$, note that

$$\begin{aligned} \partial_t^\beta \partial_x^\gamma Q_\alpha|_{\mathbb{R}^+ \times \mathcal{U}_\alpha^2}(t,x,y) &= \partial_t^\beta \partial_x^\gamma \psi_\alpha(x) \widetilde{Q}_\alpha(t,x,y)\phi_\alpha(y) \\ &= \sum_{\gamma_1+\gamma_2=\gamma} C_{\gamma_1} \partial_x^{\gamma_1}\psi_\alpha(x) \partial_t^\beta \partial_x^{\gamma_2} \widetilde{Q}_\alpha(t,x,y)\phi_\alpha(y), \end{aligned}$$

hence, $\partial_t^\beta \partial_x^\gamma Q_\alpha|_{\mathbb{R}^+ \times \mathcal{U}_\alpha^2}(0,x,y) = 0$, and as a result $\partial_t^\beta R(0,x,y) = 0$ either. When $x = y$, we consider the following two cases.

case 1. $x \in \operatorname{supp}\phi_\alpha$. Then there exists a neighborhood of x, denoted by $\mathcal{N}_x^\alpha$, such that $\psi_\alpha \equiv 1$ over $\mathcal{N}_x^\alpha$. Consequently,

$$(\partial_t + L)Q_\alpha|_{\mathbb{R}^+ \times (\mathcal{N}_x^\alpha)^2} = \psi(\partial_t + L_\alpha)\widetilde{Q}_\alpha|_{\mathbb{R}^+ \times (\mathcal{N}_x^\alpha)^2}\phi = \psi R_\alpha \phi|_{\mathbb{R}^+ \times (\mathcal{N}_x^\alpha)^2},$$

thus $\partial_t^\beta(\partial_t + L)Q_\alpha(0,x,y) = 0$ on $(\mathcal{N}_x^\alpha)^2$.

case 2. $x \notin \operatorname{supp}\phi_\alpha$, then there exists a neighborhood $\widetilde{\mathcal{N}}_x^\alpha$, such that $\phi_\alpha \equiv 0$ on

$\widetilde{\mathcal{N}}_x^\alpha$. Therefore, $Q_\alpha|_{\mathbb{R}^+\times(\widetilde{\mathcal{N}}_x^\alpha)^2} \equiv 0$, and hence $\partial_t^\beta(\partial_t + L)Q_\alpha(0,x,y) = 0$ on $(\widetilde{\mathcal{N}}_x^\alpha)^2$.

Since $\partial_t^\beta R(t,x,y)$ is a sum of finitely many terms of $\partial_t^\beta(\partial_t+L)Q_\alpha(t,x,y)$, we conclude that $\partial_t^\beta R(0,x,y) = 0$ for all β, that is $R \in \Psi_{\mathcal{H}}^{-\infty}(Y)$.

To complete the proof, just observe that

$$Q|_{t=0} = \sum_\alpha \psi_\alpha \widetilde{Q}_\alpha \phi_\alpha\big|_{t=0} = \sum_\alpha \psi_\alpha\phi_\alpha = \sum_\alpha \phi_\alpha = \mathrm{Id}\,. \qquad \square$$

Remark. Indeed, the theorem above does not require Y to be compact. This fact is used in Section 3.3.

Theorem 3.1.11. *Let L be a generalized Laplacian on Y. Then the heat operator* e^{-tL} *exists and* $\mathrm{e}^{-tL} \in \Psi_{\mathcal{H}}^{-2}(Y)$*.*

By (3.1.1), the partial derivatives of the heat kernel of L with respect to t are bounded by θt^k, where $\theta > 0$ and $k \in \mathbb{R}$ are constants depending on the order of the partial derivatives.

3.2 Fundamentals of b-calculus

We give a brief review of the Melrose's classical *b-calculus.* For more details, see [25], [20], [13] or [22].

Foe each $m \in \mathbb{R}$, the space of b-pseudodifferential operators of order m is the space of distributional right-density

$$\Psi_b^m(X) := \{A \in I^m(X_b^2, \Delta_b, \Omega_{b,R}) \bigm| A \equiv 0 \text{ at } lb \cup rb\},$$

where, in general, $\equiv 0$ at a submanifold, means that the Taylor series vanishes at the submanifold, see Figure 3.1. Note that

$$\Psi_b^m(X) \circ \Psi_b^{m'}(X) \subseteq \Psi_b^{m+m'}(X).$$

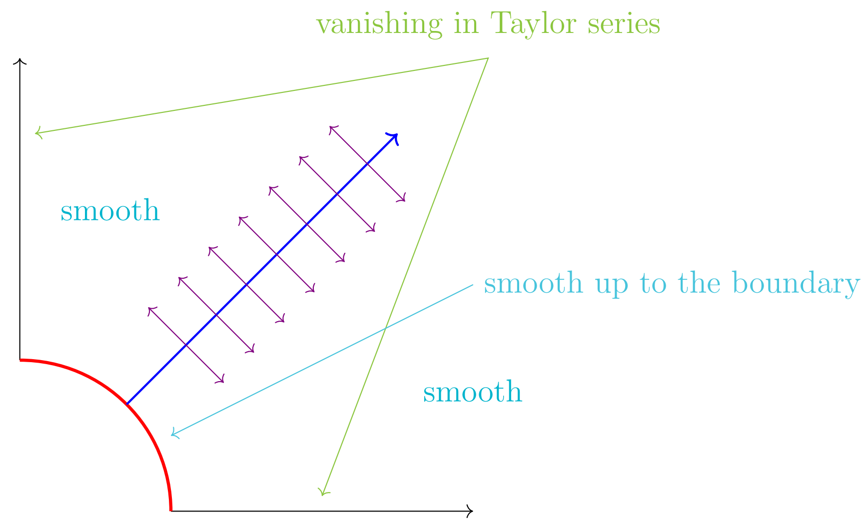


Figure 3.1: Schwartz Kernel of a b-Pseudodifferential Operator

Vanishing in Taylor series is a natural boundary decay condition to consider. This is reflected in the mapping properties of b-pseudodifferential operators. For example, given $A \in \Psi_b^m(X)$,

$$A : \dot{C}^\infty(X) \longrightarrow \dot{C}^\infty(X),$$

where $\dot{C}^\infty(X) = \{f \in C^\infty(X) \mid f \text{ vanishes in Taylor series at } \partial X\}$. More generally, for any $s \in \mathbb{R}$,

$$A : H_b^s(X) \longrightarrow H_b^{s-m}(X),$$

where $H_b^s(X)$ is the b-Sobolev space of order s.

The normal operator $\widehat{A}(\tau)$ of $A \in \Psi_b^m(X)$ may or may not be invertible for a given $\tau \in \mathbb{C}$. However, even if $\widehat{A}(\tau)$ is invertible for all τ in a strip Ω, there may not be a $B \in \Psi_b^{-m}(X)$ such that $\widehat{B}(\tau) = \widehat{A}(\tau)^{-1}$ for all $\tau \in \Omega$. To incorporate the inverse of the normal operator into the theory, one needs to enlarge the (small) b-calculus into the "calculus with bounds", which we now describe briefly.

If $\alpha = (\epsilon_1, \epsilon_2, \delta)$ is a multi-index with $\delta \geqslant 0$, then we define

$$\Psi_b^{-\infty,\alpha}(X) := \rho_{lb}^{\epsilon_1}\rho_{rb}^{\epsilon_2}\bigcup_{\epsilon>0}\rho_{lb}^{\epsilon}\rho_{rb}^{\epsilon}S_{ff}^{0,\delta}(X_b^2, \Omega_{b,R}),$$

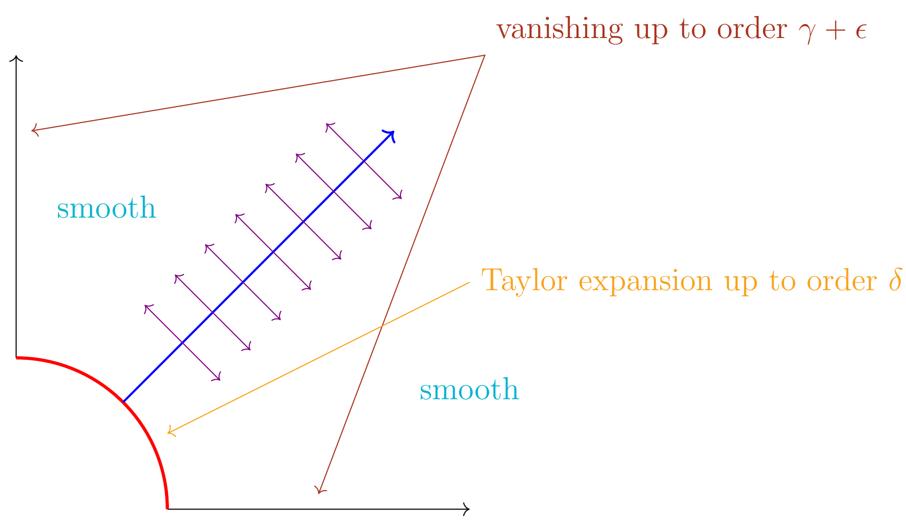


Figure 3.2: Schwartz Kernel of a b-ΨDO with bounds

where ρ_{lb} and ρ_{rb} are total boundary defining function for lb and rb respectively, and where $u \in S^{0,\delta}_{ff}(X^2_b)$ if for any product decomposition $X^2_b \cong [0,1)_r \times ff_{(\omega,y,y')}$ near ff,

$$u(r,\omega) = u_0(\omega,y,y') + ru_1(\omega,y,y') + \cdots + r^k u_k(\omega,y,y') + r^\alpha u_{k+1}(r,\omega,y,y'),$$

where $0 \leqslant \alpha - k < 1$, $u_i(\omega,y,y') \in S^0(ff)$, and $u_{k+1}(r,\omega,y,y') \in S^0([0,1) \times ff)$ is continuous with all b-derivatives up to ff. For any $m \in \mathbb{R}$ (Figure 3.2),

$$\Psi^{m,\alpha}_b(X) := \Psi^m_b(X) + \Psi^{-\infty,\alpha}_b(X)$$

Let ρ_{ff} be a boundary defining function for ff and assume that $\alpha|_{rb} + \alpha'|_{lb} \geqslant 0$. Then the following composition result holds: Provided that $\alpha|_{lb} + \alpha'|_{rb} \geqslant \gamma + \gamma' + \alpha''|_{ff}$, where $\alpha''|_{ff} := \min\{\alpha|_{ff} + \alpha'|_{ff}\}$,

$$\rho^{\gamma}_{ff}\Psi^{m,\alpha}_b(X) \circ \rho^{\gamma'}_{ff}\Psi^{m',\alpha'}_b(X) \subseteq \rho^{\gamma+\gamma'}_{ff}\Psi^{m+m',\alpha''}_b(X),$$

where $\alpha''|_{lb} := \min\{\alpha|_{lb}, \alpha'|_{lb} + \gamma\}$, $\alpha''|_{rb} := \min\{\alpha'|_{rb}, \alpha|_{rb} + \gamma'\}$.

Observe that, if $u(x,y) = x^\epsilon f(x,y)$, then

$$(x\partial_x)^k u(x,y) = \sum_{j=0}^{k} C_j x^\epsilon (x\partial_x)^j f(x,y). \tag{3.2.1}$$

Also, note that, if $\epsilon > 0$, then

$$\lim_{x\to 0} \frac{(\ln x)^m}{x^{-\epsilon}} = \lim_{x\to 0} \frac{m!}{(-\epsilon)^m x^{-\epsilon}} = 0$$

by L'Hospital's Law. From (3.2.1) and the definitions, it follows that, when $\epsilon_1, \epsilon_2 \geqslant 0$,

$$\Psi_b^m(X) \subseteq \Psi_b^{m,\alpha}(X) \subseteq \Psi_{bl}^m(X).$$

In comparison, vanishing in Taylor series is a "super" decay condition, while vanishing to infinite logarithmic order is relatively modest. The title of this work was derived from this viewpoint.

We conclude this section by remarking that the b-calculus with bounds can be characterized by the holomorphic extendibility of (local) symbols. In what follows, we identify the multi-index (α,α,α) with the real number $\alpha > 0$. Following from elementary complex analysis (see, e.g., [30]), we have (Figure 3.3)

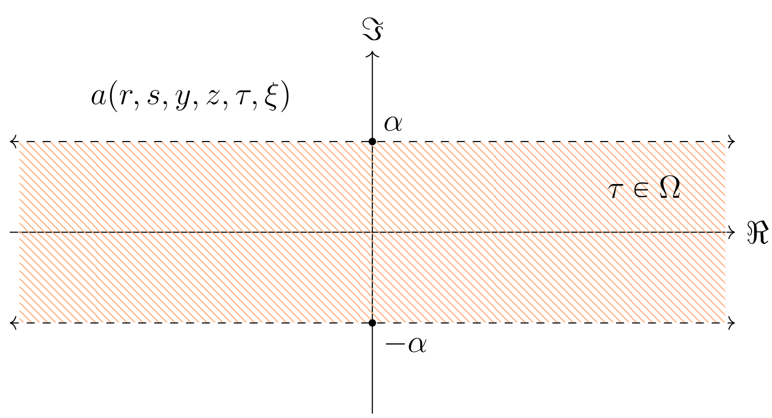


Figure 3.3: Holomorphic Extension of Local Symbol

Proposition 3.2.1. *$A \in \Psi_b^{m,\alpha}(X)$ if and only if, over any coordinate patch near Δ_b, the kernel of A is*

$$\int \mathrm{e}^{is\tau+iz\cdot\xi}\, a(r,s,y,z,\tau,\xi)\ \mathrm{d}\tau \text{đ}\xi,$$

with a a symbol of order m in (τ,ξ) holomorphically extendible in τ to a strip $\Omega_\alpha = \{\tau \in \mathbb{C} \mid |\mathrm{Im}\tau| < \alpha\}$.

On the other hand, the local symbol of an operator in $\Psi_{bl}^m(X)$ is not necessarily holomorphically extendible beyond $\tau \in \mathbb{R}$.

3.3 Heat parametrices

Definition 3.3.1. A *generalized b-Laplacian* on a vector bundle $\pi : E \to X$ is a b-differential operator $\triangle \in \mathrm{Diff}_b^2(X,E)$ with

$${}^b\sigma(\triangle)(\xi) = |\xi|^2 .$$

$\triangle$ is said to be of *product-type* on the collar $C = [0,1)_x \times Y$ if

$$\triangle\big|_C = -(x\partial_x)^2 + \triangle_0$$

where $\triangle_0 \in \mathrm{Diff}^2(Y,E_0)$ is a generalized Laplacian on E_0.

Example 3.3.2. By Lemma 2.1.1, $\eth^2$ lifts from $\widetilde{M}$ to a generalized b-Laplacian of product-type on X.

The idea of constructing the heat kernel of $\triangle$ over the whole X is gluing up the heat kernels on the collar and on the interior. To be precise, let $\phi \in C_c^\infty([0,\infty))$ such that $\phi \equiv 1$ on $[0,1/2]$, $\phi \equiv 0$ on $[3/4,\infty)$ and $\phi \geqslant 0$. Pick a $\psi_0 \in C_c^\infty([0,\infty))$ such that $\psi_0 \equiv 1$ over $[0,7/8)$ and $\psi_0 \equiv 0$ over $[1,\infty)$. Lastly, choose a $\psi_1 \in C^\infty([0,\infty))$

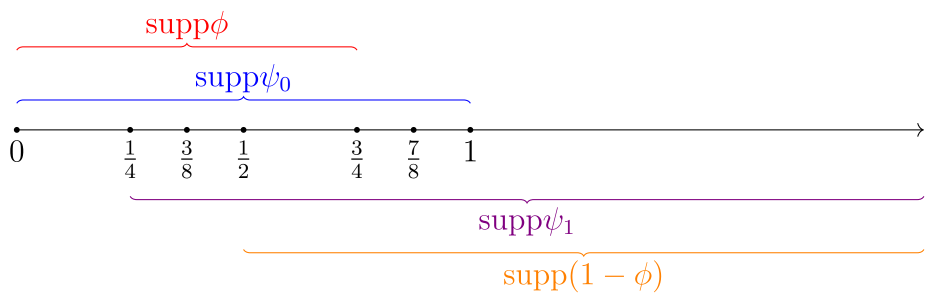


Figure 3.4: Supports of the Gluing Functions

such that $\psi_1 \equiv 1$ on $(3/8, \infty)$ and $\psi_1 \equiv 0$ on $[0, 1/4]$. These three functions will be serving as the "gluing" functions. See Figure 3.4.

Remark. Note that $\psi_0 \equiv 1$ over the support of ϕ and $\psi_1 \equiv 1$ over the support of $1 - \phi$.

We first derive an expression for the heat kernel of $-(x\partial_x)^2$ over $[0, 1)$. Observe that, making the change of variable $r = \ln x$, we have $(0, 1)_x \leftrightarrow (-\infty, 0)_r$ and $x\partial_x = \partial_r$. Note that for any $\varphi \in x^\beta C_c^\infty([0, 1))$, $\varphi(e^r)$ could be viewed as a function in $C^\infty(\mathbb{R})$ by extending beyond $r = 0$ by zero. Now recall that the heat kernel of $-\partial_r^2$ over $\mathbb{R}$ is given by

$$\frac{1}{\sqrt{4\pi t}}\, \mathrm{e}^{-|r-r'|^2/4t},$$

thus, changing back to variable x, we obtain:

$$\frac{1}{\sqrt{4\pi t}}\, \mathrm{e}^{-|\ln(x/x')|^2/4t}\,.$$

We will verify that this is just the heat kernel of $-(x\partial_x)^2$ on $[0, 1)$. In fact, we will put it in a more general context. Recall that Y is a closed manifold, hence the heat kernel of $\triangle_0$ exists. Denote both the heat operator and the heat kernel of $\triangle_0$ by $\mathrm{e}^{-t\triangle_0}$.

Theorem 3.3.3. *The function*

$$
\begin{aligned}
H_0 &= \frac{1}{\sqrt{4\pi t}}\,\mathrm{e}^{-|\ln(x/x')|^2/4t}\,\mathrm{e}^{-t\triangle_0}\\
&= \frac{1}{\sqrt{4\pi t}}\,\mathrm{e}^{-|z|^2/4t}\,\mathrm{e}^{-t\triangle_0},
\end{aligned}
\tag{3.3.1}
$$

where $z = \ln(x/x')$, can be lifted to $S^0_{bl}(C^2_b)$, where $C = [0,1) \times Y$, for every $t > 0$. In particular, it defines a map

$$H_0 : \phi \cdot S^0_{bl}([0,1)_x \times Y) \to S^0_{bl}([0,\infty)_t \times [0,1)_x \times Y).$$

Moreover, given $\varphi \in \phi \cdot S^0_{bl}([0,1)_x \times Y)$, $u = H_0\varphi$ solves the initial value problem

$$
\begin{cases}
(\partial_t - (x\partial_x)^2 + \triangle_0)u = 0\\
u(0,x,y) = \varphi(x,y)
\end{cases}
.
\tag{3.3.2}
$$

That is, the heat kernel of $-(x\partial_x)^2 + \triangle_0$ is just given by (3.3.1).

Proof. Note that for any fixed $t > 0$, $\mathrm{e}^{-|z|^2/4t}$ is Schwartz in z and smooth up to the front face of C^2_b, hence $H_0(t) \in S^0_{bl}(C^2_b)$. Therefore, $H_0(t) : \phi \cdot S^0_{bl}([0,1)_x \times Y) \to S^0_{bl}([0,1)_x \times Y)$. Extend φ beyond $x = 1$ by zero. Under this recognition, we compute

$$
\begin{aligned}
&\int_Y \int_0^1 \frac{\mathrm{e}^{-|\ln(x/x')|^2/4t}\,\mathrm{e}^{-t\triangle_0}}{\sqrt{4\pi t}}\varphi(x',y')\,\frac{\mathrm{d}x'}{x'}\nu(y')\\
=&\int_Y \int_0^\infty \frac{\mathrm{e}^{-|\ln(x/x')|^2/4t}\,\mathrm{e}^{-t\triangle_0}}{\sqrt{4\pi t}}\varphi(x',y')\,\frac{\mathrm{d}x'}{x'}\nu(y')\\
=&\int_Y \int_{-\infty}^\infty \frac{\mathrm{e}^{-|w|^2}\,\mathrm{e}^{-t\triangle_0}}{\sqrt{\pi}}\varphi(x\mathrm{e}^{-2w\sqrt{t}},y')\,\mathrm{d}w\nu(y')\\
=&u(t,x,y).
\end{aligned}
$$

Let $g(t,s,x,y,w) = \sqrt{\pi}^{-1}\int_Y \mathrm{e}^{-|w|^2}\mathrm{e}^{-t\triangle_0}\,\varphi(x\mathrm{e}^{-2ws},y')\nu(y') \in C^\infty(\overline{\mathbb{R}^+} \times \mathbb{R} \times (0,1) \times Y \times \mathbb{R})$. Note that by the properties of heat kernels on closed manifolds, g is smooth in all variables. Also, observe that

$$u(t,x,y) = \int g(t,\sqrt{t},x,y,w)\ \mathrm{d}w,$$

and

$$g(t,-s,x,y,-w) = g(t,s,x,y,w). \tag{3.3.3}$$

Applying the Taylor expansion on s, given $N \in \mathbb{N}$, we have

$$g(t,s,x,y,w) = \sum_{p=0}^{N} \frac{s^p}{p!} \partial_s^p g(t,0,x,y,w) + s^N g_N(t,s,x,y,w)$$

for some $g_N \in C^\infty(\overline{\mathbb{R}^+} \times \mathbb{R} \times (0,1) \times Y \times \mathbb{R})$. By (3.3.3), we have

$$\partial_s^p g(t,0,x,y,w) = (-1)^p \partial_s^p g(t,0,x,y,-w).$$

As a consequence, when $p = 2q+1, q \in \mathbb{N}$,

$$\int \partial_s^p g(t,0,x,y,w)\ \mathrm{d}w = 0,$$

therefore, assuming $N = 2N'$ for definitiveness,

$$\begin{aligned}\int g(t,\sqrt{t},x,y,w)\ \mathrm{d}w &= \int \sum_{p=0}^{N} \frac{(\sqrt{t})^p}{p!} \partial_s^p g(t,0,x,y,w) + (\sqrt{t})^N g_N(t,\sqrt{t},x,y,w)\ \mathrm{d}w \\ &= \int \sum_{q=0}^{N'} \frac{t^q}{2q!} \partial_s^{2q} g(t,0,x,y,w) + t^{N'} g_N(t,\sqrt{t},x,y,w)\ \mathrm{d}w.\end{aligned}$$

By the arbitrariness of N, $u(t,x,y)$ is smooth up to $t = 0$.

Lastly, we will verify that u solves (3.3.2). To see this, note that

$$\left(\partial_t - (x\partial_x)^2\right) \frac{\mathrm{e}^{-|z|^2/4t}}{\sqrt{4\pi t}} = \frac{\mathrm{e}^{-|z|^2/4t}}{\sqrt{4\pi t}} \left(-\frac{1}{2t} + \frac{|z|^2}{(4t)^2} + \frac{1}{2t} - \frac{|z|^2}{(4t)^2} \right) = 0.$$

Consequently,

$$(\partial_t - (x\partial_x)^2 + \triangle_0)H_0 = \mathrm{e}^{-t\triangle_0}\left(\partial_t - (x\partial_x)^2\right)\frac{\mathrm{e}^{-|z|^2/4t}}{\sqrt{4\pi t}} + \frac{\mathrm{e}^{-|z|^2/4t}}{\sqrt{4\pi t}}\left(\partial_t + \triangle_0\right)\mathrm{e}^{-t\triangle_0} = 0.$$

Meanwhile, setting $t = 0$, we have

$$u(0, x, y) = \int \frac{\mathrm{e}^{-|w|^2}}{\sqrt{\pi}}\varphi(x, y)\ \mathrm{d}w = \varphi(x, y),$$

which is equivalent to

$$H_0\varphi\big|_{t=0} = \varphi. \qquad \square$$

Lemma 3.3.4. *Let H_0 be defined by (3.3.1). Then $H_0 \equiv 0$ at $t = 0$ when $x \neq x'$.*

Proof. Recall from heat calculus that $\mathrm{e}^{-t\triangle_0} \in \Psi_{\mathcal{H}}^{-2}(Y)$, hence any given partial derivative with respect to t of $\mathrm{e}^{-t\triangle_0}$ is bounded by θt^k for some constant θ and k. We will see that

$$\frac{1}{\sqrt{4\pi t}}\,\mathrm{e}^{-|z|^2/4t} \equiv 0 \tag{3.3.4}$$

at $t = 0$ when $x \neq x'$. In fact, give any (x, x') with $x \neq x'$, there exists a bounded neighborhood N that does not intersect $\{x = x'\}$. Note that over N, given any α, β and γ, the partial derivative

$$\partial_t^\alpha \partial_x^\beta \partial_{x'}^\gamma \frac{\mathrm{e}^{-|\ln(x/x')|^2/4t}}{\sqrt{4\pi t}} \tag{3.3.5}$$

is a sum of terms in the general form

$$C\,\mathrm{e}^{-|\ln(x/x')|^2/4t}\ln(\frac{x}{x'})^a t^{-b} x^{-c} (x')^{-d} \tag{3.3.6}$$

for some constants C and $a, b, c, d \geqslant 0$. Since (3.3.6) converges to zero uniformly over N as $t \to 0$, so does (3.3.5), which implies (3.3.4).

Lastly, the claim follow from the product rule of differentiation. $\square$

Proposition 3.3.5. *The operator defined by $\psi_0 H_0 \phi$ maps $S^0_{bl}(X)$ to $S^0_{bl}([0,\infty)\times X)$, and*

$$\psi_0 H_0 \phi\big|_{t=0} = \phi \,\mathrm{Id}\,.$$

Proof. Let $u \in S^0_{bl}(X)$. Then ϕu is supported in the collar $C \cong [0,1)\times Y$, $\phi u \in S^0_{bl}(C)$ and $\psi_0 \phi u = \phi u$. On the other hand, given any $\rho \in S^0_{bl}([0,\infty)\times C)$, $\psi_0(x)\rho$ can be identified with a function in $S^0_{bl}([0,\infty)\times X)$ by extending beyond $x = 1$ by zero. Now the claim follows from Theorem 3.3.3. $\square$

Let $X' = X\backslash[0,1/4)\times Y$. Note that X' is a manifold without boundary. We recall the following result from heat calculus, which is a immediate consequence of Theorem 3.1.10 applied to X'.

Theorem 3.3.6 (Interior heat parametrix)**.** *There exists an $H_1 \in \Psi^{-2}_{\mathcal{H}}(X')$ such that*

$$(\partial_t + \triangle\big|_{X'})H_1 = R_1' \in \Psi^{-\infty}_{\mathcal{H}}(X').$$

and

$$H_1\big|_{t=0} = \mathrm{Id}\,.$$

Also, recall that if $R \in \Psi^{-\infty}_{\mathcal{H}}(X')$ then its Schwartz kernel K_R is in $C^\infty([0.\infty)\times X'\times X')$ and $K_R \equiv 0$ at $t = 0$.

Proposition 3.3.7. *The operator $\psi_1 H_1(1-\phi)$ maps $S^0_{bl}(X)$ to $S^0_{bl}([0,\infty)\times X)$ and*

$$\psi_1 H_1 (1-\phi)\big|_{t=0} = (1-\phi)\,\mathrm{Id}\,.$$

Moreover, the Schwartz kernel of $\psi_1 H_1(1-\phi)$ is smooth and vanishing near the boundary of X^2.

Now we "glue" H_0 and H_1 up. Precisely, we define

$$Q := \psi_0 H_0 \phi + \psi_1 H_1 (1-\phi). \tag{3.3.7}$$

We shall show that Q is a heat parametrix of $\triangle$. More precisely, we have the following result:

Theorem 3.3.8. *$(\partial_t + \triangle)\, Q = R$, where $R \in C^\infty([0,\infty) \times X^2)$ such that $R \equiv 0$ in Taylor series at $t = 0$ and ∂X^2. Moreover, $Q|_{t=0} = \mathrm{Id}$.*

Proof. Observe that $Q|_{t=0} = \phi \,\mathrm{Id} + (1-\phi)\,\mathrm{Id} = \mathrm{Id}$. The liberty in the choice of the gluing function suggests that those two parts of Q could be handled separately.

Step 1. We analyze the $\psi_0 H_0 \phi$ part first. Since $\psi_0 H_0 \phi\big|_{X \setminus C} = 0$, we will restrict on the collar and hence $\partial_t + \triangle = \partial_t - (x\partial_x)^2 + \triangle_0$. We compute

$$\begin{aligned}
\left(\partial_t - (x\partial_x)^2 + \triangle_0\right) \psi_0 H_0 \phi &= \psi_0(\partial_t H_0 \phi) - (x\partial_x)^2(\psi_0 H_0 \phi) + \psi_0(\triangle_0 H_0 \phi) \\
&= \psi_0((\partial_t - (x\partial_x)^2 + \triangle_0) H_0 \phi) \\
&\quad - ((x\partial_x)^2 \psi_0) H_0 \phi - 2(x\partial_x \psi_0)(x\partial_x H_0 \phi) \\
&= -((x\partial_x)^2 \psi_0) H_0 \phi - 2(x\partial_x \psi_0)(x\partial_x H_0 \phi) \\
&= R_0,
\end{aligned}$$

since $(\partial_t - (x\partial_x)^2 + \triangle_0) H_0 = 0$. The kernel of $((x\partial_x)^2 \psi_0) H_0 \phi$ is

$$k_{0,1}(t, x, x', y, y') = ((x\partial_x)^2 \psi_0(x)) \frac{\mathrm{e}^{-|z|^2/4t}}{\sqrt{4\pi t}} \phi(x') \,\mathrm{e}^{-t\,\triangle_0}$$

where $z = \ln(x/x')$, and the kernel of $2(x\partial_x\psi_0)(x\partial_x H_0\phi)$ is

$$\begin{aligned} k_{0,2}(t,x,x',y,y') &= 2((x\partial_x)\psi_0(x))(-\frac{z}{2t}\frac{\mathrm{e}^{-|z|^2/4t}}{\sqrt{4\pi t}})\phi(x')\,\mathrm{e}^{-t\triangle_0} \\ &= -(x\partial_x\psi_0(x))(\frac{z}{t}\frac{\mathrm{e}^{-|z|^2/4t}}{\sqrt{4\pi t}})\phi(x')\,\mathrm{e}^{-t\triangle_0}. \end{aligned}$$

Then the kernel of R_0 is $k_0 = k_{0,1} + k_{0,2}$. We claim that k_0 is supported away from the diagonal of $C \times C$. In fact, note that both the supports of $(x\partial_x)^2\psi_0(x)$ and $x\partial_x\psi_0(x)$ are contained in $[7/8, 1]$, and the support of $\phi(x')$ is in $[0, 3/4]$, hence both

$$((x\partial_x)^2\psi_0(x))\frac{\mathrm{e}^{-|z|^2/4t}}{\sqrt{4\pi t}}\phi(x')$$

and

$$2((x\partial_x)\psi_0(x))(-\frac{z}{2t}\frac{\mathrm{e}^{-|z|^2/4t}}{\sqrt{4\pi t}})\phi(x')$$

are supported in $[7/8, 1] \times [0, 3/4]$. In particular, k_0 is supported away from $\{x = x'\}$. Now by a similar argument as Lemma 3.3.4, we see that $R_0 \equiv 0$ at $t = 0$.

We also observe that k_0 vanishes identically near $\{x = x' = 0\}$. In fact, it vanishes identically near the left boundary of X^2, and near the right boundary when x is sufficiently large. Since

$$\frac{\mathrm{e}^{-|z|^2/4t}}{\sqrt{4\pi t}} \equiv 0$$

and

$$\frac{z}{t}\frac{\mathrm{e}^{-|z|^2/4t}}{\sqrt{4\pi t}} \equiv 0$$

at $\partial X^2 \backslash \{x = x' = 0\}$, we have $R_0 \equiv 0$ at ∂X^2.

Step 2. Now we look at the $\psi_1 H_1(1-\phi)$ part. Since $\psi_1 \equiv 1$ beyond $x = 1/2$, we have

$$(\partial_t + \triangle)\,\psi_1 H_1(1-\phi) = \psi_1\,(\partial_t + \triangle)\,H_1(1-\phi)$$

when $x > 1/2$. On the other hand, over the collar part, the computation is similar to that in Step 1, and we also recall that ψ_1 is supported outside $[0, 1/4)$. Putting together, we conclude that

$$\begin{aligned}
&(\partial_t + \triangle)\,\psi_1 H_1(1-\phi) \\
=&\psi_1\left(\partial_t + \triangle\big|_{X'}\right) H_1(1-\phi) - ((x\partial_x)^2\psi_1)H_1(1-\phi) - 2(x\partial_x\psi_1)(x\partial_x H_1(1-\phi)) \\
=&\psi_1 R_1'(1-\phi) - ((x\partial_x)^2\psi_1)H_1(1-\phi) - 2(x\partial_x\psi_1)(x\partial_x H_1(1-\phi)) \\
=&\psi_1 R_1'(1-\phi) + R_1'' = R_1,
\end{aligned}$$

where $R_1'' = -((x\partial_x)^2\psi_1)H_1(1-\phi) - 2(x\partial_x\psi_1)(x\partial_x H_1(1-\phi))$. We remark that the analysis of the kernel of R_1 is even simpler than R_0. Note that $\operatorname{supp}(1-\phi) \subset [1/2, \infty)$, $\operatorname{supp}\psi_1 \subset [1/4, \infty)$ and $\operatorname{supp}(x\partial_x)^2\psi_1 \cup \operatorname{supp} x\partial_x\psi_1 \subset [1/4, 3/8]$, hence the support of the kernel of R_1 is away from ∂X^2. In addition, since $R_1' \in \Psi_{\mathcal{H}}^{-\infty}(X')$, $\psi_1 R_1'(1-\phi) \equiv 0$ at $t = 0$; since the kernel of R_1'' is supported away from the diagonal and $H_1 \in \Psi_{\mathcal{H}}^{-2}(X')$, the same property holds for R_1'' as well. In conclusion, we see that $R_1 \equiv 0$ at $t = 0$ and ∂X^2. $\square$

3.4 Construction of the heat kernel

We finish up the construction of the heat kernel with the Volterra series argument in this section.

Recall the Duhamel's principle.

Theorem 3.4.1. *The following statements are equivalent.*

1. *There exists a linear map*

$$H : \dot{C}^\infty(X) \longrightarrow C^\infty(\overline{\mathbb{R}^+} \times X),$$

such that

$$\begin{cases} (\partial_t + \triangle)H = 0 \\ H\big|_{t=0} = \mathrm{Id}\,. \end{cases} \tag{3.4.1}$$

2. There exists a linear map

$$G : \dot{C}^\infty(\overline{\mathbb{R}^+} \times X) \longrightarrow C^\infty(\overline{\mathbb{R}^+} \times X),$$

such that

$$\begin{cases} (\partial_t + \triangle)G = \mathrm{Id} \\ G\big|_{t=0} = 0. \end{cases} \tag{3.4.2}$$

Proof. Assume that (3.4.1) holds. Define

$$Gu(t) := \int_0^t H(t-s)u(s)\ \mathrm{d}s \tag{3.4.3}$$

for any $u \in \dot{C}^\infty(\overline{\mathbb{R}^+} \times X)$, then G satisfies (3.4.2).

Suppose that (3.4.2) is true. Note that $\dot{C}^\infty(X)$ can be canonically identified with a subspace of $\dot{C}^\infty(\overline{\mathbb{R}^+} \times X)$. With this recognition, given any $u \in \dot{C}^\infty(X)$, define

$$Hu(t) := \partial_t Gu(t).$$

Then H satisfies (3.4.1). □

Proposition 3.4.2. *There exists a G satisfies (3.4.2).*

Proof. Consider the function space

$$\mathcal{H}^\beta([0,\infty) \times X) := \big\{x^\beta v \in x^\beta \cdot C^\infty([0,\infty) \times \mathring{X})\big| \\ \forall k \in \mathbb{N}, P \in \mathrm{Diff}_b^*(X), \partial_t^k P v \in L_\infty([0,t) \times \mathring{X})\big\}.$$

Given $u \in \mathcal{H}^\beta([0,\infty) \times X)$, define

$$C_Q u = \int_0^t Q(t-s)u(s)\ \mathrm{d}s,$$

where Q was given by (3.3.7), then $C_Q : \mathcal{H}^\beta([0,\infty) \times X) \to \mathcal{H}^\beta([0,\infty) \times X)$. Observe that

$$(\partial_t + \triangle)C_Q u = (\mathrm{Id} + C_R)u,$$

where $R = (\partial_t + \triangle)Q$. Note that if $(\mathrm{Id} + C_R)$ is invertible on $\mathcal{H}^\beta([0,\infty) \times X)$, then $C_Q(\mathrm{Id} + C_R)^{-1}$ satisfies (3.4.2). Therefore, we will first derive a formula for the inverse of $(\mathrm{Id} + C_R)$ with a Neumann series argument.

Fix $\epsilon > 0$, let $\mathscr{V} = x^\epsilon (x')^\epsilon C([0,t_0] \times X^2)$. Define a norm on $\mathscr{V}$ by

$$\|u\| := \|v\|_\infty,$$

if $u = x^\epsilon (x')^\epsilon v, v \in C([0,t_0] \times X^2)$. Then $\mathscr{V}$ is a Banach space with this norm. Given $f, g \in \mathscr{V}$, define

$$f * g := \int_0^t \int f(t-s,x,w,y,z) g(s,w,x',z,y') \frac{\mathrm{d}w}{w} \nu(z) \mathrm{d}s, t \leqslant t_0,$$

Note that $R|_{[0,t_0]\times X^2} \in \mathscr{V}$. Write $*R^2 = R * R$, and in general $*R^{\ell+1} = R * R^\ell, \ell \in \mathbb{N}$. Then $(C_R)^\ell = C_{*R^\ell}$. Denote $C_0 = \int x^{2\epsilon} \frac{\mathrm{d}x}{x} \nu(y)$. Given $f \in \mathscr{V}$, by induction, one can show that

$$\left|{*f^\ell}(t)\right| \leqslant \frac{(C_0 t)^{\ell-1}}{(\ell-1)!} \|f\|^\ell$$

for any $\ell \in \mathbb{N}$, hence

$$\left\|{*f^\ell}\right\| \leqslant \frac{(C_0 t_0)^{\ell-1}}{(\ell-1)!} \|f\|^\ell. \tag{3.4.4}$$

Now define $S := \sum_{k\geqslant 1} *(-R)^k$, then by (3.4.4), S converges in $\mathscr{V}$. Furthermore, since

$$S = -R + R * R - R * S * R,$$

$S \equiv 0$ in Taylor series at $t = 0$ and ∂X^2. From the construction, we have

$$(\mathrm{Id} + C_R)^{-1} = \mathrm{Id} + C_S.$$

Consequently, define $G := C_Q + C_{Q*S}$, then G satisfies (3.4.2). □

By the Duhamel principle, the heat kernel exists, and $\mathrm{e}^{-t\triangle} = H = Q + Q * S$. In particular, for any $t > 0$, $\mathrm{e}^{-t\triangle} \in \Psi_b^{-\infty}(X)$.

3.5 Trace expansions and Dirac operators

In this section, we assume that the reader is familiar with the *local index formula* proved by Getzler. See [19].

Theorem 3.5.1. *As $t \to 0$, we have the following asymptotic expansion:*

$$\mathrm{e}^{-t\triangle}\big|_{\Delta_b} \sim t^{-n/2} \sum_{k\geqslant 0} a_k t^k,$$

where $a_k \in C^\infty(X, \Omega_b)$.

Proof. Note that $Q * S\big|_{\Delta_b} \sim 0$. Recall that $Q = \psi_0 H_0 \phi + \psi_1 H_1 (1-\phi)$. Thus, we compute

$$\begin{aligned}\psi_0 H_0 \phi\big|_{\Delta_b} &= (4\pi t)^{-1/2} \psi_0(x)\, \mathrm{e}^{-\frac{|\ln x/x'|^2}{4t}}\, \mathrm{e}^{-t\triangle_0}\, \phi(x')|_{\Delta_b} \\ &= (4\pi t)^{-1/2}\, \mathrm{e}^{-t\triangle_0}(y,y)\phi(x).\end{aligned}$$

Since $\dim Y = n-1$, we have

$$\mathrm{e}^{-t\triangle_0} \sim t^{-(n-1)/2} \sum_{k \geqslant 0} t^k a'_k(y),$$

where $a''_k \in C^\infty(Y, \Omega)$, and consequently,

$$\psi_0 H_0 \phi\big|_{\Delta_b} \sim t^{-n/2} \sum_{k \geqslant 0} t^k (4\pi)^{-1/2} \phi(x) a'_k(y) \left| \frac{\mathrm{d}x}{x} \right|.$$

On the other hand, we have

$$\psi_1 H_1 (1-\phi)\big|_{\Delta_b} = (1-\phi) H_1\big|_{\Delta_b} \sim t^{-n/2} \sum_{k \geqslant 0} t^k a''_k,$$

where $a''_k \in C_c^\infty(\mathring{X}, \Omega_b)$, since $H_1 \in \Psi_{\mathcal{H}}^{-2}(X')$. Then the claim follows. $\square$

From the local index formula, the pointwise supertrace of $\mathrm{e}^{-t\eth^2}$ has the following asymptotic property:

$$\operatorname{str}(\mathrm{e}^{-t\eth^2}) \sim \widehat{\mathcal{A}}(TM) \operatorname{ch}_Z(E/S) + o(t). \tag{3.5.1}$$

Chapter 4

The index theorem

4.1 b-integrals and b-traces

We follow the approach in [12].

Let $a \in S^0_{bl}(X, \Omega_b)$. Over the collar $C = [0,1)_x \times Y$, write $a(x,y) = (a_0(y) + a_1(x,y)) \left|\frac{\mathrm{d}x}{x}\right| \nu(y)$, and define the b-*integral* of a by

$$ {}^b\!\!\int_X a := \int_0^1 \int_Y a_1(x,y) \frac{\mathrm{d}x}{x} \nu(y) + \int_{X \setminus C} a. $$

Definition 4.1.1. Let $A \in \Psi^{-\infty}_{bl}(X)$. The b-*trace* of A is

$$ {}^b\mathrm{Tr}(A) = {}^b\!\!\int_X A\big|_{\Delta_b}. $$

Proposition 4.1.2. *1. Let $A \in {}^1S^0_{lb,rb}(X^2, \Omega_{b,R}) \cong {}^1S^0_{lb,f\!f,rb}(X^2_b, \Omega_{b,R}) \subset \Psi^{-\infty}_{bl}(X)$. Then A is of trace-class, and ${}^b\mathrm{Tr}(A) = \mathrm{Tr}(A)$.*

2. Let $A \in \Psi^m_{bl}(X)$ and $B \in {}^1S^0_{lb,f\!f,rb}(X^2_b, \Omega_{b,R})$. Then both AB and BA are of trace-class . In particular,

$$ \mathrm{Tr}([A,B]) = 0. $$

Proof. 1. Note that $A|_\Delta \in {}^1\mathcal{S}^0_{\partial X}((X, \Omega_b))$, hence

$$\mathrm{Tr}(A) = \int_X A\big|_\Delta < \infty.$$

That ${}^b\mathrm{Tr}(A) = \mathrm{Tr}(A)$ follows immediately from the definitions.

2. Note that $\mathcal{N}(AB)(\tau) = \widehat{A}(\tau)\widehat{B}(\tau) = \widehat{A}(\tau) \circ 0 = 0$, and similarly $\mathcal{N}(AB)(\tau) = 0$, hence $AB, BA \in {}^1\mathcal{S}^0_{lb,f\!f,rb}(X^2_b, \Omega_{b,R})$, and by part 1, both AB and BA are of trace-class.

□

Lemma 4.1.3. *Let $A \in \Psi^{-\infty}_{bl}(X)$, and for any $z > 0$, define*

$$F(z) := \int_X x^z A\big|_{\Delta_b}.$$

Then

$$F(z) = \frac{1}{z}\int_{\mathbb{R}} \mathrm{Tr}\left(\widehat{A}(\tau)\right) \,đ\tau + {}^b\mathrm{Tr}(A) + o(z).$$

Proof. Near the front face, write

$$A = \big(a_0(s, y, y') + a_1(x, s, y, y')\big) \cdot \mu'$$

then

$$A\big|_{\Delta_b} = \big(a_0(0, y, y) + a_1(x, 0, y, y)\big) \cdot \left|\frac{\mathrm{d}x}{x}\nu(y)\right|.$$

Thus,

$$\begin{aligned} F(z) &= \int_0^1\int_Y x^z a_0(0, y, y)\,\frac{\mathrm{d}x}{x}\nu(y) + \int_0^1\int_Y x^z a_1(x, 0, y, y)\,\frac{\mathrm{d}x}{x}\nu(y) + \int_{X\setminus C} x^z A\big|_{\Delta_b} \\ &= \frac{1}{z}\int_Y a_0(0, y, y)\,\frac{\mathrm{d}x}{x}\nu(y) + f(z) + g(z), \end{aligned}$$

where $f(z) = \int_0^1\int_Y x^z a_1(x, 0, y, y)\,\frac{\mathrm{d}x}{x}\nu(y)$, $g(z) = \int_{X\setminus C} x^z A\big|_{\Delta_b}$. Observe that $f(z)$ and

$g(z)$ are continuous at $z = 0$.

Note that

$$\widehat{A}(\tau) = \int \mathrm{e}^{-is\tau}\, a_0(s, y, y')\ \mathrm{d}s \cdot\ \mathrm{d}y',$$

thus

$$a_0(s, y, y') \cdot\ \mathrm{d}y' = \int_{\mathbb{R}} \mathrm{e}^{is\tau}\, \widehat{A}(\tau)\, đ\tau$$

and consequently

$$a_0(0, y, y') \cdot\ \mathrm{d}y' = \int_{\mathbb{R}} \widehat{A}(\tau)\, đ\tau.$$

Therefore, we have

$$\int_Y u_0(0, y, y)\ \mathrm{d}y = \int_Y \int_{\mathbb{R}} \widehat{A}(\tau)(y, y)\, đ\tau\, \mathrm{d}y = \int_{\mathbb{R}} \int_Y \widehat{A}(\tau)(y, y)\, \mathrm{d}y đ\tau = \int_{\mathbb{R}} \mathrm{Tr}\left(\widehat{A}(\tau)\right) đ\tau,$$

which implies the lemma. ∎

Theorem 4.1.4. *If $A \in \Psi^m_{bl}(X)$ and $B \in \Psi^{-\infty}_{bl}(X)$, then*

$$^b\mathrm{Tr}([A, B]) = i \int_{\mathbb{R}} \mathrm{Tr}\left(\partial_\tau \widehat{A}(\tau) \circ \widehat{B}(\tau)\right)\ đ\tau. \tag{4.1.1}$$

Proof. Observe that

$$\begin{aligned} x^z[A, B] &= x^z AB - x^z BA \\ &= x^z AB - Ax^z B + Ax^z B - x^z BA \\ &= [x^z, A]B + [A, x^z B]. \end{aligned}$$

Note that for $z > 0$,

$$x^z K_B \in {}^1S^0_{lb,rb}(X^2, \Omega_{b,R}).$$

Hence by Proposition 4.1.2, we have

$$\int [A, x^z B]\Big|_{\Delta_b} = 0$$

for any $z > 0$. As a result,

$$\int x^z [A, B]\Big|_{\Delta_b} = \int [x^z, A]B\Big|_{\Delta_b}.$$

Now observe that

$$[x^z, A]B = x^z C_z B$$

where $C_z = A - x^{-z} A x^z$. Note that the kernel of C_z is given by

$$K_A - (x/x')^{-z} K_A = (1 - \mathrm{e}^{-zs}) K_A$$

with $s = \ln(x/x')$. Recall that

$$\mathrm{e}^{-zs} = \sum_{k \geqslant 0} \frac{(-sz)^k}{k!} = 1 - zs - z^2 g(s, z).$$

Consequently the kernel of $x^z C_z B$ is

$$z x^z (s + z g(s, z)) K_{AB}.$$

By Lemma 4.1.3, we have

$$\begin{aligned}
\int z x^z (s + z g(s,z)) K_{AB}\big|_{\Delta_b} &= z \left(\frac{1}{z} \int_{\mathbb{R}} \operatorname{Tr}\left(\widehat{D}_z(\tau) \right) \, đ\tau + {}^b\operatorname{Tr}(D_z) + o(z) \right) \\
&= \int_{\mathbb{R}} \operatorname{Tr}\left(\widehat{D}_z(\tau) \right) \, đ\tau + z \, {}^b\operatorname{Tr}(D_z) + z \cdot o(z) \\
&= \int_{\mathbb{R}} \operatorname{Tr}\left(\widehat{D}_z(\tau) \right) \, đ\tau + o(z),
\end{aligned}$$

where D_z is a family of *bl*-pseudodifferential operators defined by the kernels $(s + zg(s,z))K_{AB}$. Lastly, we conclude that

$$\begin{aligned}
{}^b\mathrm{Tr}([A,B]) &= \lim_{z\to 0}\int_X x^z[A,B]\big|_{\Delta_b} - \frac{1}{z}\int_{\mathbb{R}} \mathrm{Tr}(\mathcal{N}([A,B])(\tau))\,đ\tau \\
&= \lim_{z\to 0}\int_{\mathbb{R}} \mathrm{Tr}\left(\widehat{D}_z(\tau)\right)\,đ\tau + o(z) - \frac{1}{z}\int_{\mathbb{R}} \mathrm{Tr}([\widehat{A}(\tau),\widehat{B}(\tau)])\,đ\tau \\
&= \int_{\mathbb{R}} \mathrm{Tr}\left(\widehat{D}_0(\tau)\right)\,đ\tau = \int_{\mathbb{R}} \mathrm{Tr}(\widehat{sA}(\tau)\circ\widehat{B}(\tau))\,đ\tau \\
&= i\int_{\mathbb{R}} \mathrm{Tr}\left(\partial_\tau\widehat{A}(\tau)\circ\widehat{B}(\tau)\right)\,đ\tau. \qquad \square
\end{aligned}$$

Formula (4.1.1) is called the *trace-defect formula*.

4.2 The index theorem

At last, we finish off the proof of the index theorem. We begin with an alternative way to obtain the η-invariant adopted from [20].

Let $\lambda \in \mathbb{R}$. Assume $\lambda > 0$. For any $a > 0$, we compute

$$\begin{aligned}
\frac{1}{\pi}\int_{-a}^{a}(\lambda + i\tau)^{-1}\,đ\tau &= \frac{1}{i\pi}\left(\ln(\lambda + ia) - \ln(\lambda - ia)\right) \\
&= \frac{1}{i\pi}(i\arg(\lambda + ia) - i\arg(\lambda - ia)) \\
&= \frac{2}{\pi}\arg(\lambda + ia),
\end{aligned}$$

hence

$$\begin{aligned}
\frac{1}{\pi}\int_{-\infty}^{\infty}(\lambda + i\tau)^{-1}\,đ\tau &= \lim_{a\to\infty}\frac{1}{\pi}\int_{-a}^{a}(\lambda + i\tau)^{-1}\,đ\tau \\
&= \lim_{a\to\infty}\frac{2}{\pi}\arg(\lambda + ia) \\
&= 1.
\end{aligned}$$

For $\lambda < 0$, we make a substitution $\sigma = -\tau$ to compute

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}(\lambda + i\tau)^{-1}\,\mathrm{d}\tau &= \frac{1}{\pi}\int_{\infty}^{-\infty}(-\lambda + i\sigma)^{-1}\,\mathrm{d}\sigma \\ &= -1.\end{aligned}$$

Consequently, if $\Lambda = \{\lambda_1, \ldots, \lambda_N\} \subset \mathbb{R}\backslash\{0\}$, then

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\sum_{\lambda\in\Lambda}(\lambda + i\tau)^{-1}\,\mathrm{d}\tau = \#\{\lambda \in \Lambda | \lambda > 0\} - \#\{\lambda \in \Lambda | \lambda < 0\}. \tag{4.2.1}$$

Proposition 4.2.1. *If A is a self-adjoint matrix with eigenvalues $\Lambda \subset \mathbb{R}\backslash\{0\}$, then*

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{Tr}\left((A + i\tau)^{-1}\right)\,\mathrm{d}\tau = \mathrm{sgn}(A) := \#\{\lambda \in \Lambda | \lambda > 0\} - \#\{\lambda \in \Lambda | \lambda < 0\}.$$

Proof. The proof is a simple exercise in linear algebra. Recall that A is diagonalizable since it is self-adjoint. Let

$$A = P^{-1}[\lambda_i\delta_{ij}]P = P^{-1}DP$$

for some invertible matrix P. Observe that

$$\begin{aligned}\mathrm{Tr}\left((A + i\tau)^{-1}\right) &= \mathrm{Tr}\left(P^{-1}(D + i\tau)^{-1}P\right) \\ &= \mathrm{Tr}\left(PP^{-1}(D + i\tau)^{-1}\right) \\ &= \mathrm{Tr}\left((D + i\tau)^{-1}\right) \\ &= \sum_{\lambda\in\Lambda}(\lambda + i\tau)^{-1}.\end{aligned}$$

The claim now follows from (4.2.1). □

The above discussion motivates the following consideration. Let $\mathcal{D}$ be a Dirac operator of product-type on the collar. For $t > 0$, define

$$\bar{\eta}(t) := \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\left((\mathcal{D}_0 + i\tau)^{-1}\, \mathrm{e}^{-t(\tau^2+\mathcal{D}_0^2)} \right) \mathrm{d}\tau. \tag{4.2.2}$$

Note that $\mathrm{e}^{-t(\tau^2+\mathcal{D}_0^2)}$ serves as a regularizing factor to assure that the integral (4.2.2) converges. The value of $\bar{\eta}(t)$ at $t = 0$, if exists, should measure the spectral asymmetry of $\mathcal{D}_0$, in view of Proposition 4.2.1. To see the connection of $\bar{\eta}(t)$ with the η-invariant of $\mathcal{D}_0$, we compute

$$\begin{aligned}
\bar{\eta}(t) &= \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\left((\mathcal{D}_0 - i\tau)(\tau^2 + \mathcal{D}_0^2)^{-1}\, \mathrm{e}^{-t(\tau^2+\mathcal{D}_0^2)} \right) \mathrm{d}\tau \\
&= \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\left((\mathcal{D}_0 - i\tau) \int_t^\infty \mathrm{e}^{-s(\tau^2+\mathcal{D}_0^2)}\, \mathrm{d}s \right) \mathrm{d}\tau \\
&= \frac{1}{\pi}\int_t^\infty \int_{\mathbb{R}} \mathrm{Tr}\left((\mathcal{D}_0 - i\tau)\, \mathrm{e}^{-s(\tau^2+\mathcal{D}_0^2)} \right) \mathrm{d}\tau\, \mathrm{d}s.
\end{aligned}$$

Since $\mathrm{Tr}\left(-i\tau\, \mathrm{e}^{-s(\tau^2+\mathcal{D}_0^2)} \right)$ is odd in τ, we go on to compute

$$\begin{aligned}
\bar{\eta}(t) &= \frac{1}{\pi}\int_t^\infty \int_{\mathbb{R}} \mathrm{e}^{-s\tau^2}\, \mathrm{Tr}\left(\mathcal{D}_0\, \mathrm{e}^{-s\mathcal{D}_0^2} \right) \mathrm{d}\tau\, \mathrm{d}s \\
&= \frac{1}{\pi}\int_t^\infty \int_{\mathbb{R}} s^{-1/2}\, \mathrm{e}^{-\iota^2}\, \mathrm{Tr}\left(\mathcal{D}_0\, \mathrm{e}^{-s\mathcal{D}_0^2} \right) \mathrm{d}\iota\, \mathrm{d}s \\
&= \frac{1}{\sqrt{\pi}}\int_t^\infty s^{-1/2}\, \mathrm{Tr}\left(\mathcal{D}_0\, \mathrm{e}^{-s\mathcal{D}_0^2} \right) \mathrm{d}s.
\end{aligned}$$

Recall that $\eta(\mathcal{D}_0) := \frac{1}{\sqrt{\pi}}\int_0^\infty s^{-1/2}\, \mathrm{Tr}\left(\mathcal{D}_0\, \mathrm{e}^{-s\mathcal{D}_0^2} \right) \mathrm{d}s$ is exactly the η-invariant of $\mathcal{D}_0$.

Now assume that $\mathcal{D}_0$ is invertible. Note that $\mathcal{D}^+$ is Fredholm (Proposition 2.2.1). Our study of the index formula relies on the following fundamental result:

Theorem 4.2.2 (Hörmander-Fedosov)**.** *If D is a fully elliptic operator, and P its (full) parametrix, then, in particular, both* $\mathrm{Id} - PD$ *and* $\mathrm{Id} - DP$ *are of trace-class, and*

$$\mathrm{Ind} D = \mathrm{Tr}(\mathrm{Id} - PD) - \mathrm{Tr}(\mathrm{Id} - DP).$$

Define

$$B_0 := \mathcal{D}^- \int_0^t \mathrm{e}^{-s\mathcal{D}^+\mathcal{D}^-} \,\mathrm{d}s.$$

Then,

$$\mathcal{D}^+ B_0 = \mathrm{Id} - \mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-},\ B_0\,\mathcal{D}^+ = \mathrm{Id} - \mathrm{e}^{-t\mathcal{D}^-\mathcal{D}^+}.$$

Since $\widehat{\mathcal{D}}(\tau)^{-1}$ exists for all $\tau \in \mathbb{R}$, we can choose a $Q \in \Psi_{bl}^{-1}(X)$ such that $\widehat{Q}(\tau) = \widehat{\mathcal{D}}(\tau)^{-1} = i(i\tau + \mathcal{D}_0)^{-1}\sigma$. Define

$$B = B_0 + Q\,\mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-},$$

then

$$D^+B = \mathrm{Id} - K_1,\ BD^+ = \mathrm{Id} - K_2,$$

where

$$K_1 = (\mathrm{Id} - \mathcal{D}^+\,Q)\,\mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-},\ K_2 = (\mathrm{Id} - QD^+)\,\mathrm{e}^{-t\mathcal{D}^-\mathcal{D}^+}.$$

Observe that

$$\widehat{K}_1(\tau) = 0 = \widehat{K}_2(\tau),$$

hence K_1 and K_2 are of trace-class. Now the Hörmander-Fedosov's theorem implies that

$$
\begin{aligned}
\mathrm{Ind}\mathcal{D}^+ &= \mathrm{Tr}(K_2) - \mathrm{Tr}(K_1) \\
&= \int_X K_2\big|_\Delta - \int_X K_1\big|_\Delta \\
&= {}^b\!\!\int_X K_2\big|_{\Delta_b} - {}^b\!\!\int_X K_1\big|_{\Delta_b} \\
&= {}^b\mathrm{Tr}(K_2) - {}^b\mathrm{Tr}(K_1) \qquad (4.2.3)\\
&= {}^b\mathrm{Tr}\left((\mathrm{Id} - QD^+)\,\mathrm{e}^{-t\mathcal{D}^-\mathcal{D}^+}\right) - {}^b\mathrm{Tr}\left((\mathrm{Id} - \mathcal{D}^+ Q)\,\mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-}\right) \\
&= {}^b\mathrm{Tr}\left(\mathrm{e}^{-t\mathcal{D}^-\mathcal{D}^+}\right) - {}^b\mathrm{Tr}\left(\mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-}\right) \\
&\quad - {}^b\mathrm{Tr}\left(QD^+\,\mathrm{e}^{-t\mathcal{D}^-\mathcal{D}^+} - \mathcal{D}^+ Q\,\mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-}\right).
\end{aligned}
$$

Let $\xi(t) = 2\,{}^b\mathrm{Tr}\left(QD^+\,\mathrm{e}^{-t\mathcal{D}^-\mathcal{D}^+} - \mathcal{D}^+ Q\,\mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-}\right)$.

Theorem 4.2.3. *$\xi(t) = \bar{\eta}(t)$.*

Proof. Recall that $\mathcal{D}^+\,\mathrm{e}^{-t\mathcal{D}^-\mathcal{D}^+} = \mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-}\,\mathcal{D}^+$ by the uniqueness of solution to heat equation. Thus

$$
\begin{aligned}
\xi(t) &= -2\,{}^b\mathrm{Tr}\left(\mathcal{D}^+ Q\,\mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-} - Q\,\mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-}\,\mathcal{D}^+\right) \\
&= -2\,{}^b\mathrm{Tr}\left([\mathcal{D}^+, Q\,\mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-}]\right).
\end{aligned}
$$

Now by the trace-defect formula (4.1.1),

$$
\begin{aligned}
\xi(t) &= -2\frac{i}{2\pi}\int_{\mathbb{R}} \mathrm{Tr}\left[(\partial_\tau \widehat{\mathcal{D}^+}(\tau))\widehat{Q}(\tau)\mathcal{N}(\mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-})(\tau)\right]\mathrm{d}\tau \\
&= \frac{1}{i\pi}\int_{\mathbb{R}} \mathrm{Tr}\left[\partial_\tau\big(\frac{1}{i}\sigma(i\tau + \mathcal{D}_0)\big) \circ i(i\tau + \mathcal{D}_0)^{-1}\sigma \circ \mathcal{N}(\mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-})(\tau)\right]\mathrm{d}\tau \\
&= \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\left[\sigma \circ (i\tau + \mathcal{D}_0)^{-1}\sigma \circ \mathcal{N}(\mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-})(\tau)\right]\mathrm{d}\tau.
\end{aligned}
$$

To obtain $\mathcal{N}(\mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-})(\tau)$, we observe that

$$\mathcal{N}(\mathrm{e}^{-t\mathcal{D}^+\mathcal{D}^-})(\tau) = \widehat{H}_0(\tau)$$

where H_0 is given by (3.3.1), and compute

$$\begin{aligned}
\widehat{H}_0(\tau) &= \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} \,\mathrm{e}^{-|z|^2/4t}\,\mathrm{e}^{-t\triangle_0}\,\mathrm{e}^{-iz\tau}\ \mathrm{d}z \\
&= \mathrm{e}^{-t\tau^2}\,\mathrm{e}^{-t\triangle_0} \\
&= \sigma\,\mathrm{e}^{-t(\tau^2+\mathcal{D}_0^2)}\,\sigma.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\xi(t) &= \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\left[\sigma(i\tau+\mathcal{D}_0)^{-1}\sigma\circ\sigma\,\mathrm{e}^{-t(\tau^2+\mathcal{D}_0^2)}\,\sigma\right]\mathrm{d}\tau \\
&= \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\left[\sigma(i\tau+\mathcal{D}_0)^{-1}\circ\mathrm{e}^{-t(\tau^2+\mathcal{D}_0^2)}\,\sigma\right]\mathrm{d}\tau \\
&= \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\left[\mathrm{e}^{-t(\tau^2+\mathcal{D}_0^2)}\,\sigma\circ\sigma(i\tau+\mathcal{D}_0)^{-1}\right]\mathrm{d}\tau \\
&= \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{Tr}\left(\mathrm{e}^{-t(\tau^2+\mathcal{D}_0^2)}(i\tau+\mathcal{D}_0)^{-1}\right)\mathrm{d}\tau \\
&= \bar{\eta}(t).
\end{aligned}$$

□

Theorem 4.2.4. $\mathrm{Ind}\mathcal{D}^+ = \displaystyle\int_X \widehat{\mathcal{A}}(TM)\,\mathrm{ch}_Z(E/S) - \frac{1}{2}\eta(\mathcal{D}_0).$

Proof. By (4.2.3), (3.5.1) and Theorem 4.2.3, we have

$$\mathrm{Ind}\mathcal{D}^+ = \int_X \widehat{\mathcal{A}}(TM)\,\mathrm{ch}_Z(E/S) + o(t) - \frac{1}{2}\bar{\eta}(t).$$

In particular, $\bar{\eta}(0)$ exists, and

$$\begin{aligned}
\frac{1}{2}\eta(\mathcal{D}_0) &= \frac{1}{2}\bar{\eta}(0) = \frac{1}{2}\lim_{t\to 0}\bar{\eta}(0) \\
&= \lim_{t\to 0}\left(\int_X \widehat{\mathcal{A}}(TM)\,\mathrm{ch}_Z(E/S) + o(t)\right) - \mathrm{Ind}\mathcal{D}^+ \\
&= \int_X \widehat{\mathcal{A}}(TM)\,\mathrm{ch}_Z(E/S) - \mathrm{Ind}\mathcal{D}^+ \,. \qquad \square
\end{aligned}$$

Appendix A

Conormal distributions

In this section, we describe a calculus of conormal distributions, which is particularly suitable to the study of compact manifolds with boundary. The approach here is adopted from [21].

Denote $i^{|\alpha|}\partial_x^\alpha$ by D_x^α for any (multi-)index α.

Let $f \in \mathcal{S}(\mathbb{R}^n)$ be a Schwartz function on $\mathbb{R}^n$. Recall the Fourier transform of f

$$\widehat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\ dx$$

and the inverse transform

$$\check{f}(x) = \int e^{ix\cdot\xi} f(\xi)\, đ\xi$$

where $đ\xi = (2\pi)^{-n}\ d\xi$.

Lemma A.1. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\check{\widetilde{fg}} = \check{f} * \check{g}$$

and

$$\widehat{fg} = (2\pi)^{-n} \widehat{f} * \widehat{g}$$

Proof. Recall the convolution between φ and ψ

$$\varphi * \psi(\xi) := \int \varphi(\xi - \eta)\psi(\eta) \, \mathrm{d}\eta.$$

Note that $f * g \in \mathcal{S}(\mathbb{R}^n)$. Appling the Fourier inverse transform to the formula

$$(\check{f} * \check{g})\widehat{\ } = fg,$$

we verify the first claim. The second claim could be proved similarly. □

Recall that a is said to be in $S^m_{bl}(\mathbb{R}^{k,1} \times \mathbb{R}^n; \mathbb{R}^n)$ for $m \in \mathbb{R}$ if a can be written as $a = a_0 + a_1$, where $a_0 \in S^m(\mathbb{R}^{k-1} \times \mathbb{R}^n; \mathbb{R}^n)$, and $a_1 \in {}^1S^m_\partial(\mathbb{R}^{k,1} \times \mathbb{R}^n; \mathbb{R}^n)$, that is, given any multi-indexes α, β, γ and $\ell \in \mathbb{N}$, $x_0 > 0$,

$$\sup_{x < x_0} \left|(1 + |\xi|)^{|\gamma| - m}[(1 + |\ln x|)\,(1 + |y| + |z|)]^\ell (x\partial_x)^\alpha (\partial_y \partial_z)^\beta \partial_\xi^\gamma a_1(x, y, z, \xi)\right| < \infty. \tag{A.1}$$

Note that this definition of symbols is slightly different from the standard in that we impose decaying property on the (y, z)-variables. See also [9]. The decaying property allows us to carry out Fourier transform more freely and precisely.

Example A.2. Let $\varphi \in C_c^\infty(\mathbb{R}^{k,1})$, $\psi \in C_c^\infty(\mathbb{R}^n)$, and $p(\xi) \in P_m(\mathbb{R}^n)$ where $P_m(\mathbb{R}^n)$ is the collection of m-th degree polynomials in n variables. Then $a(x, y, z, \xi) = \varphi(x, y)\psi(z)p(\xi)$ is in $S^m_{bl}(\mathbb{R}^{k,1} \times \mathbb{R}^n; \mathbb{R}^n)$.

We will denote

$$\int \mathrm{e}^{-iy \cdot \tau} \, a(x, y, z, \xi) \, \mathrm{d}y$$

by $a(x, \hat{\tau}, z, \xi)$, and similarly

$$\int \mathrm{e}^{-iz \cdot \eta} \, a(x, y, z, \xi) \, \mathrm{d}z$$

by $a(x, y, \widehat{\eta}, \xi)$.

Lemma A.3. If $a \in S^m_{bl}(\mathbb{R}^{k,1} \times \mathbb{R}^n; \mathbb{R}^n)$, then both $a(x, \widehat{\tau}, z, \xi)$ and $a(x, y, \widehat{\eta}, \xi)$ are in $S^m_{bl}(\mathbb{R}^{k,1} \times \mathbb{R}^n; \mathbb{R}^n)$.

Proof. Let $\alpha, \beta, \gamma, \delta, \epsilon$ be arbitrary multi-indexes in $\ell \in \mathbb{N}$. Note that

$$
\begin{aligned}
&\tau^\epsilon (x\partial_x)^\alpha D^\beta_\tau \partial^\gamma_z \partial^\delta_\xi a(x, \widehat{\tau}, z, \xi) = \tau^\epsilon (x\partial_x)^\alpha D^\beta_\tau \partial^\gamma_z \partial^\delta_\xi \int \mathrm{e}^{-iy\cdot\tau}\, a(x, y, z, \xi)\ \mathrm{d}y \\
=& \int \tau^\epsilon D^\beta_\tau\, \mathrm{e}^{-iy\cdot\tau} (x\partial_x)^\alpha \partial^\gamma_z \partial^\delta_\xi a(x, y, z, \xi)\ \mathrm{d}y = \int D^\epsilon_y\, \mathrm{e}^{-iy\cdot\tau} \cdot y^\beta (x\partial_x)^\alpha \partial^\gamma_z \partial^\delta_\xi a(x, y, z, \xi)\ \mathrm{d}y \\
=&(-1)^{|\epsilon|} \int \mathrm{e}^{-iy\cdot\tau}\, D^\epsilon_y \left(y^\beta (x\partial_x)^\alpha \partial^\gamma_z \partial^\delta_\xi a(x, y, z, \xi) \right)\ \mathrm{d}y \\
=&(-1)^{|\epsilon|} \sum_{\epsilon_1+\epsilon_2=\epsilon} C_{\epsilon_1} \int \mathrm{e}^{-iy\cdot\tau}\, D^{\epsilon_1}_y y^\beta \cdot (x\partial_x)^\alpha D^{\epsilon_2}_y \partial^\gamma_z \partial^\delta_\xi a(x, y, z, \xi)\ \mathrm{d}y \\
=&(-1)^{|\epsilon|} \sum_{\epsilon_1+\epsilon_2=\epsilon} C_{\epsilon_1} \int \mathrm{e}^{-iy\cdot\tau}\, D^{\epsilon_1}_y y^\beta \cdot (x\partial_x)^\alpha D^{\epsilon_2}_y \partial^\gamma_z \partial^\delta_\xi \left(a_0(y, z, \xi) + a_1(x, y, z, \xi) \right)\ \mathrm{d}y.
\end{aligned}
$$

Thus, for $0 < x < x_0$, we have

$$
\begin{aligned}
\left| \tau^\epsilon (x\partial_x)^\alpha D^\beta_\tau \partial^\gamma_z \partial^\delta_\xi a_1(x, \widehat{\tau}, z, \xi) \right| &\leqslant \sum_{\epsilon_1+\epsilon_2=\epsilon} \int \left| D^{\epsilon_1}_y y^\beta \right| \cdot \left| (x\partial_x)^\alpha D^{\epsilon_2}_y \partial^\gamma_z \partial^\delta_\xi a_1(x, y, z, \xi) \right|\ \mathrm{d}y \\
&< C^\ell_{\alpha,\beta,\gamma,\delta,\epsilon} \int \frac{(1+|y|)^{|\beta|}}{(1+|y|)^{p+|\beta|}}\ \mathrm{d}y \cdot \frac{(1+|\xi|)^{m-|\delta|}}{[(1+|z|)(1+|lnx|)]^\ell} \\
&= \widetilde{C}^\ell_{\alpha,\beta,\gamma,\delta,\epsilon} \frac{(1+|\xi|)^{m-|\delta|}}{[(1+|z|)(1+|\ln x|)]^\ell}
\end{aligned}
$$

for some constant $C^\ell_{\alpha,\beta,\gamma,\delta,\epsilon}$, where $\widetilde{C}^\ell_{\alpha,\beta,\gamma,\delta,\epsilon} = C^\ell_{\alpha,\beta,\gamma,\delta,\epsilon} \int (1+|y|)^{-p}\ \mathrm{d}y$ and $p = 2(k-1)$. Hence $a_1(x, \hat{\tau}, z, \xi)$ satisfies the symbol estimate (A.1). Similarly we have

$$
\sup \left| [(1+|z|+|\tau|)^\ell (1+|\xi|)^{|\delta|-m} D^\beta_\tau \partial^\gamma_z \partial^\delta_\xi a_0(\widehat{\tau}, z, \xi) \right| < \infty,
$$

so we conclude that $a(x, \hat{\tau}, z, \xi) \in S^m_{bl}(\mathbb{R}^{k,1} \times \mathbb{R}^n; \mathbb{R}^n)$. The other claim is proved in the exact same way. $\square$

Let $u \in I^m_{bl}(\mathbb{R}^{k,1}\times\mathbb{R}^n;\mathbb{R}^{k,1}\times\{0\})$ be associated with $a \in S^m_{bl}(\mathbb{R}^{k,1}\times\mathbb{R}^n;\mathbb{R}^n)$, that is,

$$u := \int \mathrm{e}^{iz\cdot\xi}\, a(x,y,z,\xi)\, đ\xi.$$

We will derive the "left" symbol of u, i.e., a symbol $\widetilde{a} \in S^m_{bl}(\mathbb{R}^{k,1};\mathbb{R}^n)$ such that

$$u = \int \mathrm{e}^{iz\cdot\xi}\, \widetilde{a}(x,y,\xi)\, đ\xi.$$

Let $\varphi\cdot\mu \in \dot{C}^\infty_c(\mathbb{R}^{k,1}\times\mathbb{R}^n,\Omega_b)$, where $\mu = \left|\frac{\mathrm{d}x}{x}\mathrm{d}y\mathrm{d}z\right|$. In particular, φ vanishes to infinite order in x at the boundary of $\mathbb{R}^{k,1}\times\mathbb{R}^n$. Observe that

$$\begin{aligned}
&\int \mathrm{e}^{iz\cdot\xi}\, a(x,y,z,\xi)\varphi(x,y,z)\,\frac{\mathrm{d}x}{x}\mathrm{d}y\mathrm{d}zđ\xi\\
&= \int\left(\int a(x,y,\widecheck{\xi-\eta},\xi)\varphi(x,y,\check{\eta})\ \mathrm{d}\eta\right)\,\frac{\mathrm{d}x}{x}\mathrm{d}y\mathrm{d}\xi\\
&= \int\left(\int a(x,y,\widecheck{\xi-\eta},\xi)\ \mathrm{d}\xi\right)\varphi(x,y,\check{\eta})\,\frac{\mathrm{d}x}{x}\mathrm{d}y\mathrm{d}\eta\\
&= \int\left(\int a(x,y,\widehat{\eta-\xi},\xi)\, đ\xi\right)\varphi(x,y,\check{\eta})\,\frac{\mathrm{d}x}{x}\mathrm{d}y\mathrm{d}\eta\\
&= \int \mathrm{e}^{iz\cdot\eta}\left(\int a(x,y,\widehat{\eta-\xi},\xi)\, đ\xi\right)\varphi(x,y,z)\ \mathrm{d}z\frac{\mathrm{d}x}{x}\mathrm{d}yđ\eta.
\end{aligned}$$

Therefore, the left symbol of u shall be defined by

$$\tilde{a}(x,y,\eta) := \int a(x,y,\widehat{\eta-\xi},\xi)\, đ\xi \tag{A.2}$$

$$= \int a(x,y,\widehat{\eta-\xi},\xi)\, đ\xi \tag{A.3}$$

$$= \int a_0(y,\widehat{\eta-\xi},\xi)\, đ\xi + \int a_1(x,y,\widehat{\eta-\xi},\xi)\, đ\xi$$

$$= \tilde{a}_0(y,\eta) + \tilde{a}_1(x,y,\eta). \tag{A.4}$$

To justify the work, we need the following result.

Theorem A.4. Let $a \in S^m_{bl}(\mathbb{R}^{k.1}\times\mathbb{R}^n;\mathbb{R}^n)$. Then $\tilde{a} \in S^m_{bl}(\mathbb{R}^{k,1};\mathbb{R}^n)$, where $\tilde{a}$ is defined in (A.2).

Proof. We will establish the desired decaying property for $\widetilde{a}$. Observe that

$$\begin{aligned}
\partial_{\eta_i}^{\gamma_i}\tilde{a}(x,y,\eta) &= \int \partial_{\eta_i}^{\gamma_i}\,\mathrm{e}^{-i(\eta-\xi)z}\,a(x,y,z,\xi)\;\mathrm{d}z\text{đ}\xi \\
&= \int (-iz_i)^{\gamma_i}\,\mathrm{e}^{-i(\eta-\xi)z}\,a(x,y,z,\xi)\;\mathrm{d}z\text{đ}\xi \\
&= \int -(\partial_{\xi_i}(-iz_i)^{\gamma_i-1}\mathrm{e}^{-i(\eta-\xi)z})a(x,y,z,\xi)\;\mathrm{d}z\text{đ}\xi \\
&= -\int \partial_{\xi_i}\big((-iz_i)^{\gamma_i-1}\mathrm{e}^{-i(\eta-\xi)z}\,a(x,y,z,\xi)\big)\;\mathrm{d}z\text{đ}\xi \\
&\quad + \int (-iz_i)^{\gamma_i-1}\mathrm{e}^{-i(\eta-\xi)z}\,\partial_{\xi_i}a(x,y,z,\xi)\;\mathrm{d}z\text{đ}\xi.
\end{aligned}$$

Since by Lemma A.3 for any $\ell \in \mathbb{N}$ there exists a constant C^ℓ such that

$$\left|\int \mathrm{e}^{-i(\eta-\xi)z}(-iz_i)^{\gamma_i-1}a(x,y,z,\xi)\;\mathrm{d}z\right| < C^\ell(1+|\eta-\xi|)^{-\ell}(1+|\xi|)^m,$$

in particular for any fixed η

$$\int (-iz_i)^{\gamma_i-1}\mathrm{e}^{-i(\eta-\xi)z}\,a(x,y,z,\xi)\;\mathrm{d}z \xrightarrow{|\xi|\to\infty} 0,$$

we have

$$\begin{aligned}
&\int \partial_{\xi_i}\big((-iz_i)^{\gamma_i-1}\mathrm{e}^{-i(\eta-\xi)z}\,a(x,y,z,\xi)\big)\;\mathrm{d}z\mathrm{d}\xi_i \\
&= \left(\int (-iz_i)^{\gamma_i-1}\mathrm{e}^{-i(\eta-\xi)z}\,a(x,y,z,\xi)\;\mathrm{d}z\right)\Bigg|_{\xi_i-\infty}^{\xi_i=\infty} = 0.
\end{aligned}$$

Thus,

$$\partial_{\eta_i}^{\gamma_i}\tilde{a}(x,y,\eta) = \int (-iz_i)^{\gamma_i-1}\mathrm{e}^{-i(\eta-\xi)z}\,\partial_{\xi_i}a(x,y,z,\xi)\;\mathrm{d}z\text{đ}\xi.$$

Continue this procedure, we eventually have

$$\begin{aligned}\partial_{\eta_i}^{\gamma_i}\tilde{a}(x,y,\eta) &= \int e^{-i(\eta-\xi)z}\,\partial_{\xi_i}^{\gamma_i}a(x,y,z,\xi)\ \mathrm{d}z\text{đ}\xi \\ &= \int(\partial_{\xi_i}^{\gamma_i}a)(x,y,\widehat{\eta-\xi},\xi)\,\text{đ}\xi.\end{aligned}$$

Arguing similarly, we have

$$\partial_{\eta}^{\gamma}\tilde{a}(x,y,\eta) = \int(\partial_{\xi}^{\gamma}a)(x,y,\widehat{\eta-\xi},\xi)\,\text{đ}\xi.$$

Note that $(\partial_{\xi}^{\gamma}a)(x,y,\hat{\zeta},\xi)\in S_{bl}^{m-|\gamma|}(\mathbb{R}^{k,1}\times\mathbb{R}^n;\mathbb{R}^n)$. Since $\tilde{a}$ has a decomposition given in (A.4), now it suffices to establish the symbolic estimate

$$\left|(x\partial_x)^{\alpha}\partial_y^{\beta}\tilde{a}(x,y,\eta)\right| \leqslant \frac{\widetilde{C}_{\alpha\beta}\,(1+|\eta|)^m\left(1+(1+|\ln x|)^{-\ell}\right)}{(1+|y|)^{\ell}}$$

for any indexes α, β and $\ell\in\mathbb{N}$, when $0<x<x_0$. Consider

$$\begin{aligned}\left|(x\partial_x)^{\alpha}\partial_y^{\beta}\tilde{a}(x,y,\eta)\right| &= \left|\int(x\partial_x)^{\alpha}\partial_y^{\beta}a(x,y,\widehat{\eta-\xi},\xi)\ \text{đ}\xi\right| \\ &\leqslant \int\frac{C_{\alpha\beta}^{\ell}\left(1+(1+|\ln x|)^{-\ell}\right)(1+|\xi|)^m}{\left[(1+|y|)\,(1+|\eta-\xi|)\right]^{\ell}}\,\text{đ}\xi\end{aligned}$$

with some constant $C_{\alpha\beta}^{\ell}$. Recall the Peetre's inequality

$$(1+|\xi|)^s \leqslant (1+|\eta-\xi|)^{|s|}\,(1+|\eta|)^s\,, \tag{A.5}$$

$s \in \mathbb{R}$. Thus, let $\ell = |m| + 2n$, we have

$$\begin{aligned}\left|(x\partial_x)^\alpha \partial_y^\beta \tilde{a}(x,y,\eta)\right| &\leqslant \int \frac{C_{\alpha\beta}^\ell \left(1 + (1+|\ln x|)^{-\ell}\right)(1+|\eta-\xi|)^{|m|}(1+|\eta|)^m}{[(1+|y|)(1+|\eta-\xi|)]^\ell} \, đ\xi \\ &= \int \frac{C_{\alpha\beta}^\ell \left(1 + (1+|\ln x|)^{-\ell}\right)(1+|\eta|)^m}{(1+|y|)^\ell (1+|\eta-\xi|)^{2n}} \, đ\xi \\ &= \frac{\widetilde{C}_{\alpha\beta}(1+|\eta|)^m \left(1 + (1+|\ln x|)^{-\ell}\right)}{(1+|y|)^\ell}\end{aligned}$$

where $\widetilde{C}_{\alpha\beta}^e ll = C_{\alpha\beta}^\ell \int (1+|\eta-\xi|)^{-2n} \, đ\xi$. □

Corollary A.4.1. There is an symbolic asymptotic expansion of $\widetilde{a}$ given by

$$\widetilde{a}(x,y,\eta) \sim \sum_\alpha \frac{(D_z^\alpha \partial_\zeta^\alpha a)(x,y,0,\eta)}{(-1)^{|\alpha|}\alpha!}.$$

In particular, $\widetilde{a}(x,y,\eta) = a(x,y,0,\eta) \mod S_{bl}^m(\mathbb{R}^{k,1};\mathbb{R}^n)$.

Proof. To reduce notational ambiguity, we rename the variables in the integrand of (A.3) as $a(x,y,\widehat{\xi},\zeta)$. Applying Taylor expansion to $a(x,y,\widehat{\xi},\eta-\xi)$ in ζ, we have

$$a(x,y,\widehat{\xi},\eta-\xi) = \sum_{|\alpha|\leqslant p} \frac{(\partial_\zeta^\alpha a)(x,y,\widehat{\xi},\eta)}{\alpha!}(-\xi)^\alpha + \sum_{|\beta|=p+1} \frac{(-1)^{|\beta|}\,|\beta|}{\beta!} R_\beta(x,y,\xi,\eta)$$

where $R_\beta(x,y,\xi,\eta) = \xi^\beta \int_0^1 (1-t)^{|\beta|-1} (\partial_\zeta^\beta a)(x,y,\widehat{\xi},\eta - t\xi) \, \mathrm{d}t$ for any $p \in \mathbb{N}$. Therefore,

$$\widetilde{a}(x,y,\eta) = \int \sum_{|\alpha|\leqslant p} \frac{\xi^\alpha (\partial_\zeta^\alpha a)(x,y,\widehat{\xi},\eta)}{(-1)^{|\alpha|}\alpha!} \, đ\xi + \sum_{|\beta|=p+1} \frac{(-1)^{|\beta|}\,|\beta|}{\beta!} \widetilde{R}_\beta(x,y,\eta), \tag{A.6}$$

where $\widetilde{R}_\beta(x,y,\eta) = \int R_\beta(x,y,\xi,\eta) \, đ\xi$. From the elementary properties of Fourier transform, we can rewrite the first term in (A.6) as

$$\begin{aligned}\int \sum_{|\alpha|\leqslant p} \frac{\xi^\alpha(\partial_\zeta^\alpha a)(x,y,\widehat{\xi},\eta)}{(-1)^{|\alpha|}\alpha!}\,đ\xi &= \int \sum_{|\alpha|\leqslant p} \frac{(D_z^\alpha\partial_\zeta^\alpha a)(x,y,\widehat{\xi},\eta)}{(-1)^{|\alpha|}\alpha!}\,đ\xi \\ &= \sum_{|\alpha|\leqslant p} \frac{(D_z^\alpha\partial_\zeta^\alpha a)(x,y,0,\eta)}{(-1)^{|\alpha|}\alpha!}.\end{aligned}$$

Thus, it is only left to show that the second term in (A.6) is in $S_{bl}^{m-p-1}(\mathbb{R}^{k,1},\mathbb{R}^n)$, and it suffices to verify that so is each $\widetilde{R}_\beta$. We will establish the symbol estimate. Hence, given any multi-indexes γ,δ,ϵ,we compute

$$\begin{aligned}(x\partial_x)^\gamma\partial_y^\delta\partial_\eta^\epsilon\widetilde{R}_\beta(x,y,\eta) &= \int\int_0^1 (1-t)^{|\beta|-1}\xi^\beta(x\partial_x)^\gamma\partial_y^\delta\partial_\eta^\epsilon\left((\partial_\zeta^\beta a)(x,y,\widehat{\xi},\eta-t\xi)\right)\,\mathrm{d}tđ\xi \\ &= \int\int_0^1 (1-t)^{|\beta|-1}\xi^\beta\left((x\partial_x)^\gamma\partial_y^\delta\partial_\zeta^{\beta+\epsilon}a\right)(x,y,\widehat{\xi},\eta-t\xi)\,\mathrm{d}tđ\xi,\end{aligned}$$

then for any $\ell\in\mathbb{N}$, by Lemma A.3 and the Peetre's inequality, we estimate

$$\begin{aligned}&\left|(x\partial_x)^\gamma\partial_y^\delta\partial_\eta^\epsilon\widetilde{R}_\beta(x,y,\eta)\right| \leqslant \int\int_0^1 \left|(1-t)^p\xi^\beta\left((x\partial_x)^\gamma\partial_y^\delta\partial_\zeta^{\beta+\epsilon}a\right)(x,y,\widehat{\xi},\eta-t\xi)\right|\,\mathrm{d}tđ\xi \\ \leqslant&\frac{C\left(1+(1+|\ln x|)^{-\ell}\right)}{(1+|y|)^\ell}\int\int_0^1 \frac{(1-t)^p\,(1+|\xi|)^{|\beta|}\,(1+|\eta-t\xi|)^m}{(1+|\xi|)^q\,(1+|\eta-t\xi|)^{|\beta|+|\epsilon|}}\,\mathrm{d}tđ\xi \\ \leqslant&\frac{C\left(1+(1+|\ln x|)^{-\ell}\right)}{(1+|y|)^\ell}\int\int_0^1 \frac{(1-t)^p\,(1+|\xi|)^{|\beta|}\,(1+|\eta-t\xi|)^m}{(1+|\xi|)^q\,(1+|\eta-t\xi|)^{|\beta|+|\epsilon|}}\,\mathrm{d}tđ\xi \\ \leqslant&\frac{C\left(1+(1+|\ln x|)^{-\ell}\right)}{(1+|y|)^\ell}\int\int_0^1 \frac{(1-t)^p\,(1+|t\xi|)^{|m-|\beta|-|\epsilon||}\,(1+|\eta|)^m}{(1+|\xi|)^{q-|\beta|}\,(1+|\eta|)^{|\beta|+|\epsilon|}}\,\mathrm{d}tđ\xi \\ \leqslant&\widetilde{C}\left(1+(1+|\ln x|)^{-\ell}\right)(1+|y|)^{-\ell}\,(1+|\eta|)^{m-|\beta|+|\epsilon|},\end{aligned}$$

where $q=|(m-|\beta|-|\epsilon|)|+|\beta|+2n$, C is some constant depending on $\gamma,\delta,\epsilon,\ell$, and

$$\widetilde{C}=C\int(1-t)^p\,(1+|t\xi|)^{|m-|\beta|-|\epsilon||}\,(1+|\xi|)^{|\beta|-q}\,\mathrm{d}tđ\xi. \qquad \square$$

Example A.5. Suppose that $a\in S_{bl}^m(\mathbb{R}^{k,1};\mathbb{R}^n)$ and $\psi\in C_c^\infty(\mathbb{R}^n)$, then $\psi(z)a(x,y,\xi)\in$

$S^m_{bl}(\mathbb{R}^{k,1}\times\mathbb{R}^n;\mathbb{R}^n)$, and the left symbol of the conormal distribution u defined by

$$\int e^{iz\xi}\,\psi(z)a(x,y,\xi)\,đ\xi$$

is just

$$\widetilde{a}(x,\xi)=\int\widehat{\psi}(\xi-\eta)a(x,y,\eta)\;\mathrm{d}\eta.$$

This construction was used extensively in the study and applications of conormal distributional densities on manifolds

Theorem A.6 (Continuity principle)**.** Let $\rho(x,y,z,\xi)\in C^\infty(\mathbb{R}^{k,1}_{(x.y)}\times\mathbb{R}^n_z\times\mathbb{R}^n_\xi)$ with $\rho(x,y,z,0)$

$\equiv 1$ be bounded in (x,y,z) and Schwartz in ξ, i.e., given any multi-indexes $\alpha,\beta,\gamma,\delta,\epsilon$ there is some constant $C^\epsilon_{\alpha\beta\gamma\delta}$ such that

$$\left|\xi^\epsilon\partial^\alpha_x\partial^\beta_y\partial^\gamma_z\partial^\delta_\xi\rho(x,z,\xi)\right|<C^\epsilon_{\alpha\beta\gamma\delta}.$$

If $u\in I^m_{bl}(\mathbb{R}^{k,1}\times\mathbb{R}^n;\mathbb{R}^{k,1}\times\{0\})$ such that

$$u=\int e^{iz\xi}\,a(x,y,z,\xi)\,đ\xi$$

with $a\in S^m_{bl}(\mathbb{R}^{k,1}\times\mathbb{R}^n;\mathbb{R}^n)$, and

$$\begin{aligned}u_r&=\int e^{iz\xi}\,\rho(x,y,z,r\xi)a(x,y,z,\xi)\,đ\xi\\&=\int e^{iz\xi}\,a_r(x,y,z,\xi)\,đ\xi\end{aligned}$$

with $r>0$. Then $u_r\in I^{-\infty}_{bl}(\mathbb{R}^{k,1}\times\mathbb{R}^n)$, and given any $\varphi\cdot\mu\in\dot{C}^\infty_c(\mathbb{R}^{k,1}\times\mathbb{R}^n,\Omega_b)$ with $\mu=\left|\frac{\mathrm{d}x}{x}\mathrm{d}y\mathrm{d}z\right|$,

$$\lim_{r\to0}u_r(\varphi)=u(\varphi).$$

Proof. Observer that for any multi-indexes α, β, γ and $\ell \in \mathbb{N}$

$$
\begin{aligned}
&\left|(x\partial_x)^\alpha \partial_y^\beta \partial_z^\gamma u_r\right| = \left|\int (x\partial_x)^\alpha \partial_y^\beta \partial_z^\gamma \left(\mathrm{e}^{iz\xi}\, \rho(x,y,z,r\xi)a(x,y,z,\xi)\right)\, \text{đ}\xi\right| \\
=&\left|\int \mathrm{e}^{iz\xi} \sum_{\substack{0\leqslant\alpha'\leqslant\alpha \\ \alpha_1+\alpha_2=\alpha' \\ \beta_1+\beta_2=\beta \\ \gamma_1+\gamma_2+\gamma_3=\gamma}} C_{\alpha'}^{\alpha} i^{|\gamma_3|}\xi^{\gamma_3}(x^{\alpha_1}\partial_x^{\alpha_1}\partial_y^{\beta_1}\partial_z^{\gamma_1}\rho)(x,y,z,r\xi)(x^{\alpha_2}\partial_x^{\alpha_2}\partial_y^{\beta_2}\partial_z^{\gamma_2}a)(x,y,z,\xi)\, \text{đ}\xi\right| \\
\leqslant&\widetilde{C}_{\alpha\beta\gamma}^{\ell}\int \frac{(1+|\xi|)^m}{(1+|r\xi|)^{2n+m}}\, \text{đ}\xi \cdot [(1+|\ln x|)(1+|y|+|z|)]^{-\ell},
\end{aligned}
$$

hence $u_r \in I_{bl}^{-\infty}(\mathbb{R}^{k,1}\times\mathbb{R}^n)$. Since ρ is bounded in z and a is Schwartz in z, a_r is Schwartz in z as well for any $r > 0$. Hence by Lemma A.1, we have

$$
u(\varphi) = \int \mathrm{e}^{iz\xi}\, a(x,y,z,\xi)\varphi(x,y,z)\, \text{đ}z\text{đ}\xi\frac{\mathrm{d}x}{x}\mathrm{d}y = \int a(x,y,\widecheck{\xi-\eta},\xi)\varphi(x,\check{\eta})\, \mathrm{d}\eta\mathrm{d}\xi\frac{\mathrm{d}x}{x}\mathrm{d}y
$$

and

$$
\begin{aligned}
u_r(\varphi) &= \int \mathrm{e}^{iz\xi}\, \rho(x,y,z,r\xi)a(x,y,z,\xi)\varphi(x,y,z)\, \text{đ}z\text{đ}\xi\frac{\mathrm{d}x}{x}\mathrm{d}y \\
&= \int \mathrm{e}^{iz(\xi-\eta)}\, \rho(x,y,z,r\xi)a(x,y,z,\xi)\varphi(x,y,\check{\eta})\, \text{đ}z\mathrm{d}\eta\mathrm{d}\xi\frac{\mathrm{d}x}{x}\mathrm{d}y \\
&= \int \mathrm{e}^{iz(\xi-\eta)}\, \rho(x,y,z,r\xi)a(x,y,z,\xi)\varphi(x,y,\check{\eta})\, \text{đ}z\mathrm{d}\xi\mathrm{d}\eta\frac{\mathrm{d}x}{x}\mathrm{d}y,
\end{aligned}
$$

where the employment of Fubini's theorem is justified by Lemma A.3 and the Peetre's inequality. Write

$$
\int \mathrm{e}^{iz(\xi-\eta)}\, \rho(x,y,z,r\xi)a(x,y,z,\xi)\text{đ}z = f_r(x,y,\xi,\eta),
$$

and

$$
\int \mathrm{e}^{iz(\xi-\eta)}\, a(x,y,z,\xi)\text{đ}z = f(x,y,\xi,\eta).
$$

Note that for any fixed (x, y, ξ), $\left|\mathrm{e}^{iz(\eta-\xi)}\,\rho(x, y, z, r\xi)a(x, y, z, \xi)\right| < C\,(1 + |z|)^{-2n}$ for some constant C, then by the dominated convergence theorem, $\lim_{r\to 0} f_r(x, y, \xi, \eta) = f(x, y, \xi, \eta)$. Now observing that

$$\begin{aligned} |f_r(x, y, \xi, \eta)\varphi(x, y, \check{\eta})| &\leqslant C' x\tilde{\varphi}(x)(1 + |y|)^{-2(k-1)}(1 + |\xi - \eta|)^{-2n}(1 + |\eta|)^{-4n} \\ &\leqslant C' x\tilde{\varphi}(x)(1 + |y|)^{-2(k-1)}(1 + |\xi|)^{-2n}(1 + |\eta|)^{-2n} \end{aligned}$$

for some constant C' and $\tilde{\varphi}(x) \in C_c^\infty([0, \infty))$, by Peetre's inequality, another application of the dominated convergence theorem shows that

$$\begin{aligned} \lim_{r\to 0} u_r(\varphi) &= \lim_{r\to 0} \int f_r(x, y, \xi, \eta)\varphi(x, y, \check{\eta})\ \mathrm{d}\xi\mathrm{d}\eta\frac{\mathrm{d}x}{x}\mathrm{d}y \\ &= \int f(x, y, \xi, \eta)\varphi(x, y, \check{\eta})\ \mathrm{d}\xi\mathrm{d}\eta\frac{\mathrm{d}x}{x}\mathrm{d}y \\ &= u(\varphi). \end{aligned}$$

□

Example A.7. Let $\pi_L : \mathbb{R}^{k,1} \times \mathbb{R}^n \longrightarrow \mathbb{R}^{k,1}$ be defined by $\pi_L(x, y, z) = (x, y)$. Assume that $u \in I^m_{bl}(\mathbb{R}^{k,1} \times \mathbb{R}^n; \mathbb{R}^{k,1} \times \{0\})$ associated with $a \in S^m_{bl}(\mathbb{R}^{k,1} \times \mathbb{R}^n; \mathbb{R}^n)$, and the left symbol of u is $\widetilde{a} \in S^m(\mathbb{R}^{k,1}; \mathbb{R}^n)$. Then the push-forward of u via π_L is defined. Moreover,

$$\begin{aligned} (\pi_L)_* u(x, y) &= \int \mathrm{e}^{iz\xi}\, a(x, y, z, \xi)\ \mathrm{d}z đ\xi \\ &= \int a(x, y, \widehat{0 - \xi}, \xi)\, đ\xi \\ &= \widetilde{a}(x, y, 0). \end{aligned}$$

On the other hand, let $u_r \in I^{-\infty}_{bl}(\mathbb{R}^{k,1} \times \mathbb{R}^n)$ be associated with $\widetilde{a}_r := \rho(r\xi)\widetilde{a} \in S^{-\infty}_{bl}(\mathbb{R}^{k,1}; \mathbb{R}^n)$, where $\rho \in \mathcal{S}(\mathbb{R}^n)$ with $\rho(0) = 1$. Then by the continuity principle,

$u_r \xrightarrow{r\to 0} u$, and hence $(\pi_L)_* u_r \xrightarrow{r\to 0} (\pi_L)_* u$, as distributions. More explicitly, we have

$$\begin{aligned}
(\pi_L)_* u = \lim_{r\to 0} (\pi_L)_* u_r &= \lim_{r\to 0} \int \mathrm{e}^{iz\xi}\, \widetilde{a}_r(x,y,\xi)\, đ\xi \mathrm{d}z \\
&= \lim_{r\to 0} \widetilde{a}_r(x,y,0) = \lim_{r\to 0} \rho(0)\widetilde{a}(x,y,0) \\
&= \widetilde{a}(x,y,0),
\end{aligned}$$

consistent with the previous computation. Note that, formally, we have

$$\int \mathrm{e}^{iz\xi}\, \widetilde{a}(x,y,\xi)\, đ\xi \mathrm{d}z = \widetilde{a}(x,y,0), \tag{A.7}$$

a well-known formula in Fourier analysis for Schwartz functions.

To study conormal distributions on compact manifolds with boundary, we need to work with boundary points as well as interior points. Henceforth, we will also denote the collection of ordinary m-th order symbols defined on $\mathbb{R}^{k+n}$ by $S^m_{bl}(\mathbb{R}^k;\mathbb{R}^n)$ in lieu of $S^m(\mathbb{R}^k;\mathbb{R}^n)$. The symbol estimates to use shall be clear from the domain of the symbols. Also, a subset $U \subset \mathbb{R}_x \times \mathbb{R}^{k-1}_y \times \mathbb{R}^n_z$ may be denoting $\mathbb{R}^k \times \mathbb{R}^n$ or $\mathbb{R}^{k,1} \times \mathbb{R}^n$, depending on the nature of the coordinate patch in question. Note that the first coordinate x may or may not be a boundary defining coordinate.

Theorem A.8. For any $m \in \mathbb{R}$, there is a linear map

$$\sigma_m : I^m_{bl}(X,Z) \to S^{[m]}_{bl}(N^*Z, \Omega_f(N^*Z))$$

such that the sequence

$$0 \longrightarrow I^{m-1}_{bl}(X,Z) \longrightarrow I^m_{bl}(X,Z) \xrightarrow{\sigma_m} S^{[m]}_{bl}(N^*Z, \Omega_f(N^*Z)) \longrightarrow 0$$

is exact.

Proof. We first define σ_m. Suppose that $u \in I^m_{bl}(X,Z)$. Let $\{\mathcal{U}_j \cong U_j \times \mathbb{R}^n \subset \mathbb{R}^k \times \mathbb{R}^n, \Phi_j\}$ be a (finite) coordinate chart cover of Z with compatible coordinates (as submanifold). Let $\{\varphi_j\}$ be a partition of unity for Z subordinate to $\{\mathcal{U}_j\}$, that is, a collection of smooth functions on X such that $\operatorname{supp}\varphi_j \subset \mathcal{U}_j$ and $\sum_j \varphi_j \equiv 1$ on Z. Then by definition we have

$$\varphi_j u = \int \mathrm{e}^{iz\cdot\xi}\, a_j(x,y,\xi)\, đ\xi$$

for some $a_j \in S^m_{bl}(U_j;\mathbb{R}^n)$. Let a^*_j be the pullback of $a_j(x,y,\xi)\cdot|\mathrm{d}\xi|$ to $S^m_{bl}(N^*Z\big|_{\mathcal{U}_j}, \Omega_f(N^*Z))$, and define

$$a := \sum_j a^*_j.$$

Then $\sigma_m(u)$ is defined by

$$\sigma_m(u) := [a] \in S^{[m]}_{bl}(N^*Z, \Omega_f(N^*Z)).$$

We must check that $\sigma(K)$ is well defined in the sense that it is independent of the choice of $\{\mathcal{U}_j, \varphi_j\}$. Therefore, Let $\{\mathcal{U}'_k, \Phi'_k\}$ be another coordinate chart cover of Z and $\{\varphi'_k\}$ be a partition of unity for Z subordinate to $\{\mathcal{U}'_k\}$, and b_k be the local left symbol of $\varphi'_k u$. Let $b := \sum_k b^*_k$. Note that

$$a = \sum_{j,k} (\varphi'_k)^* a^*_j,\ \ b = \sum_{j,k} \varphi^*_j b^*_k,$$

where φ^*_j and $(\varphi'_k)^*$ are the lifting of $\varphi_j\big|_Z$ and $\varphi'_k\big|_Z$ to N^*Z. If $\{\mathcal{U}_j \cap \mathcal{U}'_k \cap Z = \varnothing\}$, then $(\varphi'_k)^* a^*_j = 0 = \varphi^*_j b^*_k$; if $\{\mathcal{U}_j \cap \mathcal{U}'_k \cap Z \neq \varnothing\}$, then $\Phi'_k \Phi^{-1}_j(x,y,z) = (f,g,h)$ and $\Phi_j(\Phi'_k)^{-1}(u,v,w) = (f',g',h')$ with $h(x,y,0) = 0 = h'(u,v,0)$, and

$$\int \mathrm{e}^{iz\cdot\xi}\, \varphi'_k(x,y,z) a_j(x,y,\xi)\, đ\xi = \varphi_j \varphi'_k K = \int \mathrm{e}^{iw\cdot\xi}\, \varphi_j(u,v,w) b_k(u,v,\eta)\, đ\eta.$$

Hence, by the coordinate invariance of conormal distributions (e.g., see Proposition 1.2.12),

$$(\varphi_k')^* a_j^* - \varphi_j^* b_k^* \in S_{bl}^{m-1}(N^*Z\big|_{\mathcal{U}_j \cap \mathcal{U}_k'}, \Omega_f(N^*Z)).$$

By the arbitrariness of j, k, we conclude that $[a] = [b]$.

It is clear that $\ker(\sigma_m) = I_{bl}^{m-1}(X, Z)$. To see the surjectivity of σ_m, we reverse the previous process. Fix an arbitrary $a \in S_{bl}^m(N^*Z, \Omega_f(N^*Z))$. Recall that $\{\mathcal{U}_j\}$ covers Z and $\{\varphi_j\}$ is a partition of unity of Z subordinate to $\{\mathcal{U}_j\}$. Then for each j, there exists an $a_j \in S_{bl}^m(U_j; \mathbb{R}^n)$, such that the pullback of $\varphi_j^* a$ to $U_j \times \mathbb{R}^n$ is equal to a_j. In particular, the support of a_j in U_j is compact. Let u_j be the distribution conormal to Z with compact support in $\mathcal{U}_j$ determined by a_j as the local left symbol. Lastly, define a distribution on X in the following way: Given any $f \in C^\infty(X, \Omega(X))$,

$$\langle u, f \rangle := \sum_j \langle u_j, f \rangle.$$

Then in particular we have $\varphi_j u = u_j$ and $u \in I_{bl}^m(X, Z)$. Moreover, we have $\sigma_m(u) = [a]$, which implies that σ_m is surjective. $\square$

Appendix B

b-geometry and blow-ups

In this section, we review some basics of Melrose's b-geometry and blown-up spaces. Let X be a smooth manifold with boundary $\partial X = Y$.

B.1 b-type geometric objects

We commence with the *b-cotangent bundle* ${}^bT^*X$ of X. We will construct ${}^bT^*X$ by specifying its transition functions between local trivializations. Near a point $p \in \mathring{X}$, the transition functions are essentially the ones for the ordinary cotangent bundle. If $\{\mathcal{U}_O, \phi_O\}$, where $O = \alpha, \beta$, are coordinate patches of X with $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \partial X \neq \varnothing$, and denote

$$\phi_\alpha \circ \phi_\beta^{-1}(x, y_1, \ldots, y_{n-1}) = (x' A(x', \ldots, y'_{n-1}), \ldots, y_{n-1}(x', \ldots, y'_{n-1})) = (x, \ldots, y_{n-1}),$$

with $A(x', y') > 0$, then we define

$$g_{\alpha\beta} = \begin{bmatrix} 1 + \frac{x'}{A}\frac{\partial A}{\partial x'} & \frac{1}{A}\frac{\partial A}{\partial y_1'} & \cdots & \frac{1}{A}\frac{\partial A}{\partial y_{n-1}'} \\ x'\frac{\partial y_1}{\partial x'} & \frac{\partial y_1}{\partial y_1'} & \cdots & \frac{\partial y_1}{\partial y_{n-1}'} \\ \vdots & \vdots & \ddots \vdots & \\ x'\frac{\partial y_{n-1}}{\partial x'} & \frac{\partial y_{n-1}'}{\partial y_1'} & \cdots & \frac{\partial y_{n-1}}{\partial y_{n-1}'} \end{bmatrix} \tag{B.1}$$

Hence, ${}^bTX$ is defined as the unique smooth vector bundle over X determined by the transition functions prescribed above, and the *b-cotangent bundle* ${}^bT^*X$ is defined as the dual bundle of ${}^bTX$. In particular, the transition function of ${}^bT^*X$ over $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ is given by $(g_{\alpha\beta}^{-1})^T$. The b-cotangent bundles are the carriers of the principle symbols of the b-type pseudodifferential operators. Note that ${}^bTX$ was constructed in the way so that in the interior of X, ${}^bTX$ is isomorphic to TX, while near ∂X, $(x\partial_x, \partial_y) = (x\partial_x, \partial_{y_1}, \ldots, \partial_{y_{n-1}})$ is serving as a local ordered basis[1], and (B.1) just reflects this purpose. This assertion can be made precise by considering the following "realization"[2] of ${}^bTX$. Recall that, over $\mathcal{U}_\alpha$, an element in ${}^bTX$ can be represented by an equivalent class $[(x, y), v]$ where $v \in \mathbb{R}^n$. Let

$$\tau([(x, y), v]) = (x, y, v \cdot (x\partial_x, \partial_y)).$$

Then by (B.1), τ is a well-defined map from ${}^bTX$ to TX, and it is actually a vector bundle homomorphism. Furthermore, in $\mathring{X}$, τ is a vector bundle isomorphism. The transpose $\tau^t : T^*X \to {}^bT^*X$ is therefore also a bundle homomorphism restricting to an isomorphism in the interior of X. Moreover, in the interior of the coordinate patch $\mathcal{U}_\alpha$, $\big(\tau^t(\frac{\mathrm{d}x}{x}), \tau^t(\mathrm{d}y)\big)$ is a local ordered basis of ${}^bT^*X\big|_{\mathring{\mathcal{U}}_\alpha}$, which extends by continuity to a local basis $\big(\bar{\tau}^t(\frac{\mathrm{d}x}{x}), \bar{\tau}^t(\mathrm{d}y)\big)$ of ${}^bT^*X\big|_{\mathcal{U}_\alpha}$ over the entire $\mathcal{U}_\alpha$. Under this recognition, we will henceforth also denote $\big(\bar{\tau}^t(\frac{\mathrm{d}x}{x}), \bar{\tau}^t(\mathrm{d}y)\big)$ as $\big(\frac{\mathrm{d}x}{x}, \mathrm{d}y\big)$, and the following useful

[1]Similar notations like $(\frac{\mathrm{d}x}{x}, \mathrm{d}y) = (\frac{\mathrm{d}x}{x}, \mathrm{d}y_1, \ldots, \mathrm{d}y_{n-1})$, etc., are used below.
[2]For an intrinsic realization of ${}^bTX$ and more, see [25].

fact is readily verified:

Lemma B.1. Let X, X' be manifolds with boundary and $f : X \to X'$ a diffeomorphism. Let $\pi_E : E \to X'$ be a vector bundle over X' and $\varphi : T^*X \to E$ a bundle homomorphism covering f, that is, φ preserves the fiber structure and the diagram below commutes:

$$\begin{tikzcd} T^*X \arrow[r, "\varphi"] \arrow[d, "\pi_{T^*X}"'] & E \arrow[d, "\pi_E"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

Assume that $\{\varphi\big|_{\mathring{X}}(\frac{\mathrm{d}x}{x}),\ \varphi\big|_{\mathring{X}}(\mathrm{d}y)\}$ can be extended smoothly to ∂X. Denote the extended version by $\{\overline{\varphi\big|_{\mathring{X}}(\frac{\mathrm{d}x}{x})},\ \overline{\varphi\big|_{\mathring{X}}(\mathrm{d}y)}\}$. Define a map $\psi : {}^bT^*X \to E$ by setting

$$\psi(\bar{\tau}^t(\frac{\mathrm{d}x}{x})) = \overline{\varphi\big|_{\mathring{X}}(\frac{\mathrm{d}x}{x})},\ \psi(\bar{\tau}^t(\mathrm{d}y)) = \overline{\varphi\big|_{\mathring{X}}(\mathrm{d}y)}$$

and extending through linearity fiber-wisely. Then

$$\begin{tikzcd} {}^bT^*X \arrow[r, "\psi"] \arrow[d, "\pi_{{}^bT^*X}"'] & E \arrow[d, "\pi_E"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

is a bundle homomorphism diagram.

The b-density bundle over X is $\Omega_b(X) := \coprod_{p\in X} \Omega({}^bT^*_pX)$, where

$$\Omega({}^bT^*_pX) := \{c\,|\omega|\,\big|\omega \in \textstyle\bigwedge^n {}^bT^*X \text{ and } c \in \mathbb{R}\}.$$

The transition function $h_{\alpha\beta}$ of $\Omega_b(X)$ over $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ is just

$$h_{\alpha\beta} = \left|\det(g_{\alpha\beta}^{-1})^T\right| = \left|\det g_{\alpha\beta}\right|^{-1},$$

in particular, in the interior of $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, $h_{\alpha\beta} = A\left|\det J\right|^{-1}$, where J is the Jacobian matrix of $\phi_\alpha \circ \phi_\beta^{-1}$. On the other hand, in the interior of $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, as elements of (the ordinary density bundle) $\Omega(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$, we also have

$$\left|\frac{\mathrm{d}x'}{x'}\mathrm{d}y'\right| = A\left|\det J\right|^{-1}\left|\frac{\mathrm{d}x}{x}\mathrm{d}y\right|.$$

Hence, in view of the discussion of b-cotangent bundles above, it makes consistent sense to integrate sections of the b-density bundle over the entire manifold X. In particular, if $\mu \in C^\infty(X, \Omega_b(X))$ is supported inside a coordinate patch $\mathcal{U}_\alpha$ near ∂X, we can right $\mu = \mu(x,y) \cdot \left|\frac{\mathrm{d}x}{x}\mathrm{d}y\right|$, where $\left|\frac{\mathrm{d}x}{x}\mathrm{d}y\right|$ is technically $\left|\tau^t(\frac{\mathrm{d}x}{x}) \wedge \tau^t(\mathrm{d}y)\right|$, and the integration of μ is

$$\int_X \mu := \int_{\phi_\alpha(\mathcal{U}_\alpha)} \mu(x,y)\,\frac{\mathrm{d}x}{x}\mathrm{d}y. \tag{B.2}$$

Note that the value of (B.2) may or may not be finite.

Since ${}^bT^*X$ is a manifold with boundary on its own, one can define $\Omega_b({}^bT^*X)$ accordingly. Suppose that (x,y,ξ) are local coordinates associated to a coordinate patch, say, $\{\mathcal{U}_\alpha, (x,y)\}$, of X, then $\left|\frac{\mathrm{d}x}{x}\mathrm{d}y\mathrm{d}\xi\right|$ is a local trivialization of $\Omega_b({}^bT^*X)$. The observation is that arguing exactly like the case for the ordinary density bundle $\Omega(T^*X)$, one can show that $\left|\frac{\mathrm{d}x}{x}\mathrm{d}y\mathrm{d}\xi\right|$ is defined invariantly. Hence, there is a canonical global trivialization of $\Omega_b({}^bT^*X)$, and in particular, sections of b-density on ${}^bT^*X$ are identified with functions on ${}^bT^*X$.

B.2 Conormal bundles

Now we turn to the conormal bundles of submanifolds, which play a fundamental role in the study of general conormal distributions.

Let $Z \subset X$ be an embedded submanifold. Define the vector bundle $\pi : N^*Z \to Z$ with

$$N_p^*Z := \{\xi \in T_p^*X \big| \xi(v) = 0, \forall v \in T_pZ\}.$$

and $N^*Z := \coprod_{p\in Z} N_p^*Z$. Note that N^*Z is a sub-bundle of $T^*X\big|_Z$ and is called the *conormal bundle* of Z. It is, of course, just the dual bundle of $NZ := TX/TZ$, the *normal bundle* of Z.

Assume that Z intersects $\partial X = Y$ transversally. This condition can be interpreted in terms of local coordinates: there exists a coordinate patch $\{\mathcal{U}, (x, y, z)\} = \{\mathcal{U}, (x, y_1, \ldots, y_k,$
$z_1, \ldots, z_\ell)\}$ near $Z \cap Y$ with x the boundary defining coordinate, such that $Z \cap \mathcal{U} = \{(x, y, 0)\}$. In such a coordinate system, an element ξ in $N_p^* Z$ where $p = (x, y, 0)$ can be represented by

$$\xi = \sum_{j=1}^{\ell} \xi_j \, \mathrm{d}z_j.$$

Hence, a local coordinates system of N^*Z associated to $\{\mathcal{U}, (x, y, z)\}$ is given by

$$\{\pi^{-1}(\mathcal{U} \cap Z), (x, y, \xi)\} = \{\pi^{-1}(\mathcal{U} \cap Z), (x, y_1, \ldots, y_k, \xi_1, \ldots, \xi_\ell)\},$$

where ξ_i satisfies

$$\xi_i(\sum_j \eta_j \, \mathrm{d}z_j) = \eta_i.$$

Now assume that $\{\mathcal{U}', (x', y', z')\}$ is another coordinate patch of X such that $\mathcal{U} \cap \mathcal{U}' \cap Z \cap \partial X \neq \varnothing$, and moreover, $Z \cap \mathcal{U}' = \{(x', y', 0)\}$. Then, the change of coordinates formula over $\pi^{-1}(\mathcal{U} \cap \mathcal{U}' \cap Z)$ has the following form

$$x' = x'(x, y, 0), y' = y'(x, y, 0), \xi' = (H^{-1})^T(x, y, 0)\xi,$$

where $x'(x, y, z), y'(x, y, z), z'(x, y, z)$ are the transition maps from $\mathcal{U}$ to $\mathcal{U}'$ and

$$H(x, y, z) = \begin{bmatrix} \frac{\partial z_1'}{\partial z_1} & \cdots & \frac{\partial z_1'}{\partial z_\ell} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_\ell'}{\partial z_1} & \cdots & \frac{\partial z_\ell'}{\partial z_\ell} \end{bmatrix}. \tag{B.1}$$

Note that the coordinate transition formula near an interior point of X in Z can be analyzed in the same way.

Recall that the (1-)density bundle of N_p^*Z as manifold is

$$\Omega(N_p^*Z) := \coprod_q \Omega(T_q^*(N_p^*Z)),$$

where

$$\Omega(T_q^*(N_p^*Z)) := \{c\,|\omega|\,\big|\omega \in \textstyle\bigwedge^\ell T_q^*(N_p^*Z) \text{ and } c \in \mathbb{R}\}.$$

Since (ξ_i) are (global) coordinates of N_p^*Z for any fixed $p \in Z$, $\mathrm{d}\xi := |\mathrm{d}\xi_1 \wedge \cdots \wedge \mathrm{d}\xi_\ell|$ is a basis of $\Omega(N_p^*Z)$. We then construct a line bundle $\Omega_f(N^*Z)$ on N^*Z whose fiber over the point $(p,\xi) = (x,y,\xi)$ is just $\Omega(N_p^*Z)$, that is,

$$\Omega_f(N^*Z) := \coprod_{(p,\xi)\in N^*Z} \Omega(N_p^*Z).$$

The subscript "f" here stands for "fiberwise", against the genuine density bundle $\Omega(N^*Z) := \coprod_{p\in N^*Z} \Omega(T_p^*(N^*Z))$. Through the discussion above, one sees that the transition function (of the naturally associated local trivializations) from $\pi^{-1}(\mathcal{U}' \cap Z)$ to $\pi^{-1}(\mathcal{U} \cap Z)$ is just

$$g_{\mathcal{U},\mathcal{U}'}(p,\xi) = |\det H(p)|^{-1},$$

where $(p,\xi) \in \pi^{-1}(\mathcal{U}' \cap Z) \cap \pi^{-1}(\mathcal{U} \cap Z)$ and H was defined in (B.1).

Note that $\Omega_f(N^*Z)$ can be interpreted as the pullback of the "normal density bundle" over Z via the projection $\pi : N^* \to Z$. Similarly in the spirit, we can consider the pullback of the "tangent b-density bundle" over Z. Hence, define

$$\Omega_{b,t}(N^*Z) := \coprod_{(p,\xi)\in N^*Z} \Omega({}^bT_p^*Z), \tag{B.2}$$

and we remark that

$$\Omega_f(N^*Z) \otimes \Omega_{b,t}(N^*Z) \cong \Omega_b(N^*Z). \tag{B.3}$$

B.3 Blow-ups

We review the notion of blow-ups introduced by R. Melrose.

When taking finite products of manifolds with boundary, the notion of manifolds with corners arises naturally. A manifold with corners is locally modeled $\mathbb{R}^{n,k} \cong [0,\infty)^k \times \mathbb{R}^{n-k}$, where $0 \leqslant k \leqslant n$. However, the theory is not universally agreed, depending on the concrete applications, and there are a couple of inequivalent definitions of manifolds with corners and smooth map, see [10]. We follow the sense of Melrose in [25]. See also [13], [14], [22]. The prominent feature of this definition is that the (topological) boundary of a manifold with corners is a finite union of hypersurfaces, i.e., co-dimension 1 embedded submanifolds. Moreover, the boundary faces, that is, the intersections of boundary hypersurfaces, are also embedded submanifolds. Note that if a boundary face is an embedded submanifold, then it is indeed a p-submanifold, i.e., it is locally $\mathbb{R}^{n',k'} \times \{0\} \subset \mathbb{R}^{n,k}$ with $n' \leqslant n$ and $k' \leqslant \min\{n', k\}$. We remark that this is important in the discussion of blow-ups. Main examples are provided by finite products of manifolds with boundary.

Lemma B.2. If

$$\begin{aligned} \Phi : \mathbb{R}^{n,k} &\longrightarrow \mathbb{R}^{n,k} \\ (x_1, \cdots, x_n) &\longmapsto (\phi_1, \cdots, \phi_n) \end{aligned}$$

is a diffeomorphism such that $\phi_i(\cdots, x_{i-1}, 0, x_{i+1}, \cdots) = 0$ for $1 \leqslant i \leqslant k$, then

$$\phi_i = x_i \tilde{\phi}_i$$

for some $\tilde{\phi}_i \in C^\infty(\mathbb{R}^{n,k})$ with $\tilde{\phi}_i > 0$.

Proof. By Taylor's theorem, it is clear that $\phi_i = x_i\tilde{\phi}_i$ with $\tilde{\phi}_i \in C^\infty(\mathbb{R}^{n,k})$. Note that when $x_i > 0, 0 \leqslant i \leqslant k$, then $\phi_i > 0$, hence $\tilde{\phi}_i > 0$.

Observe that the Jacobian matrix of Φ is

$$J_\Phi = \begin{bmatrix} x_1\partial_{x_1}\tilde{\phi}_1 + \tilde{\phi}_1 & \cdots & x_1\partial_{x_k}\tilde{\phi}_1 & x_1\partial_{x_{k+1}}\tilde{\phi}_1 & \cdots & x_1\partial_{x_n}\tilde{\phi}_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_k\partial_{x_1}\tilde{\phi}_k & \cdots & x_k\partial_{x_k}\tilde{\phi}_k + \tilde{\phi}_k & x_k\partial_{x_{k+1}}\tilde{\phi}_k & \cdots & x_k\partial_{x_n}\tilde{\phi}_k \\ \partial_{x_1}\tilde{\phi}_{k+1} & \cdots & \partial_{x_k}\tilde{\phi}_{k+1} & \partial_{x_{k+1}}\tilde{\phi}_{k+1} & \cdots & \partial_{x_n}\tilde{\phi}_{k+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1}\tilde{\phi}_n & \cdots & \partial_{x_k}\tilde{\phi}_n & \partial_{x_{k+1}}\tilde{\phi}_n & \cdots & \partial_{x_n}\tilde{\phi}_n \end{bmatrix}.$$

Since $\det(J_\Phi) \neq 0$, by extending the determinant along the i-th row, it follows that

$$\tilde{\phi}_i(\cdots, x_{i-1}, 0, x_{i+1}, \cdots) \neq 0.$$

By continuity, we have

$$\tilde{\phi}_i(\cdots, x_{i-1}, 0, x_{i+1}, \cdots) > 0. \qquad \square$$

Corollary B.2.1. The b-derivatives are preserved by diffeomorphisms on $\mathbb{R}^{n,k}$. In particular,

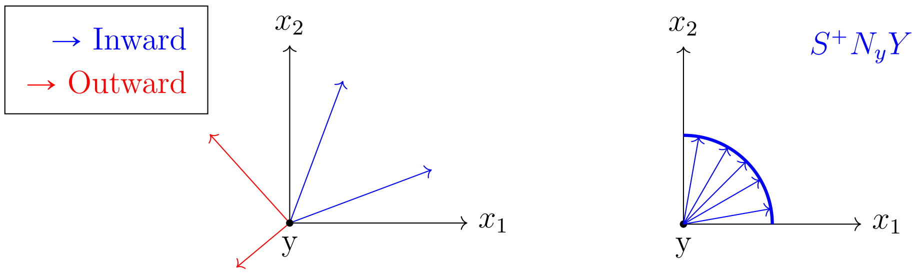
- for $1 \leqslant i \leqslant k$,

$$x_i\partial_{x_i} = \left(1 + \frac{x_i}{\tilde{\phi}_i}\frac{\partial\tilde{\phi}_i}{\partial x_i}\right)x_i'\partial_{x_i'} + \sum_{j\neq i, j=1}^{k}\frac{x_i}{\tilde{\phi}_j}\frac{\partial\tilde{\phi}_j}{\partial x_i}x_j'\partial_{x_j'} + \sum_{j=k+1}^{n} x_i\frac{\partial\phi_j}{\partial x_i}\partial_{x_j'};$$

- for $k+1 \leqslant i \leqslant n$,

$$\partial_{x_i} = \sum_{j=1}^{k}\frac{1}{\tilde{\phi}_j}\frac{\partial\tilde{\phi}_j}{\partial x_i}x_j'\partial_{x_j'} + \sum_{j=k+1}^{n}\frac{\partial\phi_j}{\partial x_i}\partial_{x_j'}.$$

Let M be a n-dimensional manifold with corners. Suppose that N is a closed l-dimensional p-submanifold of M. In particular, near any point $y \in N$, $M \cong \mathbb{R}^{l,k} \times \mathbb{R}^{j,i}$ centered at y and $N \cong \mathbb{R}^{l,k} \times \{0\}$. One can define "$M$ blown-up at N ", $[M; N]$,



(a) Inward/Outward Pointing Vectors (b) Spherical Normal Vectors

Figure B.1: Abstract Sphere

by introducing polar coordinate about N. As a set, $[M;N]$ is obtained by replacing each point in N by an "abstract sphere". Precisely, we define, at y (see Figure B.1):

inward-pointing tangent vectors in M: $T_y^+M := \{\sum_{j=1}^{n} a_j \partial_{x_j}; a_j \geqslant 0 \text{ for } j \leqslant l\}$.

inward-pointing normal vectors to N: $N_y^+N := T_y^+M/T_y^+N$.

inward-pointing spherical normal vectors to N: $S^+N_yN := (N_y^+N \backslash \{0\})/\mathbb{R}^+$.

inward-pointing spherical normal bundle to N: $S^+NN := \coprod_{y\in N} S^+N_yN$.

M blown-up along N: $[M;N] := S^+NN \coprod (M\backslash N)$.

The introduction of polar coordinates turns $[M;N]$ into a manifold with corners. To illustrate this procedure, we investigate a pertinent example.

Let X be a manifold with boundary and $Y = \partial X$. Then Y^2 is a boundary face of X^2, hence a p-submanifold. The *stretched double product of X* is

$$X_b^2 := [X^2; Y^2].$$

Over $\Omega' = \big(\Omega\backslash Y\big) \coprod \big(S^+NY|_{\Omega\cap Y}\big)$ in X_b^2, a chart $\phi : \Omega' \to \mathbb{R}^+ \times S^{1,2} \times \mathbb{R}^{2(n-1)}$ is

Figure B.2: Stretched Double Product

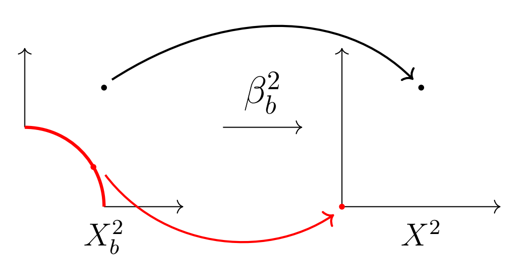
given by

$$\begin{cases} \phi([a\partial_{x_1} + b\partial_{x_2}], y) = (0, \dfrac{(a,b)}{\sqrt{a^2+b^2}}, y), & \text{if } ([a\partial_{x_1} + b\partial_{x_2}], y) \in S^+NY, \\ \phi(x_1, x_2, y) = (\sqrt{x_1^2+x_2^2}, \dfrac{(x_1,x_2)}{\sqrt{x_1^2+x_2^2}}, y), & \text{if } (x_1, x_2, y) \in \Omega\backslash Y. \end{cases}$$

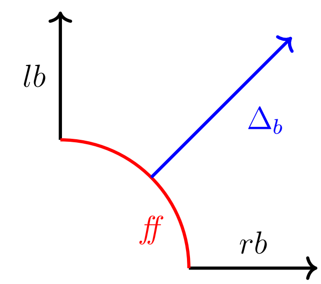
Assume that $\Phi(x_1, x_2, y) = (x_1 A, x_2 B, C)$ is a diffeomorphism on $\mathbb{R}^{2n,2}$, then the transition map induced by Φ is

$$\widetilde{\Phi}(r, \omega, y) = (r\sqrt{(A\omega_1)^2 + (B\omega_2)^2}, \frac{(A\omega_1, B\omega_2)}{\sqrt{(A\omega_1)^2 + (B\omega_2)^2}}, C).$$

Note that A, B, C are smooth functions with $A, B > 0$ by Lemma B.2, hence $\widetilde{\Phi}$ is also smooth. Thus, X_b^2 is equipped with a smooth structure, which makes it into a manifold with corners. Other coordinate systems than the polar coordinates can be introduced on X_b^2. For instance, the local functions $\rho(r, \omega, y) = r\omega_1 + r\omega_2$ and $s(r, \omega, y) = \ln(\frac{\omega_1}{\omega_2})$ are smooth, and (ρ, s, y) is a coordinate system on X_b^2 mapping an



(a) Blow-down map



(b) Submanifolds in X_b^2

Figure B.3: Blow-down Map and Submanifolds

open subset to $[0,\infty)\times\mathbb{R}\times\mathbb{R}^{2(n-1)}$. The map

$$\begin{aligned}\beta_b^2 : X_b^2 &\longrightarrow X^2\\ x &\longmapsto \begin{cases} x, & \text{if } x \in X^2\backslash Y^2\\ y, & \text{if } x \in S^+N_yY^2\end{cases}\end{aligned}$$

is called the *blow-down map*. See Figure B.3a. Note that $\beta_b^2 \in C^\infty(X_b^2, X^2)$. Important p-submanifolds of X_b^2 include (Figure B.3b):

- left boundary, $lb = \mathrm{cl}\Big((\beta_b^2)^{-1}(\partial X \times \mathring{X})\Big)$
- right boundary, $rb = \mathrm{cl}\Big((\beta_b^2)^{-1}(\mathring{X} \times \partial X)\Big)$
- front face, $ff = S^+NY^2$
- b-diagonal, $\Delta_b = \mathrm{cl}((\beta_b^2)^{-1}(\Delta\backslash Y^2))$

Note that $\partial X_b^2 = lb \cup rb \cup ff$.

Lemma B.3. Let $\pi_{L,b}, \pi_{R,b} : \Delta_b \to X$ be the restriction of the b-projections at the b-diagonal. Then $(\pi_{L,b}|_{\mathring{\Delta}_b})^* - (\pi_{R,b}|_{\mathring{\Delta}_b})^*$ extends to a bundle isomorphism

$$\pi_{L,b}^* - \pi_{R,b}^* : {}^bT^*X \longrightarrow N^*\Delta_b.$$

Proof. We use local coordinates. Near ∂X, we have $X \cong \mathbb{R}^{n,1}$ and use the coordinates (x, y); near $\Delta_b \cap \mathit{ff}$, we have $X_b^2 \cong \mathbb{R}^{n,1} \times \mathbb{R}^n$ and use coordinates $(x, y, s, z) = (x, y, \ln \frac{x'}{x}, y - y')$. Note that $\Delta_b \cap \mathcal{O} = \{(x, y, 0, 0)\}$, and $T\Delta_b\big|_{\Delta_b \cap \mathcal{O}} = \mathrm{span}\{\partial_x, \partial_y\}$. Hence,

$$(\pi_{L,b}\big|_{\mathring{\Delta}_b})^* - (\pi_{R,b}\big|_{\mathring{\Delta}_b})^*(\frac{\mathrm{d}x}{x}) = \mathrm{d}s, \ (\pi_{L,b}\big|_{\mathring{\Delta}_b})^* - (\pi_{R,b}\big|_{\mathring{\Delta}_b})^*(\mathrm{d}y) = \mathrm{d}z.$$

Since $\{\mathrm{d}s, \, \mathrm{d}z\}$ is a local frame of $N^*\Delta_b$ near $\mathit{ff} \cap \Delta_b$, the claim then follows from Lemma B.1. □

Corollary B.3.1. $C^\infty(N^*\Delta_b, \Omega_f(N^*Z) \otimes \Omega_{b,t}(N^*Z)) \cong C^\infty({}^bT^*X, \Omega_b({}^bT^*X)) \cong C^\infty({}^bT^*X)$.

Proof. It follows from the lemma and (B.3). □

The *stretched triple product*, X_b^3, is defined as the iterated blow-up

$$X_b^3 := [X^3; \mathcal{T}; \{\mathcal{B}_F, \mathcal{B}_S, \mathcal{B}_C\}],$$

where $\mathcal{T} = Y^2 \subset X^3$, $\mathcal{B}_F = Y \times Y \times X = \pi_F^{-1}(Y \times Y)$, $\mathcal{B}_S = X \times Y \times Y = \pi_S^{-1}(Y \times Y)$, and $\mathcal{B}_C = Y \times X \times Y = \pi_C^{-1}(Y \times Y)$. It arises essentially when one studies the composition of pseudodifferential operators. We rely heavily on the following result, whose formulation was adapted from [15].

Theorem B.4. There exist unique continuous functions $\pi_{O,b} : X_b^3 \longrightarrow X_b^2$, where $O = F, S$ or C, such that

$$\pi_O \circ \beta_b^3 = \beta_b^2 \circ \pi_{O,b} \tag{B.1}$$

Proof. (Sketch) Note that, in the interiors, the left hand side of B.1 is just the ordinary projection onto the correspondent components, and β_b^2 acts as identity, hence $\pi_{O,b}$ has to be defined canonically by π_O. The uniqueness of $\pi_{O,b}$ then follows from the continuity. The existence could be established by exhibiting local formulas in terms of coordinates. See, e.g., the formulas in the following Corollary. $\square$

Corollary B.4.1. Let $\mathcal{V} \cong \mathbb{R}^{n-1}$ be a coordinate patch of $Y = \partial X$. Near the boundary of X, we have $X \cong [0,1)_x \times \mathcal{V}_y$. Moreover, $X^2 \cong [0,1)^2_{(x,x')} \times \mathcal{V}^2_{(y,y')}$ and $X^3 \cong [0,1)^3_{(x,x',x'')} \times \mathcal{V}^3_{(y,y',y'')}$. Then, in local coordinates, we have the following representations of $\pi_{O,b}$, where the coordinates in X^2_b near $rb(X^2_b) \cap \mathit{ff}(X^2_b)$, $\Delta_b(X^2_b) \cap \mathit{ff}(X^2_b)$ and $lb(X^2_b) \cap \mathit{ff}(X^2_b)$ are given by $(x, \omega, y, y') := (x, \ln \frac{x'}{x}, y, y')$, $(x, \omega, y, z) := (x, \ln \frac{x'}{x}, y, y - y')$ and $(\gamma, x', y, y') := (\ln \frac{x}{x'}, x', y, y')$, respectively:

I. Near $fs \cap \mathit{ff} \cap mb \in X^3_b$, we use coordinates

$$(s, t, x'', y, y', y'') := (\ln \frac{x}{x''}, \ln \frac{x'}{x}, x'', y, y', y'') \tag{B.2}$$

then $(s, t, x'', y, y', y'') \in (-\infty, 0) \times (-\infty, 0) \times [0, 1) \times \mathbb{R}^3$ and

$$\begin{aligned}
\pi_{F,b}(s, t, x'', y, y', y'') &= (x'' e^s, t, y, y') \text{ near } rb \cap \mathit{ff} \in X^2_b;\\
\pi_{S,b}(s, t, x'', y, y', y'') &= (s + t, x'', y', y'') \text{ near } lb \cap \mathit{ff} \in X^2_b;\\
\pi_{C,b}(s, t, x'', y, y', y'') &= (s, x'', y, y') \text{ near } rb \cap \mathit{ff} \in X^2_b.
\end{aligned} \tag{B.3}$$

II. Near $fs \cap lb \cap \mathit{ff} \in X^2_b$, we use coordinates

$$(s, t, x'', y, y', y'') := (\ln \frac{x}{x'}, \ln \frac{x'}{x''}, x'', y, y', y'') \tag{B.4}$$

then $(s, t, x'', y, y', y'') \in (-\infty, 0) \times (-\infty, 0) \times [0, 1) \times \mathbb{R}^3$ and

$$\begin{aligned}
\pi_{F,b}(s, t, x'', y, y', y'') &= (s, x'' e^t, y, y') \text{ near } lb \cap \mathit{ff} \in X^2_b;\\
\pi_{S,b}(s, t, x'', y, y', y'') &= (t, x'', y', y'') \text{ near } lb \cap \mathit{ff} \in X^2_b;\\
\pi_{C,b}(s, t, x'', y, y', y'') &= (s + t, x'', y, y') \text{ near } lb \cap \mathit{ff} \in X^2_b.
\end{aligned} \tag{B.5}$$

III. Near $\mathit{ff} \cap cs \cap lb \in X_b^2$, we use coordinates

$$(s, x', t, y, y', y'') := (\ln \frac{x}{x''}, x', \ln \frac{x''}{x'}, y, y', y'') \tag{B.6}$$

then $(s, x', t, y, y', y'') \in (-\infty, 0) \times [0, 1) \times (-\infty, 0) \times \mathbb{R}^3$ and

$$\begin{aligned}
\pi_{F,b}(s, x', t, y, y', y'') &= (s + t, x', y, y') \text{ near } lb \cap \mathit{ff} \in X_b^2; \\
\pi_{S,b}(s, x', t, y, y', y'') &= (x', t, y', y'') \text{ near } rb \cap \mathit{ff} \in X_b^2; \\
\pi_{C,b}(s, x', t, y, y', y'') &= (s, x'e^t, y, y') \text{ near } lb \cap \mathit{ff} \in X_b^2.
\end{aligned} \tag{B.7}$$

IV. Near $\pi_{F,b}^{-1}(\Delta_b) \cap \pi_{S,b}^{-1}(\Delta_b)$, we use coordinates

$$(x, s, t, y, z, w) := (x, \ln \frac{x'}{x}, \ln \frac{x''}{x}, y, y - y', y - y'') \tag{B.8}$$

then

$$\begin{aligned}
\pi_{F,b}(x, s, t, y, z, w) &= (x, s, y, z) \text{ near } \Delta_b \cap \mathit{ff} \in X_b^2; \\
\pi_{S,b}(x, s, t, y, z, w) &= (xe^s, t - s, y - z, w - z) \text{ near } \Delta_b \cap \mathit{ff} \in X_b^2; \\
\pi_{C,b}(x, s, t, y, z, w) &= (x, t, y, w) \text{ near } \Delta_b \cap \mathit{ff} \in X_b^2.
\end{aligned} \tag{B.9}$$

V. Near $\pi_{F,b}^{-1}(\Delta_b) \cap fs$. In X_b^2 near $\mathit{ff}(X_b^2) \cap lb$, we use the coordinates

$$(x', y, \gamma, z) := (x', y, \ln \frac{x}{x'}, y - y'),$$

and in X_b^3, we use coordinates

$$(x'', y, s, w, t, z) := (x'', y, \ln \frac{x}{x''}, y - y'', \ln \frac{x}{x'}, y - y'). \tag{B.10}$$

Note that $X_b^3 \cong X_b^2 \times \mathbb{R}^n_{(t,z)}$ and $\pi_{F,b}^{-1}(\Delta_b(X_b^2)) \cong X_b^2 \times \{0\}$. Then

$$\begin{aligned}
\pi_{F,b}(x'', y, s, w, t, z) &= (x'' \mathrm{e}^{s-t}, y, t, z) \text{ near } \Delta_b \cap \mathit{ff} \in X_b^2, \\
\pi_{S,b}(x'', y, s, w, t, z) &= (x'', y - z, s - t, w - z) \text{ near } lb \cap \mathit{ff} \in X_b^2, \\
\pi_{C,b}(x'', y, s, w, t, z) &= (x'', y, s, w) \text{ near } rb \cap \mathit{ff} \in X_b^2.
\end{aligned} \tag{B.11}$$

VI. Near $\pi_{F,b}^{-1}(\Delta_b) \cap rb$. In X_b^2 near $\mathit{ff}(X_b^2) \cap rb$, we use the coordinates

$$(x, y, \omega, z) := (x, y, \ln \frac{x'}{x}, y - y'),$$

and in X_b^3, we use coordinates

$$(x, y, s, w, t, z) := (x, y, \ln \frac{x''}{x}, y - y'', \ln \frac{x'}{x}, y - y'). \tag{B.12}$$

Note that $X_b^3 \cong X_b^2 \times \mathbb{R}^n_{(t,z)}$ and $\pi_{F,b}^{-1}(\Delta_b(X_b^2)) \cong X_b^2 \times \{0\}$. Then

$$\begin{aligned}
\pi_{F,b}(x, y, s, w, t, z) &= (x, y, t, z) \text{ near } \Delta_b \cap \mathit{ff} \in X_b^2, \\
\pi_{S,b}(x, y, s, w, t, z) &= (x\,\mathrm{e}^{t}, y - z, s - t, w - z) \text{ near } lb \cap \mathit{ff} \in X_b^2, \\
\pi_{C,b}(x, y, s, w, t, z) &= (x, y, s, w) \text{ near } rb \cap \mathit{ff} \in X_b^2.
\end{aligned} \tag{B.13}$$

VII. Near $\pi_{S,b}^{-1}(\Delta_b) \cap ss$. In X_b^2 near $\mathit{ff} \cap rb$, we use the coordinates

$$(x, y, \omega, z) := (x, y, \ln \frac{x'}{x}, y - y'),$$

and in X_b^3, we use coordinates

$$(x, y, s, w, t, z) := (x, y, \ln \frac{x''}{x}, y - y'', \ln \frac{x''}{x'}, y - y'). \tag{B.14}$$

Note that $X_b^3 \cong X_b^2 \times \mathbb{R}^n_{(t,z)}$ and $\pi_{S,b}^{-1}(\Delta_b(X_b^2)) \cong X_b^2 \times \{0\}$. Then

$$\begin{aligned}
\pi_{F,b}(x, y, s, w, t, z) &= (x, y, t - s, z) \text{ near } rb \cap \mathit{ff} \in X_b^2, \\
\pi_{S,b}(x, y, s, w, t, z) &= (x\,\mathrm{e}^{t-s}, y - z, t, w - z) \text{ near } \Delta_b \cap \mathit{ff} \in X_b^2, \\
\pi_{C,b}(x, y, s, w, t, z) &= (x, y, s, w) \text{ near } rb \cap \mathit{ff} \in X_b^2.
\end{aligned} \tag{B.15}$$

VIII. Near $\pi_{S,b}^{-1}(\Delta_b) \cap lb$. In X_b^2 near $\mathit{ff} \cap lb$, we use the coordinates

$$(x', y, \gamma, z) := (x', y, \ln\frac{x}{x'}, y - y'),$$

and in X_b^3, we use coordinates

$$(x'', y, s, w, t, z) := (x'', y, \ln\frac{x}{x''}, y - y'', \ln\frac{x'}{x''}, y - y'). \tag{B.16}$$

Note that $X_b^3 \cong X_b^2 \times \mathbb{R}^n_{(t,z)}$ and $\pi_{S,b}^{-1}(\Delta_b(X_b^2)) \cong X_b^2 \times \{0\}$. Then

$$\begin{aligned}
\pi_{F,b}(x'', y, s, w, t, z) &= (x''\,\mathrm{e}^{t}, y, s - t, z) \text{ near } lb \cap \mathit{ff} \in X_b^2, \\
\pi_{S,b}(x'', y, s, w, t, z) &= (x'', y - z, t, w - z) \text{ near } \Delta_b \cap \mathit{ff} \in X_b^2, \\
\pi_{C,b}(x'', y, s, w, t, z) &= (x, y, s, w) \text{ near } lb \cap \mathit{ff} \in X_b^2.
\end{aligned} \tag{B.17}$$

Appendix C

Fredholm without Banach and Hilbert

We give a detailed description of an unconventional approach which, as far as we know, is due to Paul Loya and not published in literature yet, to establishing Fredholm property of operators on various function spaces.

Let X be a compact manifold with or without boundary. Denote $Y = \partial X$. Note that Y might be a empty set. Let μ be a measure on X. $C(X)$ is the Banach space of complex-valued, continuous functions on X with the sup-norm. Let Φ be a vector subspace of $C(X) \cap L^2(X;\mu)$. We assume that Φ is closed under complex conjugate, hence Φ is naturally equipped with an (Hermitian) inner product, namely, the L^2-inner product. However, we do not assume that the metric on Φ induced by the inner product is complete, and we remark that this is the major advantage of this approach and justifies the section title. Denote the collection of linear operators on Φ by $\mathscr{L}(\Phi)$. Suppose that $\Psi(X)$ is a vector subspace of $\mathscr{L}(\Phi)$, and $\Psi_0^{-\infty}(X) \subset \Psi(X)$ a subalgebra of $\mathscr{L}(\Phi)$, such that $\Psi(X) \cdot \Psi_0^{-\infty}(X) \subset \Psi_0^{-\infty}(X)$. In applications, operators in $\Psi_0^{-\infty}(X)$ are usually identified with continuous integral kernels against the measure μ. The major examples are provided, of course, by the various types of pseudodifferential operators. Henceforth, we will not distinguish the operators in $\Psi_0^{-\infty}(X)$ from their integral kernels and assume that $\Psi_0^{-\infty}(X) \subset C(X^2)$. The subscript 0 in $\Psi_0^{-\infty}(X)$ alludes to some "regularities", in addition to being "residual". whose meaning depends on the concrete calculus in question. For example, in b- or *bl*-calculus, the additional condition is the vanishing of the normal operator. Analogous regularity conditions are available for other b-type calculi, or heat calculus on closed manifolds.

Assume that $A \in \mathscr{L}(\Phi)$, $B \in \Psi(X)$ and $R \in \Psi_0^{-\infty}(X)$ such that $AB = \mathrm{Id} - R$. The first and main step of our approach is to refine the remainder R, which we will carry out momentarily. The ideal situation is that the remainder term R is given by some "finite rank" operator. We begin with a definition.

Definition C.1. Let $\mathcal{B} \subset \Psi$ be an algebra closed under complex conjugate. A function K in $\mathscr{F}_{\mathcal{B}} := \mathcal{B} \otimes \mathcal{B} \subset C(X^2)$ is called *$\mathcal{B}$-finite rank.*

Explicitly, K is $\mathcal{B}$-finite rank if it is of the form

$$K(x, x') = \sum_{j=1}^{k} f_j(x) g_j(x')$$

where $f_j, g_j \in \mathcal{B}$. Note that $\mathscr{F}_{\mathcal{B}}$ is a subalgebra of $C(X^2)$ with the multiplication induced by the one in $\mathcal{B}$. The elements in $\mathscr{F}_{\mathcal{B}}$ are meant to be serving as the integral kernel of some operators with finite dimensional range. In fact, given any $\varphi \in \Phi$, since $\mathcal{B} \subset \Phi \subset L^2(X; \mu)$, we define

$$K\varphi := \int \sum_{j=1}^{k} f_j(x) g_j(x') \varphi(x') \mu(x') = \sum_{j=1}^{k} f_j(x) \int g_j(x') \varphi(x') \mu(x').$$

In particular, $\mathrm{Im} K \subset \mathrm{span}\{f_j\}$. On the other hand, the operators with a finite rank integral kernel are not *the* finite-rank operators usually defined only on Banach spaces. In the contrast, A continuous linear operator on some subspace of $C(X)$ with finite-dimensional image is not necessarily given by a finite rank integral kernel. A simple counterexample is given by $T := \mathbf{1}\delta_p$ on $C^\infty(X)$, where $\mathbf{1}$ is the constant function with value 1 on X and δ_p is the Dirac delta function at a fixed point $p \in X$. Clearly T is not even an integral operator.

Under some compatibility conditions, the finite-rank-refinement can be achieved by choosing a new right-parametrix of A wisely.

Lemma C.1. Assume that

1. $\mathscr{F}_{\mathcal{B}} \subset \Psi_0^{-\infty}(X) \subset C(X^2)$;
2. $\Psi_0^{-\infty}(X) \cdot C(X^2) \cdot \Psi_0^{-\infty}(X) \subset \Psi_0^{-\infty}(X)$; and
3. given any $Q \in \Psi_0^{-\infty}(X)$ and $f \in \mathcal{B}$, $\int f(x)Q(x,x')\mu(x) \in \mathcal{B}$.

If there is an $F_0 \in \mathscr{F}_{\mathcal{B}}$ with

$$\int |(R - F_0)(x,x')|\, \mu(x') \leqslant \delta < 1 \tag{C.1}$$

for all $x \in X$, then there exists a $\widetilde{B} \in \Psi(X)$ and $F \in \mathscr{F}_{\mathcal{B}}$ such that $A\widetilde{B} = \mathrm{Id} - F$.

Proof. Denote $R - F_0$ by S. Note that $S \in \Psi_0^{-\infty}(X)$ by Assumption (1). Denote S by $S^{(1)}$. For any integer $k \geqslant 2$, let

$$\begin{aligned} S^{(k)}(x,x') &= \int S(x,x_2)S(x_2,x_3)\ldots S(x_k,x')\mu(x_2)\mu(x_3)\ldots\mu(x_k) \\ &= \int S(x,y)S^{(k-1)}(y,x')\mu(y). \end{aligned}$$

Note that $S^{(k)} \in \Psi_0^{-\infty}(X)$ since $\Psi_0^{-\infty}(X)$ is a subalgebra. By (C.1), we have

$$\begin{aligned} \left\|S^{(2)}\right\|_\infty &= \sup\left|S^{(2)}(x,x')\right| = \sup\left|\int S(x,y)S(y,x')\mu(y)\right| \\ &\leqslant \|S\|_\infty \cdot \sup\int |S(x,y)|\, \mu(y) \leqslant \delta\, \|S\|_\infty\,. \end{aligned}$$

Proceed with induction, one could show that

$$\left\|S^{(k)}\right\|_\infty \leqslant \delta^{k-1} \left\|S\right\|_\infty .$$

In particular, the series

$$Q = \sum_{k \geqslant 1} S^k$$

converges in $C(X^2)$ under the sup-norm topology. Note that

$$\begin{aligned} Q(x,x') &= S^{(1)}(x,x') + S^{(2)}(x,x') + SQS(x,x') \\ &= S(x,x') + \int S(x,y)S(y,x')\mu(y) + \int S(x,y)Q(y,z)S(z,x')\mu(y)\mu(z), \end{aligned}$$

hence by Assumption (2), $Q \in \Psi_0^{-\infty}(X)$. In particular, $\mathrm{Id} + Q$ is just the inverse of $\mathrm{Id} - S$ as operators on Φ. Observe that

$$\begin{aligned} AB(\mathrm{Id} + Q) &= (\mathrm{Id} - (R - F_0) - F_0)(\mathrm{Id} + Q) \\ &= (\mathrm{Id} - S)(\mathrm{Id} + Q) - F_0(\mathrm{Id} + Q) \\ &= \mathrm{Id} - F_0 - F_0 Q. \end{aligned}$$

Assume that $F_0(x,x') = \sum_j f_j(x) g_j(x')$ with $f_j, g_j \in \mathcal{B}$. Then

$$\begin{aligned} F_0 Q(x,x') &= \int F_0(x,x'')Q(x'',x')\mu(x'') \\ &= \sum_j f_j(x) \int g_j(x'')Q(x'',x')\mu(x'') \\ &= \sum_j f_j(x) \widetilde{g}_j(x'), \end{aligned}$$

where $\widetilde{g}_j(x') = \int g_j(x'')Q(x'',x')\mu(x'') \in \mathcal{B}$, by assumption (3). Consequently, we have $F := F_0 + F_0 Q \in \mathscr{F}_{\mathcal{B}}$. Let $\widetilde{B} = B + BQ$, then $\widetilde{B} \in \Psi(X)$, and $A\widetilde{B} = \mathrm{Id} - F$. $\square$

Henceforth, we will take the hypotheses in Lemma C.1 for granted.

The next simple fact, which says that a "big" vector subspace is indeed an (orthogonal) direct summand, from elementary linear algebra, turns out to be, surprisingly, a crucial piece in proving the main result we desire. The proof was included to keep this article self-contained.

Lemma C.2. Let V be a inner product space and $U \subseteq V$ a subspace. Assume that there is a finite dimensional subspace W such that $W^\perp \subseteq U$, then

(a) $U^\perp$ is finite dimensional; and

(b) $V = U \oplus U^\perp$.

Proof. Since W is finite dimensional, for any $\mathbf{v} \in V$, $\mathbf{v} = \mathbf{v}' + (\mathbf{v} - \mathbf{v}')$, where $\mathbf{v}'$ is the orthogonal projection of $\mathbf{v}$ onto W, thus $V = W \oplus W^\perp$. This implies furthermore that $(W^\perp)^\perp = W$, hence $U^\perp \subseteq W$ and (a) follows.

Observe that $U/W^\perp$ is isomorphic to a subspace of W, hence is finite dimensional. Also, the orthogonal projection onto $W^\perp$ is obtained via the direct sum decomposition $V = W \oplus W^\perp$, thus a practice essentially the same as the Gram-Schmidt process produces a finite dimensional subspace $P \subseteq U$ orthogonal to $W^\perp$ such that $U = W^\perp \oplus P$. The orthogonal projection onto U now can be defined by summing up the projections onto $W^\perp$ and P, whose existence implies (b). $\square$

Here we point out that we made no assumption on the dimension of V and the metric induced by the inner product. In particular, V was not intended to be a Hilbert space.

We arrive at the main result of this section, which is readily obtained at this point.

Theorem C.3. $\mathrm{Im}(A)^\perp$ is finite dimensional, and

$$\Phi = \mathrm{Im}(A) \oplus \mathrm{Im}(A)^\perp.$$

In particular, $\dim \operatorname{coker}(A)$ is finite.

Proof. By Lemma C.1, there is a $\widetilde{B} \in \Psi(X)$ and an $F \in \mathscr{F}_{\mathcal{B}}$, such that $A\widetilde{B} = \mathrm{Id} - F$. Assume that

$$F = \sum_j \varphi_j(x)\psi_j(x'),$$

then we define

$$W = \operatorname{span}\{\bar{\psi}_j\} \subseteq \Phi,$$

where the bar over a complex-valued function stands for taking conjugate. Given any $u \in W^{\perp}$, we compute

$$\begin{aligned} A(\widetilde{B}u) &= (\mathrm{Id} - F)u \\ &= u - \int \sum_i \varphi_j(x)\psi_j(x')u(x')\mu(x') \\ &= u, \end{aligned}$$

therefore $W^{\perp} \subseteq \operatorname{Im}(A)$. Now the desired result is implied by Lemma C.2. □

Corollary C.3.1. If, in addition, A is self-adjoint, then A is Fredholm.

Proof. Just note that $\ker(A) = \ker(A^*) = \operatorname{Im}(A)^{\perp}$, which is finite dimensional. □

Note that under the self-adjoint-ness assumption, the analytical index of A is trivial. We record also the result about Fredholm property for not-necessarily-self-adjoint operators.

Theorem C.4. Let A be a linear operators on Φ. If there exist linear operators $B_i \in \Psi(X)$ and $S_i \in \Psi_0^{-\infty}(X)$, $i = 1, 2$, such that $AB_1 = I - S_1$ and $B_2A = I - S_2$, then A is Fredholm.

Proof. It suffices to show $\dim \ker(A)$ is finite dimensional. An argument parallel to the one for Lemma C.1 (with the additional hypotheses that given any $Q \in \Psi_0^{-\infty}(X)$

and $g \in \mathcal{B}$, $\int Q(x, x')g(x')\mu(x') \in \mathcal{B}$) shows that there exists an operator $\widetilde{B_2} \in \Psi(X)$ and a finite rank operator $F_2 \in \mathscr{F}_{\mathcal{B}}$, such that $\widetilde{B_2}A = I - F_2$.

Assume that $u \in \ker(A)$, then

$$0 = \widetilde{B_2}(Au) = Iu - F_2 u = u - F_2 u$$

thus $\ker(A) \subseteq \operatorname{Im}(F_2)$, which implies the claim, since $\dim \operatorname{Im}(F_2)$ is finite. See the remark below Definition C.1. □

The assumptions here are readily available for, e.g., the elliptic pseudodifferential operators on closed manifolds. For the verification to b-pseudodifferential operators of all the hypotheses involved, see [20]. The argument to the Fredholm property is similar in spirit to the standard one via compact remainders (see [31]). However, thanks to the presence of the finite rank remainders, Loya's approach is much more accessible.

Lastly, to enlighten the readers about how hypotheses in Lemma C.1 are possibly verified, we recall a celebrated result from point set topology and a few consequences of it. See Section 1.6 for the genuine usage on manifolds.

Theorem C.5 (The Complex Stone-Weierstrass Theorem)**.** Let X be a compact Hausdorff space. If $\mathscr{A}$ is a closed complex subalgebra of $C(X, \mathbb{C})$ that separates points and is closed under complex conjugation, then either $\mathscr{A} = C(X, \mathbb{C})$ or $\mathscr{A} = \{f \in C(X, \mathbb{C}) : f(x_0) = 0 \text{ for some } x_0 \in X\}$.

See [7] for a proof.

Corollary C.5.1. Let X and Y be compact Hausdorff spaces. Let $\mathscr{F}$ be the algebra generated by functions of the form $f(x, y) = g(x)h(y)$, where $g \in C(X, \mathbb{C})$ and $h \in C(Y, \mathbb{C})$, that is, $\mathscr{F} = C(X, \mathbb{C}) \otimes C(Y, \mathbb{C})$. Then $\mathscr{F}$ is dense in $C(X \times Y, \mathbb{C})$.

Proof. The closure of $\mathscr{F}$ in $C(X \times Y, \mathbb{C})$ is a subalgebra closed under complex conjugation, since all operations involved (that is, conjugation, addition and multiplication) are continuous in the uniform convergence topology. Also, $\mathscr{F}$ contains all constant functions. Recall that compact Hausdorff spaces are normal. Now the fact that $\mathscr{F}$ (hence its closure) separates points is an easy consequence of Urysohn's lemma. $\square$

We are mostly interested in compact manifolds, so the property of separating points for subalgebras of functions with various order of differentiability follows readily from the existence of bump functions, hence the result could be extended to those settings.

We conclude this section by remarking that what we have proved in Theorem C.3 is essentially "Hodge without Banach and Hilbert", because it essentially produces the Hodge decomposition with respect to A. See [19] for the details of a complete proof of the classical Hodge theorem in this spirit, and more. Finally, this approach could be easily adapted to the study of linear operators on sections of vector bundles.

Appendix D

Heat kernel and η-invariant

Let $A \in \Psi^m(Y)$ with $m > 0$. Assume that A is elliptic and self-adjoint. Then $(A-\lambda)^{-1}$ is a continuous (in fact, holomorphic) family of bounded operator in $L^2(Y)$ (denoted by $B(L^2)$ in this section) for $\lambda \notin \operatorname{spec}(A)$. We first record the following preliminary observation from functional analysis.

Lemma D.1. If A is self-adjoint, then

$$\left\|(A-x-iy)^{-1}\right\| \leqslant \frac{1}{|y|}, \ \forall x, y \in \mathbb{R} \text{ with } y \neq 0.$$

Proof. In this argument, we denote $\|\cdot\|_{L^2}$ by $\|\cdot\|$ for any element in $L^2(Y)$. (In fact, this works for any inner product space.) Since $A-x$ is also self-adjoint for any $x \in \mathbb{R}$, it suffices to show that

$$\left\|(A-iy)^{-1}\right\| \leqslant \frac{1}{|y|}, \ \forall y \in \mathbb{R} \text{ with } y \neq 0.$$

It is clear that

$$\operatorname{Im}\left(\langle\, (A-iy)\varphi, \varphi\,\rangle\right) = -y\,\|\varphi\|^2 .$$

We will also need the following inequality

$$\left|\left\langle (A-iy)^{-1}\varphi,\, \varphi \right\rangle\right| \leqslant |y|^{-1}\,\|\varphi\|^2\,,\ \forall y\in\mathbb{R} \text{ with } y\neq 0,$$

or equivalently,

$$\left|\left\langle -iy(A-iy)^{-1}\varphi,\, \varphi \right\rangle\right|^2 \leqslant \|\varphi\|^4\,.$$

To see this, denote $\psi = (A-iy)^{-1}\varphi$, then, by the self-adjoint-ness of A and the Cauchy-Schwarz inequality, we compute

$$\begin{aligned}
&\left|\left\langle -iy(A-iy)^{-1}\varphi,\, \varphi \right\rangle\right|^2 \\
&= \left|\left\langle \varphi,\, \varphi \right\rangle - \left\langle A(A-iy)^{-1}\varphi,\, \varphi \right\rangle\right|^2 \\
&= |\langle \varphi, \varphi \rangle - \langle A\psi, (A-iy)\psi \rangle|^2 \\
&= \left|\|\varphi\|^2 - \|A\psi\|^2 - iy\,\langle A\psi, \psi \rangle\right|^2 \\
&= \left(\|\varphi\|^2 - \|A\psi\|^2\right)^2 + y^2\,\langle A\psi, \psi \rangle^2 \\
&= \|\varphi\|^4 + \left(\|A\psi\|^4 + y^2\,\langle A\psi, \psi \rangle^2 - 2\,\|(A-iy)\psi\|^2\,\|A\psi\|^2\right) \\
&= \|\varphi\|^4 + \left(\|A\psi\|^4 + y^2\,\langle A\psi, \psi \rangle^2 - 2\left(\|A\psi\|^2 + y^2\,\|\psi\|^2\right)\|A\psi\|^2\right) \\
&= \|\varphi\|^4 - \|A\psi\|^4 - y^2\left(2\,\|\psi\|^2\,\|A\psi\|^2 - \langle A\psi, \psi \rangle^2\right) \\
&\leqslant \|\varphi\|^4\,.
\end{aligned}$$

Lastly, combining those two relations above, we have

$$\begin{aligned}
\left|-y\,\|\psi\|^2\right| &= \left|\mathrm{Im}\left(\left\langle (A-iy)\psi, \psi \right\rangle\right)\right| \\
&\leqslant \left|\left\langle (A-iy)^{-1}\varphi,\, \varphi \right\rangle\right| \\
&\leqslant |y|^{-1}\,\|\varphi\|^2\,,
\end{aligned}$$

which implies the claim. $\square$

Theorem D.2. $(A-\tau)^{-1}$ has only simple poles at a discrete set $D \subset \mathbb{R}$. Moreover, near any point $\tau_1 \in D$,

$$(A-\tau)^{-1} = B(\tau) - \frac{F}{\tau - \tau_1}, \tag{D.1}$$

where $B(\tau)$ is holomorphic near τ_1 with

$$B(\tau_1) = \begin{cases} \mathbf{0} & \text{on } \ker(A-\tau_1) \\ (A-\tau_1)^{-1} & \text{on } \ker(A-\tau_1)^{\perp} \end{cases}, \tag{D.2}$$

and

$$F = \pi_{\ker(A-\tau_1)}, \tag{D.3}$$

the orthogonal projection onto the eigenspace of τ_1.

Remark. See also [25, PROPOSITION 6.27].

Proof. From analytic Fredholm theory(see [25]), we know that $(A-\tau)^{-1}$ has a meromorphic extension with discrete singularities of finite rank. Since A is self-adjoint, the singularities are contained in $\mathbb{R}$. Assume $x \in \mathbb{R}$ to be an eigenvalue of A, then, from the preliminary lemma above, we have

$$\lim_{y \to 0} (iy)^2 (A - x - iy)^{-1} = \mathbf{0}$$

so by the meromorphy of $(A-\tau)^{-1}$, it has a simple pole at x. To show that the residue of $(A-\tau)^{-1}$ at x is just $-\Pi_{\ker(A-x)}$, consider the identity

$$\begin{aligned} y\varphi &= y(A-x-iy)(A-x-iy)^{-1}\varphi \\ &= y(A-x-iy)(B(x+iy) + \frac{F}{iy})\varphi. \end{aligned}$$

Taking derivative with respect to y then letting y approach 0, we have

$$\varphi = (A - x)(B(x) + \partial_y B(x))\varphi - F\varphi$$

or,

$$\operatorname{Id} + F = (A - x)(B(x) + \partial_y B(x)) = (A - x)\widetilde{B},$$

that says, $(\operatorname{Id} + F)\varphi \in \mathcal{R}(A - x)$. Since $(A - x)$ is self-adjoint, $\ker(A - x)^{\perp} = \mathcal{R}(A - x)$, and the claim follows.

Now observe that, on one hand,

$$\begin{aligned}(A - \tau_1)(A - \tau)^{-1} &= (A - \tau + \tau - \tau_1)(A - \tau)^{-1} \\ &= \operatorname{Id} + (\tau - \tau_1)(A - \tau)^{-1} \\ &= (A - \tau)^{-1}(A - \tau_1),\end{aligned}$$

and furthermore, inserting formula (D.1) to the second equality above, we get

$$(A - \tau)^{-1}(A - \tau_1) = \operatorname{Id} + (\tau - \tau_1)B(\tau) - F$$

which shows that $B(\tau) = \mathbf{0}$ on $\ker(A - \tau_1)$ when $\tau \neq \tau_1$, but from continuity we also have

$$B(\tau_1) = \mathbf{0} \text{ on } \ker(A - \tau_1).$$

On the other hand,

$$\begin{aligned}(A - \tau_1)(A - \tau)^{-1} &= (A - \tau_1)B(\tau) - \frac{(A - \tau_1)F}{\tau - \tau_1} \\ &= (A - \tau_1)B(\tau).\end{aligned}$$

Letting τ approach τ_1, we have

$$(A-\tau_1)B(\tau_1) = \mathrm{Id} - F$$

which implies

$$B(\tau_1) = (A-\tau_1)^{-1} \text{ on } \ker(A-\tau_1)^{\perp}$$

and the proof is completed. $\square$

For a neat application of this important result, see [18, Theorem 4.1].

Recall that for $\lambda \in \mathbb{C}\backslash\mathbb{R}$ with $\mathrm{Im}\lambda \neq 0$,

$$\left\|(A-\lambda)^{-1}\varphi\right\|_{L^2} \leqslant |\mathrm{Im}\lambda|^{-1}\|\varphi\|_{L^2} \tag{D.4}$$

for any $\varphi \in L^2(Y)$. Assume in addition that A is non-negative, we define the following contour integral

$$T := \frac{i}{2\pi}\int_\Gamma \lambda^{-1}(A-\lambda)^{-1}\,\mathrm{d}\lambda \tag{D.5}$$

where $\Gamma = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda = \varepsilon\}$ oriented downward with $0 < \varepsilon < \lambda_1$, where λ_1 is the smallest positive eigenvalue of A. Note that $\|\lambda^{-1}(A-\lambda)^{-1}\| = O(|\lambda|^{-2})$ as $|\lambda| \to \infty$ for $\lambda \in \Gamma$, hence T is well defined. We shall first improve the estimate in (D.4).

Lemma D.3. Let A be self-adjoint and non-negative. If $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda < 0$, then

$$\left\|(A-\lambda)^{-1}\varphi\right\|_{L^2} \leqslant |\lambda|^{-1}\|\varphi\|_{L^2}$$

for any $\varphi \in L^2(Y)$. That is, in terms of operator norm, we have

$$\left\|(A-\lambda)^{-1}\right\| \leqslant |\lambda|^{-1}. \tag{D.6}$$

Proof. The proof is an obvious adaptation of the one to Lemma D.1. To avoid repetition, we leave the details to the interested readers. $\square$

Corollary D.3.1. For any $\lambda \in \mathbb{C} \backslash \operatorname{spec}(A)$, let

$$\chi(\lambda) = \max \left\{ \frac{1}{|\lambda - \lambda_j|} \;\middle|\; \lambda_j \in \operatorname{spec}(A) \right\},$$

then

$$\left\| (A - \lambda)^{-1} \right\| \leqslant \chi(\lambda).$$

Proof. Assume that $\lambda_j < \lambda < \lambda_{j+1}$. Let $V = \operatorname{span} \left\{ \varphi_k \;\middle|\; A\varphi_k = \lambda_k \varphi_k, \lambda_k \leqslant \lambda_j \right\}$. Then $L^2(Y) = V \oplus V^{\perp}$. Since $(A|_{V^{\perp}} - \lambda_{j+1})$ is also a self-adjoint, non-negative operator, we have

$$\left\| (A|_{V^{\perp}} - \lambda)^{-1} \right\| = \left\| [(A|_{V^{\perp}} - \lambda_{j+1}) - (\lambda - \lambda_{j+1})]^{-1} \right\| \leqslant \frac{1}{|\lambda - \lambda_{j+1}|}.$$

On the other hand, over the finite-dimensional vector space, $A|_V$ is diagonalizable, hence

$$\left\| (A|_V - \lambda)^{-1} \right\| = \frac{1}{|\lambda - \lambda_j|}.$$

Consequently, we have

$$\left\| (A - \lambda)^{-1} \right\| \leqslant \max \left\{ \frac{1}{|\lambda - \lambda_j|}, \frac{1}{|\lambda - \lambda_{j+1}|} \right\}.$$

Now the claim follows from the arbitrariness of λ. $\square$

Proposition D.1. Let G be the Green's operator for A. Then $T = G$.

Proof. Recall that for $\lambda \in \mathbb{C} \backslash \operatorname{spec}(A)$,

$$(A - \lambda)^{-1} = G(\lambda) - \frac{\pi_{\ker A}}{\lambda},$$

where $G(\lambda)$ is holomorphic at $\lambda = 0$, and $G(0) = G$. Hence,

$$\lambda^{-1}(A-\lambda)^{-1} = \frac{G(\lambda)}{\lambda} - \frac{\pi_{\ker A}}{\lambda^2}, \text{ and}$$

$$\frac{i}{2\pi}\oint_{\Gamma'} \lambda^{-1}(A-\lambda)^{-1} = G(0),$$

where Γ' is any simple closed curve enclosing the origin, oriented clockwise. Given $a > \epsilon$, consider the boundary of a square with side length equal a enclosing the origin: $\Gamma'_a = \Gamma_a \cup \Gamma^1_a \cup \Gamma^2_a \cup \Gamma^3_a$ (Figure D.1), where

$$\begin{aligned}
\Gamma_a &= \Gamma \cap \{\lambda \in \mathbb{C} \mid -a/2 \leqslant \mathrm{Im}\lambda \leqslant a/2\},\\
\Gamma^1_a &= \{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda = -a/2\} \cap \{\lambda \in \mathbb{C} \mid \epsilon - a \leqslant \mathrm{Re}\,\lambda \leqslant \epsilon\},\\
\Gamma^2_a &= \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda = \epsilon - a\} \cap \{\lambda \in \mathbb{C} \mid -a/2 \leqslant \mathrm{Im}\lambda \leqslant a/2\},\\
\Gamma^3_a &= \{\lambda \in \mathbb{C} \mid \mathrm{Im}\lambda = a/2\} \cap \{\lambda \in \mathbb{C} \mid \epsilon - a \leqslant \mathrm{Re}\,\lambda \leqslant \epsilon\},
\end{aligned}$$

oriented respectively such that Γ'_a is oriented clockwise. Denote

$$T^j_a = i(2\pi)^{-1}\oint_{\Gamma^j_a} \lambda^{-1}(A-\lambda)^{-1}\,\mathrm{d}\lambda,$$

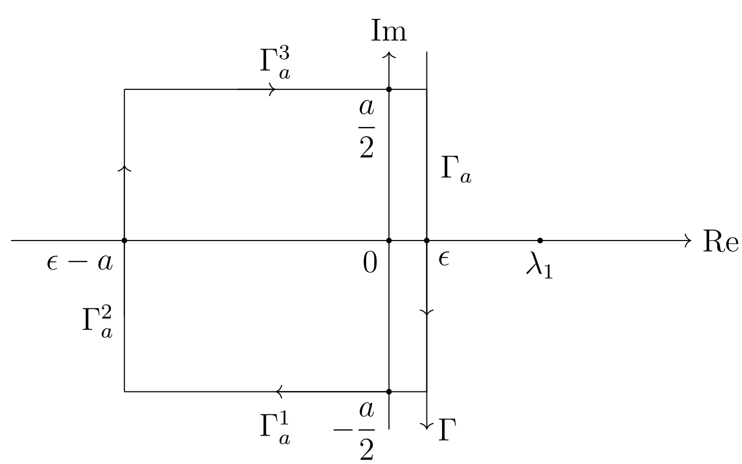


Figure D.1: Contour Γ'_a

then by (D.4) and (D.6), $\lim_{a\to\infty} T_a^j = 0$, thus,

$$\begin{aligned} T &= \lim_{a\to\infty} \frac{i}{2\pi} \oint_{\Gamma_a} \lambda^{-1}(A-\lambda)^{-1}\, \mathrm{d}\lambda \\ &= \lim_{a\to\infty} \frac{i}{2\pi} \oint_{\Gamma_a'} \lambda^{-1}(A-\lambda)^{-1}\, \mathrm{d}\lambda \\ &= \lim_{a\to\infty} G(0) \\ &= G. \end{aligned}$$ $\square$

More generally, denote $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \lambda_j \xrightarrow{j\to\infty} \infty$ as the eigenvalues of A, and let $\{\varphi_j\}$ be a set of orthonormal eigenvectors associated to $\{\lambda_j\}$. Note that near λ_j,

$$(A-\lambda)^{-1} = G_j(\lambda) - \frac{\Pi_j}{\lambda - \lambda_j},$$

where $\Pi_j = \pi_{\ker(A-\lambda_j)}$, and $G_j(\lambda)$ is holomorphic (near λ_j) with $G_j(\lambda_j)$ the generalized inverse (or Green's operator) of $(A-\lambda_j)$. In particular, when $\lambda_j > 0$,

$$\lambda^{-1}(A-\lambda)^{-1} = \frac{G_j(\lambda)}{\lambda} - \left(\frac{\lambda_j^{-1}\Pi_j}{\lambda-\lambda_j} - \frac{\lambda_j^{-1}\Pi_j}{\lambda} \right),$$

therefore, the residue at $\lambda = \lambda_j$ is just $-\lambda_j^{-1}\Pi_j$. Consequently, for $r > 0$ such that $r \notin \operatorname{spec}(A)$,

$$G = \sum_{\lambda_j < r} \frac{\Pi_j}{\lambda_j} + T_r = \sum_{\lambda_j < r} \frac{\varphi_j \otimes \bar{\varphi}_j}{\lambda_j} + T_r,$$

where

$$T_r = \frac{i}{2\pi} \int_{\Gamma_r} \lambda^{-1}(A-\lambda)^{-1}\, \mathrm{d}\lambda$$

with $\Gamma_r = \left\{\lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\lambda = r\right\}$ oriented downwardly. By Corollary D.3.1, we have $T_r \xrightarrow{r\to\infty} 0$.

Lemma D.4. For $r > 0$, let $V_r = \operatorname{span}\{\varphi_j \big| \lambda_j < r\}$, and $V_r^\perp$ be the orthogonal complement in $L^2(Y)$. Given any $N \in \mathbb{N}^+$, there exists $\delta > 0$ such that when $r > \delta$,

$\|G\phi\|_{L^2} \leqslant \lambda_{N+1}^{-1} \|\phi\|_{L^2}$ for any $\phi \in V_r^{\perp} \cap C^{\infty}(Y)$.

Proof. Write $G = \sum_{\lambda_j<s} \lambda_j^{-1}\varphi_j \otimes \bar{\varphi}_j + G_s$, then $G_s \to 0$ as $s \to \infty$. Thus there is some $\delta > 0$ such that when $s > \delta$, $\|G_s\| \leqslant \lambda_{N+1}^{-1}$. Therefore, when $r > \delta$, for any $\phi \in V_r^{\perp} \cap C^{\infty}(Y)$,

$$\|G\phi\|_{L^2} = \left\| \sum_{\lambda_j<r} \frac{1}{\lambda_j} \langle \phi, \varphi_j \rangle \varphi_j + G_r\phi \right\|_{L^2} = \|G_r\phi\|_{L^2} \leqslant \frac{\|\phi\|_{L^2}}{\lambda_{N+1}}. \qquad \square$$

Note that $\ker A$ is finite dimensional, since A is Fredholm. Let $\{\psi_k\}$ be an orthonormal basis of $\ker A$.

Proposition D.2. $\{\psi_k\} \cup \{\varphi_j\}$ forms a complete orthonormal basis of $L^2(Y)$.

Proof. Since $C^{\infty}(Y)$ is dense in $L^2(Y)$, it suffices to prove the result for the space of smooth functions under the L^2-norm. Assume that $\phi \in C^{\infty}(Y)$, and $A\phi = u$. Then $Gu = \phi - \pi_{\ker A}\phi$, where $v \in \ker A$. For any $r > 0$, we compute

$$\begin{aligned}
\phi - \pi_{\ker A}\phi - \sum_{\lambda_j<r} \langle \phi, \varphi_j \rangle \varphi_j &= Gu - \sum_{\lambda_j<r} \langle \phi, \varphi_j \rangle \lambda_j\varphi_j \\
&= G\left(u - \sum_{\lambda_j<r} \langle \phi, \lambda_j\varphi_j \rangle \varphi_j \right) \\
&= G\left(u - \sum_{\lambda_j<r} \langle \phi, A\varphi_j \rangle \varphi_j \right) \\
&= G\left(u - \sum_{\lambda_j<r} \langle A\phi, \varphi_j \rangle \varphi_j \right) \\
&= G\left(u - \sum_{\lambda_j<r} \langle u, \varphi_j \rangle \varphi_j \right) \\
&= Gv.
\end{aligned}$$

Since $v = u - \sum_{\lambda_j<r} \langle u, \varphi_j \rangle \varphi_j \in V_r^{\perp}$, and $\|v\|_{L^2}^2 = \|u\|_{L^2}^2 - \sum_{\lambda_j<r} \langle u, \varphi_j \rangle^2 \leqslant \|u\|_{L^2}^2$,

by Lemma D.4, $\|Gv\|_{L^2} \xrightarrow{r\to\infty} 0$. $\square$

Notation: Henceforth, we will rewrite $\{\psi_k\} \cup \{\varphi_j\}$ as $\{\varphi_j\}$.

Lemma D.5. Given any $k \in \mathbb{N}$, for sufficiently large $\ell \in \mathbb{N}$, there exists some constant C_k^ℓ, such that

$$\|\varphi\|_{C^k} \leqslant C_k^\ell \left(\left\|A^\ell\varphi\right\|_{L^2} + \|\varphi\|_{L^2}\right),$$

for any $\varphi \in C^\infty(Y)$.

Proof. Let G be the Green's operator of A. Recall that $GA = AG = \mathrm{Id} - \Pi$, where Π is the orthogonal projection onto $\ker A$, since A is self-adjoint. Recall that $G \in \Psi^{-m}(Y)$ and $\Pi \in \Psi^{-\infty}(Y)$. Note that for any $\ell \in \mathbb{N}^+$,

$$G^\ell A^\ell = (GA)^\ell = (\mathrm{Id} - \Pi)^\ell = \mathrm{Id} - \Pi.$$

Given any $P \in \mathrm{Diff}^k(Y)$, choose $\ell \in \mathbb{N}^+$ such that $-m\ell < -k - \dim Y - 1$. Then $PG^\ell \in \Psi^{k-m\ell}(Y) \subseteq \Psi^{-(\dim Y+1)}(Y)$, in particular, the Schwartz kernel of PG^ℓ is continuous. Denote the kernel of PG^ℓ by $K_1(x,y)$ and of $P\Pi$ by $K_2(x,y)$. Since

$$P\varphi = P(G^\ell A^\ell + \Pi)\varphi = (PG^\ell)A^\ell\varphi + P\Pi\varphi,$$

by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|P\varphi\|_{C^0} &\leqslant \left\|(PG^\ell)A^\ell\varphi\right\|_{C^0} + \|P\Pi\varphi\|_{C^0} \\
&\leqslant \left\|\int_Y K_1(x,y)(A^\ell\varphi)(y)\ \mathrm{d}y\right\|_{C^0} + \left\|\int_Y K_2(x,y)\varphi(y)\ \mathrm{d}y\right\|_{C^0} \\
&\leqslant \left\|\|K_1(x,\cdot)\|_{L^2(Y)} \left\|A^\ell\varphi\right\|_{L^2(Y)}\right\|_{C^0} + \left\|\|K_2(x,\cdot)\|_{L^2(Y)} \|\varphi\|_{L^2(Y)}\right\|_{C^0} \\
&\leqslant \left(\sup_{x\in Y}\|K_1(x,\cdot)\|_{L^2(Y)}\right)\left\|A^\ell\varphi\right\|_{L^2(Y)} + \left(\sup_{x\in Y}\|K_2(x,\cdot)\|_{L^2(Y)}\right)\|\varphi\|_{L^2(Y)}.
\end{aligned}$$

The lemma now follows from the arbitrariness of k and P. $\square$

Proposition D.3. Given any $\phi \in C^\infty(Y)$, the (Fourier) series $\sum \langle \phi, \varphi_j \rangle \varphi_j$ converges to ϕ in the C^∞-topology.

Proof. For any $k \in \mathbb{N}$, by Lemma D.5, and the self-adjointness of A,

$$\begin{aligned}
&\left\| \phi - \sum_j^N \langle \phi, \varphi_j \rangle \varphi_j \right\|_{C^k} \\
\leqslant & C_k^\ell \left(\left\| A^\ell \left(\phi - \sum_j^N \langle \phi, \varphi_j \rangle \varphi_j \right) \right\|_{L^2} + \left\| \phi - \sum_j^N \langle \phi, \varphi_j \rangle \varphi_j \right\|_{L^2} \right) \\
\leqslant & C_k^\ell \left(\left\| A^\ell \phi - \sum_j^N \langle \phi, \varphi_j \rangle A^\ell \varphi_j \right\|_{L^2} + \left\| \phi - \sum_j^N \langle \phi, \varphi_j \rangle \varphi_j \right\|_{L^2} \right) \\
\leqslant & C_k^\ell \left(\left\| A^\ell \phi - \sum_j^N \langle \phi, \lambda_j^\ell \varphi_j \rangle \varphi_j \right\|_{L^2} + \left\| \phi - \sum_j^N \langle \phi, \varphi_j \rangle \varphi_j \right\|_{L^2} \right) \\
\leqslant & C_k^\ell \left(\left\| A^\ell \phi - \sum_j^N \langle A^\ell \phi, \varphi_j \rangle \varphi_j \right\|_{L^2} + \left\| \phi - \sum_j^N \langle \phi, \varphi_j \rangle \varphi_j \right\|_{L^2} \right).
\end{aligned}$$

Therefore, from Proposition D.2, $\left\| \phi - \sum_j^N \langle \phi, \varphi_j \rangle \varphi_j \right\|_{C^k} \xrightarrow{N \to \infty} 0$. $\square$

Corollary D.5.1. If $F \in C^\infty(Y \times Y)$, then $\sum_{j,k} \langle F, \varphi_j \otimes \bar{\varphi}_k \rangle \varphi_j \otimes \bar{\varphi}_k$ converges to F in $C^\infty(Y \times Y)$.

In the rest of this section, we further restrict to $A = \triangle$, a generalized Laplacian on Y. By heat calculus, the heat kernel $\mathrm{e}^{-t\triangle}$ exists and for any $t > 0$, $\mathrm{e}^{-t\triangle} \in C^\infty(Y \times Y)$. With this *a priori* knowledge of the heat kernel, the following result is easily obtained.

Theorem D.6. Given any $t > 0$, $\mathrm{e}^{-t\triangle} = \sum_j \mathrm{e}^{-t\lambda_j} \varphi_j \otimes \bar{\varphi}_j$ in the C^∞-topology.

Proof. Denote $\left\langle \mathrm{e}^{-t\triangle}, \varphi_j \otimes \bar{\varphi}_k \right\rangle$ by $h_{jk}(t)$. From Corollary D.5.1,

$$\mathrm{e}^{-t\triangle} = \sum_{j,k} h_{jk}(t) \varphi_j \otimes \bar{\varphi}_k,$$

and consequently, $\mathrm{e}^{-t\triangle}\varphi_\ell = \sum_j h_{j\ell}(t)\varphi_j$. Thus, we have

$$0 = (\partial_t + \triangle)\sum_j h_{j\ell}(t)\varphi_j = \sum_j (h'_{j\ell}(t) + \lambda_j h_{j\ell}(t))\varphi_j,$$

and therefore, $h'_{j\ell}(t) + \lambda_j h_{j\ell}(t) = 0$. Solving the equations, we obtain that $h_{j\ell}(t) = C_{j\ell}\,\mathrm{e}^{-t\lambda_j}$, for some constant $C_{j\ell}$. Recall that $\varphi_\ell = \lim_{t\to 0}\mathrm{e}^{-t\triangle}\varphi_\ell$, and the heat kernel is unique, hence we must have $C_{j\ell} = \delta_{j\ell}$, the Kronecker delta, and the proof is completed. $\square$

Corollary D.6.1. Let $f(t) = \mathrm{Tr}(\mathrm{e}^{-t\triangle})$. Then $f^{(m)}(t) = \sum_j (-\lambda_j)^m\,\mathrm{e}^{-t\lambda_j}$.

Proof. Recall that

$$f^{(m)}(t) = \int \partial_t^m\,\mathrm{e}^{-t\triangle}\big|_{\{y=y'\}}\nu(y).$$

Note that $\partial_t^m\,\mathrm{e}^{-t\triangle} = (-\triangle)^m\,\mathrm{e}^{-t\triangle}$, and

$$(-\triangle)^m\left(\sum_j \mathrm{e}^{-t\lambda_j}\varphi_j\otimes\bar{\varphi}_j\right) = \sum_j \mathrm{e}^{-t\lambda_j}\left((-\triangle)^m\varphi_j\right)\otimes\bar{\varphi}_j = \sum_j (-\lambda_j)^m\,\mathrm{e}^{-t\lambda_j}\varphi_j\otimes\bar{\varphi}_j,$$

hence

$$f^{(m)}(t) = \int\sum_j (-\lambda_j)^m\,\mathrm{e}^{-t\lambda_j}\varphi_j(y)\bar{\varphi}_j(y)\nu(y) = \sum_j (-\lambda_j)^m\,\mathrm{e}^{-t\lambda_j}. \qquad\square$$

Corollary D.6.2. $\lim_{t\to\infty}\mathrm{e}^{-t\triangle} = \pi_{\ker\triangle}$.

Proposition D.4. Let $\mathcal{D}_0$ be a Dirac-type operator on Y with $\dim Y = n$, and $(\lambda_j)_{j\geqslant 1}$ with $|\lambda_j| \leqslant |\lambda_{j+1}|$ be the eigenvalues of $\mathcal{D}_0$. Then

$$\mathrm{Tr}(\mathcal{D}_0\,\mathrm{e}^{-t\mathcal{D}_0^2}) = \sum_{j\geqslant 1}\lambda_j\,\mathrm{e}^{-t\lambda_j^2} = \sum_{\lambda_j\neq 0}\lambda_j\,\mathrm{e}^{-t\lambda_j^2}. \tag{D.7}$$

Furthermore, as $t\to 0$, the following asymptotic expansion holds:

$$\mathrm{Tr}(\mathcal{D}_0\,\mathrm{e}^{-t\mathcal{D}_0^2}) \sim t^{-n/2}\sum_{k=0}^{\infty} t^k a_k. \tag{D.8}$$

Proof. By definition, $\mathcal{D}_0$ is elliptic and self-adjoint, hence there is a complete orthonormal basis of $L^2(Y)$ consisting of eigenfunctions of $\mathcal{D}_0$, denoted by $\{\psi_j\}$. Assume that $\mathcal{D}_0\,\psi_j = \lambda_j\psi_j$, then $\mathrm{e}^{-t\mathcal{D}_0^2} = \sum_j \mathrm{e}^{-t\lambda_j^2}\,\psi_j \otimes \bar{\psi}_j$. Therefore

$$\mathcal{D}_0\,\mathrm{e}^{-t\mathcal{D}_0^2} = \sum_j \mathrm{e}^{-t\lambda_j^2}(\mathcal{D}_0\,\psi_j)\otimes\bar{\psi}_j = \sum_j \lambda_j\,\mathrm{e}^{-t\lambda_j^2}\,\psi_j\otimes\bar{\psi}_j,$$

which implies (D.7).

To see (D.8), it suffices to work locally. Recall that $\mathrm{e}^{-t\mathcal{D}_0^2} \in \Psi_{\mathcal{H}}^{-2}(Y)$. Assume that over a local coordinate patch,

$$\mathrm{e}^{-t\mathcal{D}_0^2} = t^{-n/2}q(t^{1/2}, x, \frac{x-y}{t^{1/2}})\nu(y) + R,$$

where $q(s,x,\omega) \in \mathscr{S}(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$ with $q(-s,x,-\omega) = q(s,x,\omega)$ and $R \in \Psi_{\mathcal{H}}^{-\infty}(\mathbb{R}^n)$. Also assume that over the same coordinate patch, $D_0 = \sum_{j=1}^n a_j(x)\partial_{x_j}$. Then we compute

$$\begin{aligned}
a_j(x)\partial_{x_j}\,\mathrm{e}^{-t\mathcal{D}_0^2} &= t^{-n/2}a_j(x)\left(q_{x_j}(t^{1/2},x,\frac{x-y}{t^{1/2}}) + \frac{q_{\omega_j}(t^{1/2},x,\frac{x-y}{t^{1/2}})}{t^{1/2}}\right)\nu(y) + a_j(x)\partial_{x_j}R\\
&= \frac{a_j(x)}{t^{n/2+1/2}}\left(q_{x_j}(t^{1/2},x,\frac{x-y}{t^{1/2}}) + q_{\omega_j}(t^{1/2},x,\frac{x-y}{t^{1/2}})\right) + a_j(x)\partial_{x_j}R\\
&= t^{-n/2-1/2}\widetilde{q}(t^{1/2},x,\frac{x-y}{t^{1/2}})\nu(y) + a_j(x)\partial_{x_j}R,
\end{aligned}$$

where

$$\widetilde{q}(s,x,\omega) = a_j(x)(sq_{x_j}(s,x,\omega) + q_{\omega_j}(s,x,\omega)).$$

Note that $\widetilde{q}(s,x,\omega) \in \mathscr{S}(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$ with

$$\begin{aligned}
q(-s,x,-\omega) &= a_j(x)\left((-s)q_{x_j}(-s,x,-\omega) + q_{\omega_j}(-s,x,-\omega)\right)\\
&= -a_j(x)\left(sq_{x_j}(s,x,\omega) + q_{\omega_j}(s,x,\omega)\right)\\
&= -\widetilde{q}(s,x,\omega),
\end{aligned}$$

and $a_j(x)\partial_{x_j} R \in \Psi_{\mathcal{H}}^{-\infty}(\mathbb{R}^n)$. In summary, one could show that $\mathcal{D}_0 \, \mathrm{e}^{-t\mathcal{D}_0^2} \in \Psi_{\mathcal{H}}^{-1}(Y)$. Now observe that

$$\begin{aligned}\int t^{-n/2-1/2}\widetilde{q}(t^{1/2}, x, 0)\nu(x) &\sim t^{-n/2-1/2}\int \sum_{\ell \geqslant 0} \frac{t^{\ell+1/2}\partial_s^{2\ell+1}\widetilde{q}(0,x,0)}{(2\ell+1)!}\nu(x) \\ &\sim t^{-n/2}\sum_{\ell \geqslant 0} t^{\ell} a_{\ell},\end{aligned}$$

hence the claim follows. □

Theorem D.7. For $z \in \mathbb{C}, \operatorname{Re} z > \dim Y - 1/2 = n - 1/2$, the integral

$$\eta(z) := \frac{1}{\Gamma(\frac{z+1}{2})}\int_0^\infty t^{(z-1)/2} \operatorname{Tr}(\mathcal{D}_0 \, \mathrm{e}^{-t\mathcal{D}_0^2}) \, \mathrm{d}t$$

exists, is holomorphic, and

$$\eta(z) = \sum_{\lambda_j \neq 0} \frac{\operatorname{sign} \lambda_j}{|\lambda_j|^z}.$$

Moreover, $\eta(z)$ extends to be a meromorphic function over $\mathbb{C}$.

Proof. Recall that $1/\Gamma((z+1)/2)$ is holomorphic, hence we only need to analyze

$$\eta_0(z) = \int_0^\infty t^{(z-1)/2} \operatorname{Tr}(\mathcal{D}_0 \, \mathrm{e}^{-t\mathcal{D}_0^2}) \, \mathrm{d}t.$$

According to(D.8), for any $N \in \mathbb{N}$, write

$$g_N(t) = \operatorname{Tr}(\mathcal{D}_0 \, \mathrm{e}^{-t\mathcal{D}_0^2}) - t^{-n/2}\sum_{k=0}^{N} t^k a_k, \tag{D.9}$$

then near $t = 0$, we have

$$|g_N(t)| \leqslant C_N t^{N+1-n/2}.$$

Note that

$$\eta_0(z) = \left(\int_0^1 t^{(z-1)/2} \operatorname{Tr}(\mathcal{D}_0 \, \mathrm{e}^{-t\mathcal{D}_0^2}) \, \mathrm{d}t + \int_1^\infty t^{(z-1)/2} \operatorname{Tr}(\mathcal{D}_0 \, \mathrm{e}^{-t\mathcal{D}_0^2}) \, \mathrm{d}t \right) = \eta_1(z) + \eta_2(z),$$

where

$$\eta_1(z) = \int_0^1 t^{(z-1)/2} \operatorname{Tr}(\mathcal{D}_0 \, \mathrm{e}^{-t\mathcal{D}_0^2}) \, \mathrm{d}t,$$

and

$$\eta_2(z) = \int_1^\infty t^{(z-1)/2} \operatorname{Tr}(\mathcal{D}_0 \, \mathrm{e}^{-t\mathcal{D}_0^2}) \, \mathrm{d}t.$$

From Weyl's law, $\operatorname{Tr}(\mathcal{D}_0 \, \mathrm{e}^{-t\mathcal{D}_0^2})$ decays exponentially as $t \to \infty$, hence $\eta_2(z)$ is holomorphic over $\mathbb{C}$. With (D.9), we have

$$\begin{aligned} \eta_1(z) &= \int_0^1 t^{(z-1)/2} \left(t^{-n/2} \sum_{k=0}^{N} t^k a_k + g_N(t) \right) \mathrm{d}t \\ &= \sum_{k=0}^{N} \int_0^1 t^{\frac{z-n-1}{2}+k} a_k \, \mathrm{d}t + \int_0^1 t^{\frac{z-1}{2}} g_N(t) \, \mathrm{d}t \\ &= \sum_{k=0}^{N} \frac{a_k}{(z-n-1)/2 + k + 1} + \int_0^1 t^{\frac{z-1}{2}} g_N(t) \, \mathrm{d}t. \end{aligned}$$

Hence, $\eta(z)$ is holomorphic over $\{\operatorname{Re} z > n - 1/2\}$, and extends to a meromorphic functions.

By (D.7), we compute

$$
\begin{aligned}
\eta(z) &= \frac{1}{\Gamma(\frac{z+1}{2})} \int_0^\infty t^{(z-1)/2} \operatorname{Tr}(\mathcal{D}_0 \, \mathrm{e}^{-t\mathcal{D}_0^2}) \, \mathrm{d}t \\
&= \frac{1}{\Gamma(\frac{z+1}{2})} \int_0^\infty t^{(z-1)/2} \sum_{\lambda_j \neq 0} \lambda_j \, \mathrm{e}^{-t\lambda_j^2} \, \mathrm{d}t \\
&= \frac{1}{\Gamma(\frac{z+1}{2})} \sum_{\lambda_j \neq 0} \int_0^\infty \lambda_j^{-1} \left(\frac{u}{\lambda_j^2} \right)^{(z-1)/2} \mathrm{e}^{-u} \, \mathrm{d}u \\
&= \frac{1}{\Gamma(\frac{z+1}{2})} \sum_{\lambda_j \neq 0} \frac{\lambda_j^{-1}}{|\lambda_j|^{z-1}} \int_0^\infty u^{(z+1)/2-1} \, \mathrm{e}^{-u} \, \mathrm{d}u \\
&= \sum_{\lambda_j \neq 0} \frac{\operatorname{sign} \lambda_j}{|\lambda_j|^z}. \qquad \square
\end{aligned}
$$

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