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On a Generalization of the Hanoi Towers Group

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ON A GENERALIZATION OF THE HANOI TOWERS GROUP

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DISSERTATION

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Abstract

In 2012, Bartholdi, Siegenthaler, and Zalesskii computed the rigid kernel for the only known group for which it is non-trivial, the Hanoi towers group. There they determined the kernel was the Klein 4 group. We present a simpler proof of this theorem. In the course of the proof, we also compute the rigid stabilizers and present proofs that this group is a self-similar, self-replicating, regular branch group.

We then construct a family of groups which generalize the Hanoi towers group and study the congruence subgroup problem for the groups in this family. We show that unlike the Hanoi towers group, the groups in this generalization are just infinite and have trivial rigid kernel. We also put strict bounds on the branch kernel. Additionally, we show that these groups have subgroups of finite index with non-trivial rigid kernel, adding infinitely many new examples. Finally, we show that the topological closures of these groups have Hausdorff dimension arbitrarily close to 1.

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Contents

List of Figures

Introduction

The study of branch groups has its origin in the introduction of the Grigorchuk group in 1980. Grigorchuk originally defined this group as a group of Lebesgue measure preserving transformations of $[0,1] \setminus \{\frac{k}{2^m} \mid k, m \in \mathbb{Z}\}$. It was later understood that this group could be viewed as a group of automorphisms of an infinite, rooted binary tree and this viewpoint became the dominant one. The Grigorchuk group was the first example of a group of intermediate word growth and of an amenable but not elementary amenable group. Additionally, it is an infinite, finitely generated group where every element has order a power of 2, adding a very concrete example to the groups which provide a solution to the General Burnside Problem. Later, the Gupta-Sidki branch groups were introduced as examples of infinite p-groups for primes $p \geq 3$. To this day, the only tractable examples of Burnside groups come from this class.

The importance of branch groups comes not just from their potential to have exotic properties but also from their role in the classification of just infinite groups. In 2000, Grigorchuk showed, building on the results of Wilson, that every just infinite group falls into one of three distinct categories, branch groups being one of them. Every finitely generated group quotients onto a just infinite group. Thus, if one wishes to find a finitely generated infinite group with some property and this property is maintained under taking quotients, then it suffices to search among groups that are just infinite. Just infinite groups arise in the study of profinite groups as well. Since profinite groups have many finite quotients, and therefore can not be simple, just infinite groups serve as the analogue of simple groups for the class of profinite groups.

In this dissertation, we study a sequence of branch groups $\{G_n\}_{n=3}^{\infty}$ which generalize the Hanoi towers group, a group first introduced and studied by Grigorchuk and Šunić in 2006. We focus on these groups to obtain a better understanding of the congruence subgroup problem for branch groups which derives its name from the classical congruence subgroup problem for $SL(n,\mathbb{Z})$. The congruence subgroup problem for $SL(n,\mathbb{Z})$ asks if every subgroup of finite index contains a principal congruence subgroup, the kernel of the surjection $SL(n,\mathbb{Z}) \twoheadrightarrow SL(n,\mathbb{Z}/m\mathbb{Z})$ for some m. This is false for $n = 2$ but was answered affirmatively for $n \geq 3$ by Bass, Lazard, and Serre in 1964. Branch groups are defined in terms of their action on a rooted tree and so we say a branch group G has the congruence subgroup property if every subgroup of finite index contains the subgroup consisting of elements which stabilize some level of the tree, i.e. $Stab_G(m)$ for some $m \geq 1$. Branch groups have another family of finite index subgroups, namely the rigid stabilizers which we denote by $Rist_G(m)$ for $m \geq 1$, and so a branch group has the congruence subgroup property if and only if

1. every subgroup of finite index contains a rigid stabilizer and

2. every rigid stabilizer contains a level stabilizer.

The congruence subgroup problem can be restated in terms of profinite groups and therefore profinite group theory will be influential here. A branch group G has the congruence subgroup property if and only if the congruence kernel

$$\ker(\varprojlim_{N\in\mathcal{N}} G/N \twoheadrightarrow \varprojlim_{m\geq 1} G/Stab_G(m))$$

is trivial where $\mathcal{N}$ is the set of all finite index normal subgroups.

For a branch group, the congruence kernel has two parts namely the branch kernel

$$\ker(\varprojlim_{N\in\mathcal{N}} G/N \twoheadrightarrow \varprojlim_{m\geq 1} G/Rist_G(m))$$

and the rigid kernel

$$\ker(\varprojlim_{m\geq 1} G/Rist_G(m) \twoheadrightarrow \varprojlim_{m\geq 1} G/Stab_G(m)).$$

Thus the congruence subgroup problem for branch groups asks not only does a group have the congruence subgroup property but to quantitatively describe the congruence, branch, and rigid kernels.

Up until this point, the Hanoi towers group, which we refer to here as G_3, was the only group shown to fail property 2 and have a non-trivial rigid kernel. Hence, a motivating factor for this work was to construct new examples of branch groups which fail property 2. Surprisingly, all of the groups in the generalization except the Hanoi towers group have trivial rigid kernels. The structure of the group G_n depends on n and so this theorem and others are proved in cases.

Theorem 2.9, 3.23, 3.26, 3.28. *The group G_n has non-trivial rigid kernel if and only if $n = 3$.*

Likewise, the branch kernel for G_n with $n \neq 3$ differs significantly from the branch kernel of the Hanoi towers group which contains free profinite abelian subgroups.

Theorem 3.30. *For $n \neq 3$, the branch kernel, and thus the congruence kernel, for G_n is the inverse limit*

$$\varprojlim_{m \geq 1} M_n^m$$

where M_n is a finite abelian group. When $n \geq 5$ is even, M_n is cyclic of order $n-1$ and when $n = 4$ or $n \geq 5$ is odd, M_n has exponent bounded between $(n-1)$ and $2(n-1)$.

The groups in the generalization differ from the Hanoi towers group not just in their subgroup structure but also in the types of quotients they exhibit.

Theorem 3.32. *G_n is just infinite if and only if $n \neq 3$.*

Although the groups in the generalization do not fail the congruence subgroup property in the same way that G_3 does, we nevertheless find infinitely many new examples of groups failing property 2. These new examples live as finite index subgroups of G_n.

Theorem 4.1. *For $n \geq 4$, let $1 \neq d > 2$ be such that $d \mid (n-1)$ and let $H_{n,d}$ be the set of elements g of G_n with $\epsilon(g) \equiv 0 \bmod d$. The $H_{n,d}$ is a subgroup of index d in G_n and is a branch group with non-trivial rigid kernel.*

As a consequence of the work in proving these theorems, we are able to calculate the Hausdorff dimension of G_n and show that for n sufficiently large the Hausdorff dimension of G_n is arbitrarily close to 1.

Theorem 3.33. *For $n \geq 3$, the Hausdorff dimension for $\overline{G}_n$ is*

$$dim_{\mathcal{H}}(\overline{G}_n) = \begin{cases} 1 - \frac{\log(48)}{\log(331776)} & \text{if } n = 4 \\ 1 - \frac{\log(2)}{\log(n!)} & \text{if } n \geq 5 \text{ is even} \\ 1 - \frac{\log(2)}{n \log(n!)} & \text{if } n \text{ is odd} \end{cases}$$

Corollary 3.34. *For all $\epsilon > 0$, there exists n such that $dim_{\mathcal{H}}(\overline{G}_n) > 1 - \epsilon$.*

We organize this work as follows. In Chapter 1, we provide all the necessary background for the remaining chapters. In Chapter 2, we study the Hanoi towers group and provide an alternative, more constructive proof of the 2012 result of Bartholdi, Siegenthaler, and Zalesskii that it fails condition 2 by having the Klein 4 group as its rigid kernel. Chapter 3 contains the majority of the new results. There we study the generalization of the Hanoi towers group. We find a solution to the word problem that also allows us to determine the abelianization of G_n. We show that these groups are branch groups and also compute precisely the rigid and level stabilizers of these groups. This is then used to quantitatively describe the rigid, branch, and congruence kernels for these groups. From there we obtain that G_n is just infinite if and only if $n \neq 3$. The computation of the rigid and level stabilizers is ultimately applied to find the Hausdorff dimension. In Chapter 4, we provide the infinitely many new examples of branch groups with non-trivial rigid kernels. Finally, in Chapter 5, we provide some questions which remain open in this subject.

Chapter 1 Background

1.1 Notation

For two group elements g and h we write g^h to indicate $h^{-1}gh$ and $[g,h]$ for $g^{-1}h^{-1}gh$. For a group G, G' will mean the commutator subgroup of G. For a subset $S \subseteq G$, we write $\langle\langle S \rangle\rangle$ for the normal closure of S in G.

1.2 Profinite groups and profinite completions

A main focus in this dissertation will be on understanding topological completions of groups with various profinite topologies and using them to answer questions about the group. In this section, we introduce profinite groups and profinite completions. For more information the reader is directed to [19] or [23].

A **topological group** is a group endowed with a topology such that the group operations are continuous, i.e. the map $(g,h) \mapsto gh^{-1}$ is a continuous function from $G \times G$ to G. To define a topology on a group G it suffices to define a basis for the neighborhoods of $\{1\}$ as a basis for the neighborhoods around any other point can be obtained via left multiplication.

A partially ordered set $I = (I, \lesssim)$ is called **directed** if for all $i, j \in I$, there exists $k \in I$ with $i, j \lesssim k$. Two directed sets I and I' which are subsets of a potentially larger partially ordered set are said to be cofinal if for every $i \in I$ there is an $i' \in I'$ such that $i \lesssim i'$ and likewise for every $i' \in I'$ there exists $i \in I$ with $i' \lesssim i$.

An **inverse system** of finite groups is a collection of $\{H_i \mid i \in I\}$ indexed by a directed set I and along with a collection of homomorphisms $\rho_{ij} : H_i \to H_j$ defined for any pair i and j with $j \lesssim i$ and such that the homomorphisms are compatible, i.e. whenever $k \lesssim j \lesssim i$ we have $\rho_{ik} = \rho_{jk} \circ \rho_{ij}$.

The **inverse limit**

$$\varprojlim_{i \in I} H_i$$

is the subgroup of $\prod_{i \in I} H_i$ consisting of tuples $h = (h_i)_{i \in I}$ where $h_j = \rho_{ij}(h_i)$ whenever $j \lesssim i$.

A group G is a **profinite group** if it is the inverse limit of finite groups. If we endow finite groups with the discrete topology and $\prod_{i \in I} H_i$ with the product topology, then the inverse limit is a closed subgroup of $\prod_{i \in I} H_i$. As a result, we get that as a topological group G is compact, Hausdorff, and totally disconnected group. The converse is also true. A group G is profinite if and only if it is compact, Hausdorff, and totally disconnected.

It is not hard to see that a subgroup of a profinite group G is open in G if and only if it is closed and has finite index.

Proposition 1.1. *[19] Suppose $\{H_i \mid i \in I\}$ and $\{H_{i'} \mid i' \in I'\}$ are two inverse systems where I and I' are cofinal subsets of the same partially ordered set. Then*

$$\varprojlim_{i \in I} H_i \cong \varprojlim_{i' \in I'} H_{i'}$$

When I is countable, then there is a set I' cofinal to I that is countable and totally ordered. This occurs in particular when G is finitely generated (as a topological group). Thus when I is countable we may assume it is totally ordered. In this case, there is a metric compatible with the topology on

$$\varprojlim_{i \in I} H_i$$

given by $d(g, h) = \frac{1}{|H_m|}$ where m is the least value with $g_m h_m^{-1} \in H_m$.

For a group G, let $\mathcal{C}$ be a collection of normal, finite index subgroups such that for all N and $\tilde{N}$ in $\mathcal{C}$ there is M in $\mathcal{C}$ with $M \leq N \cap \tilde{N}$. Then $\mathcal{C}$ provides a basis for the neighborhoods of $\{1\}$ and the collection $\{G/N \mid N \in \mathcal{C}\}$ forms a directed set where $G/N \lesssim G/M$ whenever $M \leq N$ where the maps $\rho_{MN} : G/M \to G/N$ are the natural surjections.

The inverse limit

$$\varprojlim_{N \in \mathcal{C}} G/N$$

is called the **profinite completion of** G **with respect to** $\mathcal{C}$ because it is the topological completion with respect to the basis $\mathcal{C}$.

1.3 Tree automorphisms

The groups studied here are defined in terms of their actions on a rooted tree and so for this reason we introduce some initial vocabulary and notation to aid in the discussion of groups of this type.

A **rooted tree** is an acyclic connected graph with a designated vertex called the root. For any two vertices in the tree, the **distance** between them is the length of the (unique) geodesic between them. For any vertex u in the tree, its **level** will be defined as the length of the path from the root to u and denoted $|u|$. A rooted tree is called **spherically homogeneous** if every vertex on the same level has the same (finite) degree.

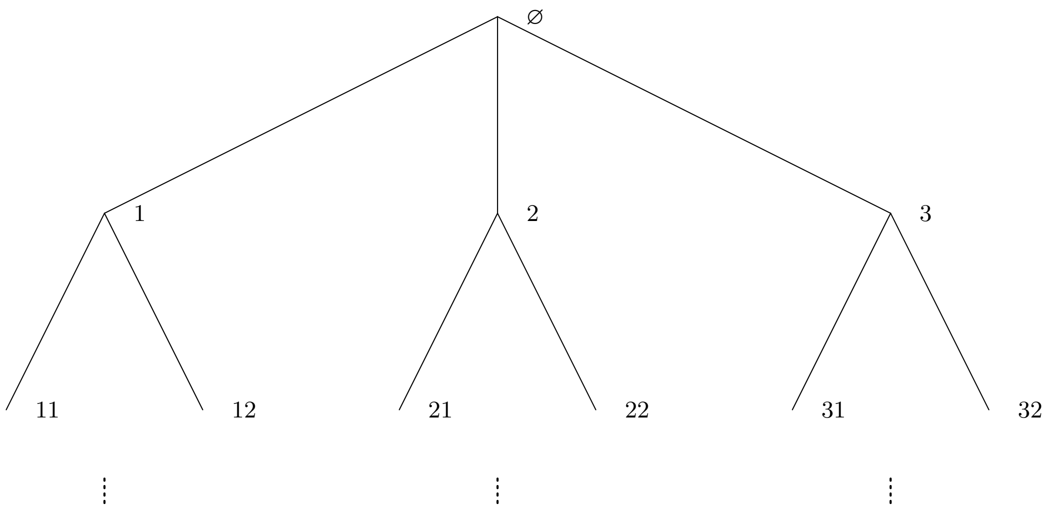

Figure 1.1: Rooted tree with $X_1 = \{1,2,3\}$ and $X_2 = \{1,2\}$

An infinite, spherically homogeneous, rooted tree is fully determined by a sequence of integers $\overline{n} = (n_1, n_2, \dots)$ where each vertex of level $m-1$ has n_m adjacent vertices of level m. We will write $\mathcal{T}_{\overline{n}}$ to denote the tree with defining sequence $\overline{n}$. When the defining sequence is either arbitrary or clear from the context, the subscript will be dropped. With this notation, we will write $\varnothing$ for the root and we will identify a vertex v of level m with a sequence $v = v_1 v_2 \cdots v_m$ where $v_i \in X_i$, an arbitrary set of size n_i, and where the prefixes of the sequence correspond to the vertices on the geodesic between v and $\varnothing$. By placing an order on each X_i, the set of vertices of level m in $\mathcal{T}$ (i.e. the elements in $X_1 X_2 \cdots X_m$) can be ordered linearly using the lexicographical ordering. Note that in general, the set X_i will be assumed to be the set $\{1, 2, \dots, n_i\}$ with the natural ordering and when convenient, the vertices of level m will be numbered by the indexing set $\{1, 2, \dots, n_1 \cdots n_m\}$. See Figure 1.1.

The set of all vertices of $\mathcal{T}_{\overline{n}}$ will be denoted $V(\mathcal{T}_{\overline{n}})$.

Definition 1.2. *The automorphism group of $\mathcal{T}_{\overline{n}}$, denoted Aut$(\mathcal{T}_{\overline{n}})$, is the set of bijections from $V(\mathcal{T}_{\overline{n}})$ to $V(\mathcal{T}_{\overline{n}})$ that fix the root and preserve edge incidences.*

Thus, under an automorphism, vertices of the same level in $\mathcal{T}_{\overline{n}}$ can only be permuted among themselves. Further, if two vertices share a prefix, then their image under an automorphism will share a prefix of the same length. Because of this, an element g in $\mathrm{Aut}(\mathcal{T}_{\overline{n}})$ can be regarded as a labeling of the vertices of $\mathcal{T}_{\overline{n}}$ by permutations, $\{g(v)\}_{v \in V(\mathcal{T}_{\overline{n}})}$, where if $|v| = m-1$ then $g(v) \in S_{n_m}$, the symmetric group on n_m letters. Then for a vertex $v = v_1 v_2 \cdots v_m$, the action of g is computed as

$$v^g = v_1^{g(\varnothing)} v_2^{g(v_1)} \cdots v_m^{g(v_1 \cdots v_{m-1})}.$$

Rules for composing and finding inverses are given by the following proposition.

Proposition 1.3. *[13]*

1. *Let* $h = fg$ *where* $f, g, h \in Aut(\mathcal{T}_{\overline{n}})$. *Then* $h(u) = f(u)g(u^f)$ *for all* $u \in V(\mathcal{T}_{\overline{n}})$.

2. *Let* $f = g^{-1}$ *where* $g \in Aut(\mathcal{T}_{\overline{n}}))$. *Then* $f(u) = (g(u^{g^{-1}}))^{-1}$ *for all* $u \in V(\mathcal{T}_{\overline{n}})$.

We say a vertex $v \in V(\mathcal{T}_{\overline{n}})$ is a **descendant** of u if the geodesic from v to $\varnothing$ includes the geodesic from u to $\varnothing$. The set of descendants of u forms a subtree rooted at u, denoted $\mathcal{T}_u$. If $\mathcal{T}_{\overline{n}}$ is a spherically homogeneous, rooted tree then for any m, each subtree of $\mathcal{T}_{\overline{n}}$ rooted at a vertex of level m is canonically isomorphic to $\mathcal{T}_{\psi^m(\overline{n})}$, where $\psi(\overline{n}) = (n_2, n_3, \dots)$ and ψ^m is the m-fold iteration of the function ψ. As a result, there is a natural isomorphism $\mathrm{Aut}(\mathcal{T}_{\overline{n}}) \cong \mathrm{Aut}(\mathcal{T}_{\psi^m(\overline{n})}) \wr M_m = (\prod \mathrm{Aut}(\mathcal{T}_{\psi^n(\overline{n})})) \rtimes M_m$ where $M_m = (\cdots(S_{n_m} \wr S_{n_{m-1}}) \wr \cdots) \wr S_{n_1}$. The iterated wreath product M_m is the automorphism group of the finite subtree of $\mathcal{T}_{\overline{n}}$ consisting of vertices of level less than or equal to m.

Throughout this work, we will canonically identify tree automorphisms g with their decomposition $(g_1, \dots, g_{n_1})\sigma$ under the isomorphism $\mathrm{Aut}(\mathcal{T}_{\overline{n}}) \cong \mathrm{Aut}(\mathcal{T}_{\psi(\overline{n})}) \wr S_{n_1}$.

For $g \in \mathrm{Aut}(\mathcal{T}_{\overline{n}})$ and for v a vertex of level m, we will denote by g_v the vth coordinate of g in the canonical identification $\mathrm{Aut}(\mathcal{T}_{\overline{n}}) \cong (\mathrm{Aut}(\mathcal{T}_{\psi^m(\overline{n})}) \wr M_m$, and we will call it the **state of** g **at** v. We will also take π_v to be the projection map $g \mapsto g_v$. Note that for a subgroup $G \leq \mathrm{Aut}(\mathcal{T}_{\overline{n}})$, π_v may not be a homomorphism.

Likewise, for a vertex v of level m and an element $g \in \mathrm{Aut}(\mathcal{T}_{\psi^m(\overline{n})})$, $v * g$ will be used to denote the automorphism of $\mathcal{T}_{\overline{n}}$ which acts as g on the subtree rooted at v and trivially outside of it. For a subgroup $G \leq \mathrm{Aut}(\mathcal{T}_{\psi^m(\overline{n})})$,

$$v * G = \{v * g \mid g \in G\}$$

and

$$X_1 \dots X_m * G = \{v * g \mid g \in G, |v| = m\} \cong \prod_{n_1 \cdots n_m} G.$$

1.4 Branch groups

For any subgroup G of $\mathrm{Aut}(\mathcal{T}_{\overline{m}})$, certain families of subgroups arise naturally.

Definition 1.4. *For a vertex* $v \in V(\mathcal{T}_{\overline{m}})$, *the vertex stabilizer of* v, $Stab_G(v)$, *is the set of elements in* G *which fix the vertex* v.

In terms of the labeling of the vertices by elements in a symmetric group, this consists of the elements that necessarily have trivial labeling on all vertices on the path between u and $\varnothing$, except possibly at u.

Definition 1.5. *For a non-negative integer m, the mth level stabilizer, $Stab_G(m)$, is the normal subgroup $\bigcap_{|u|=m} Stab_G(u)$.*

In terms of the labelings, this consists of the elements of G with trivial labeling on all vertices v where $|v| \leq m-1$. Note that for all m, $Stab_G(m)$ has finite index in G. Under the decomposition, $\mathrm{Aut}(\mathcal{T}_{\overline{n}}) \cong (\mathrm{Aut}(\mathcal{T}_{\psi^m(\overline{n})}) \wr M_m$, these are exactly the elements whose coordinate in M_m has the trivial element. For that reason, elements in $g \in Stab_G(m)$ will be described by a tuple of the form $(g_1, g_2, \dots, g_{n_1 \cdots n_m})_m$ where each g_i is the state of g for a vertex on the mth level. Moreover, such elements appear in the quotient $Stab_G(m)/Stab_G(m+1)$ as the direct product of $n_1 \cdots n_m$ permutations in $S_{n_{m+1}}$, where the ith coordinate is the permutation at the root of g_i.

The full automorphism group of the tree is itself a profinite group with a basis for the neighborhoods of $\{1\}$ consisting of the level stabilizers. In other words,

$$\mathrm{Aut}(\mathcal{T}) \cong \varprojlim_{m \geq 1} M_m$$

where again M_m is the automorphism group of the finite subtree of $\mathcal{T}$ consisting of the first m-levels.

A group is said to be **residually finite** if the intersection of its normal, finite index subgroups is trivial. Since

$$\bigcap_{m \geq 1} Stab_{\mathrm{Aut}(\mathcal{T})}(m) = \{1\}$$

$\mathrm{Aut}(\mathcal{T})$ is residually finite as are all of its subgroups.

A subgroup of $\mathrm{Aut}(\mathcal{T})$ is called **spherically transitive** if it acts transitively on every level of the tree. We will be interested in a particular class of spherically transitive groups, namely branch groups.

Definition 1.6. *The rigid stabilizer of a vertex u, $Rist_G(u)$, consists of the elements of G which act trivially outside of the subtree rooted at u.*

In terms of the labeling, this consists of elements that have trivial labeling on all vertices outside of $\mathcal{T}_u$. If G is spherically transitive then for any two vertices u and v such that $|u| = |v|$, $Rist_G(u) \cong Rist_G(v)$ (and in fact are conjugate in G).

Definition 1.7. *For a non-negative integer m, the mth level rigid stabilizer is the normal subgroup $Rist_G(m) = \langle Rist_G(u) \mid |u| = m \rangle = \prod_{|u|=m} Rist_G(u)$, the internal direct product of the rigid stabilizers of the vertices of level m.*

For any group G acting faithfully on $\mathcal{T}_{\overline{n}}$, $Rist_G(m) \leq Stab_G(m)$. In terms of the decomposition $\mathrm{Aut}(\mathcal{T}_{\overline{n}}) \cong \mathrm{Aut}(\mathcal{T}_{\psi^m(\overline{n})}) \wr M_m$, $Stab_G(m)$ sits inside the direct product of $n_1 \cdots n_m$ copies of $\mathrm{Aut}(\mathcal{T}_{\psi^m(\overline{n})})$. With this interpretation, $Rist_G(m)$ is the largest subgroup of $Stab_G(m)$ which actually decomposes as a direct product.

Definition 1.8. *A group G is a branch group if there is an embedding of G into $Aut(\mathcal{T})$ such that G is spherically transitive and for all m, $Rist_G(m)$ has finite index in G. A profinite group G is a branch group if in addition it embeds as a closed subgroup of $Aut(\mathcal{T})$ and $Rist_G(m)$ is an open subgroup of G for all m.*

It is worth noting that in some papers such as [6], the transitivity assumption is dropped.

Although, this definition relies on the existence of a tree in which the group is acting as a branch group, the existence of such a tree is actually a group theoretic property.

A group is **virtually abelian** if it has a abelian subgroup of finite index.

Definition 1.9. *A subgroup B of a group G is basal if B has only finitely many distinct conjugates and $\langle\langle B\rangle\rangle$ is their internal direct product.*

The rigid stabilizers are examples of subgroups of basal type. The following theorem was proved in [15].

Theorem 1.10. *[15] Let G be an abstract or profinite group. Then G is a branch group if and only if each of the following conditions hold:*

1. *G is just non-(virtually abelian) with no non-trivial virtually abelian normal subgroups;*
2. $\bigcap_{B \text{ basal}} N_G(B) = \{1\}$;
3. *For each non-trivial basal subgroup A, the normal closure in G of the subgroup*

$$\bigcap_{\substack{B \text{ basal} \\ A\cap B=\{1\}}} N_G(B)$$

has finite index in G.

Branch groups derive their importance from the exotic properties groups in this class can possess. The first known example of an amenable but not elementary amenable group and of a group of intermediate growth is a branch group. The tractable examples of Burnside groups, finitely generated infinite torsion groups, come from this class as well.

Branch groups also arise in the classification of just infinite groups which serve as the analogue of simple groups in the class of profinite groups.

An (abstract or profinite) group G is **just infinite** if G is infinite and every proper quotient of G is finite. A residually finite group is **hereditarily just infinite** if every normal subgroup of finite index is just infinite. A profinite residually finite group is **hereditarily just infinite** if every open subgroup is just infinite.

Proposition 1.11. *[13] Let G be a finitely generated infinite group. Then G has a just infinite quotient.*

Thus if one wishes to study some property of infinite groups and this property is maintained under taking quotients, then it often reduces to studying the property for the class of just infinite groups.

Grigorchuk constructed a classification of just infinite groups, applying previous work of Wilson [24].

Theorem 1.12. *[13]*

1. *Let G be an abstract just infinite group. Then either G is a branch group or G contains a normal subgroup of finite index which is isomorphic to the direct product of a finite number of copies of a group L, where L is either simple or hereditarily just infinite.*

2. *Let G be a profinite just infinite group. Then either G is a branch group or G contains an open normal subgroup which is isomorphic to the direct product of a finite number of copies of some hereditarily just infinite profinite group.*

1.5 Regular trees and self-similar groups

Whenever the defining sequence for a spherically homogeneous rooted tree is constant the tree is called **regular**. In particular, if $n = n_1 = n_2 = \cdots$, we call $\mathcal{T}$ an **n-ary tree**.

In this case, we fix a set X of size n, called the **alphabet**, and let $X = X_1 = X_2 = \cdots$. Then the set of vertices on the mth level corresponds exactly to the set of words of length m in the alphabet. We will denote this by X^m. The set of all finite words, i.e. the set of vertices of $\mathcal{T}$, will thus be denoted by X^*.

For the regular n-ary tree $\mathcal{T}$, $\mathrm{Aut}(\mathcal{T}) \cong \mathrm{Aut}(\mathcal{T}) \wr S_n$ and so $\mathrm{Aut}(\mathcal{T})$ contains naturally occurring isomorphic copies of itself and it can have subgroups with similar properties.

Definition 1.13. *A group $G \leq Aut(\mathcal{T})$ is called self similar if for all $g = (g_1, \ldots, g_n)\sigma \in G$ and for each i, $g_i \in G$.*

A self-similar group is called **self-replicating** if $\pi_v(Stab_G(v)) = G$.

Definition 1.14. *For $\mathcal{T}$ a regular tree, a subgroup $G \leq Aut(\mathcal{T})$ is said to be regular branch if it is sphericaly transitive and there is a subgroup K with finite index in G such that $v * K \leq K$ for all $v \in X^*$ and such that $X^m * K$ has finite index in G for all m. In this case, K is called a branching subgoup.*

If a group is regular branch then it is also branch as $X^m * K \leq Rist_G(m)$.

1.6 The congruence subgroup problem for branch groups

The congruence subgroup property for branch groups derives its name from the congruence subgroup problem for $SL(n,\mathbb{Z})$ which asks if every subgroup of finite index in $SL(n,\mathbb{Z})$ contains a principal congruence subgroup, the kernel of the map $SL(n,\mathbb{Z}) \to SL(n,\mathbb{Z}/m\mathbb{Z})$ for some m. This is false for $n = 2$ but was answered affirmatively for $n \geq 3$ in [7].

Definition 1.15. *A subgroup $G \leq Aut(\mathcal{T})$ has the* ***congruence subgroup property*** *if every subgroup of finite index contains $Stab_G(m)$ for some m.*

Since for a branch group the rigid stabilizers are subgroups of the level stabilizers and since the rigid stabilizers have finite index, a branch group has the congruence subgroup property if and only if every subgroup of finite index contains a rigid stabilizer and every rigid stabilizer contains a level stabilizer.

We can restate this in terms of profinite completions as follows. Since $Stab_G(m)$ has finite index in G for all m and since this collection forms a descending collection of normal subgroups, taking $\{Stab_G(m) | m \in \mathbb{N}\}$ as a basis for the neighborhoods of $\{1\}$ produces a topology on G called the **congruence topology**. Likewise $Rist_G(m)$ has finite index for all n, and in the same way produces a topology called the **branch topology**. Further, G has a third natural topology, the **full profinite topology** where $\mathcal{N} = \{N \trianglelefteq G \mid |G : N| < \infty\}$ is taken as a basis for the neighborhoods of $\{1\}$. Observe that the congruence topology is weaker than the branch topology which is weaker than the full profinite topology. We can complete G in terms of these topologies and obtain three profinite groups:

$$\overline{G} = \varprojlim_{m\geq 1} G/Stab_G(m) \qquad \textbf{the congruence completion}$$

$$\widetilde{G} = \varprojlim_{m\geq 1} G/Rist_G(m) \qquad \textbf{the branch completion}$$

$$\widehat{G} = \varprojlim_{N\in\mathcal{N}} G/N \qquad \textbf{the profinite completion}$$

As G is a subgroup of $\mathrm{Aut}(\mathcal{T})$, we see that $\bigcap_{m\geq 1} Stab_G(m) = \{1\}$, G is residually finite and embeds into $\overline{G}$, $\widetilde{G}$, and $\widehat{G}$.

Thus G has the congruence subgroup property if and only if the **congruence kernel**, $\ker(\widehat{G} \to \overline{G})$, is trivial. The **congruence subgroup problem** for branch groups asks not only whether a branch group has the congruence subgroup property but also to quantitatively describe the congruence kernel. Since there is a third topology at play, namely the branch topology, we can instead study two pieces of the congruence kernel, namely the **branch kernel**, $\ker(\widehat{G} \to \widetilde{G})$, and the **rigid kernel**, $\ker(\widetilde{G} \to \overline{G})$.

Although a group may have many realizations as a branch group, each of these kernels are invariants of the isomorphism class of the group and are not dependent on the choice of realization.

Theorem 1.16. *[12] Let G be a group and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two spherically homogeneous trees. Suppose G embeds into both $Aut(\mathcal{T}_1)$ and $Aut(\mathcal{T}_2)$ as a branch group. Then the branch, rigid, and congruence kernels corresponding to $\mathcal{T}_1$ and $\mathcal{T}_2$ are the same.*

Many of the most studied branch groups have been shown to have trivial congruence kernel, including the Fabrykowsky-Gupta group and the Gupta-Sidki groups [3], [11], the Grigorchuk group and an infinite family of generalizations of the Fabrykowski-Gupta group [13], and GGS-groups with non-constant accompanying vectors [18], [9].

Pervova [18] constructed the first branch groups without the congruence subgroup property. Nevertheless, the groups in her infinite family, periodic EGS groups with non-symmetric accompanying vector, have non-trivial branch kernel but trivial rigid kernel. Likewise, the twisted twin of the Grigorchuk group was found to have non-trivial branch kernel but trivial rigid kernel [4].

Despite the existence of infinite families of groups having either trivial branch and trivial rigid kernel or non-trivial branch kernel but trivial rigid kernel, only one group appearing previously in the literature has been shown to have non-trivial rigid kernel [6], [22]. This kernel is studied in detail in Chapter 2 and new examples of groups with this property are produced in Chapter 4.

Some general properties of the kernels are known in certain cases.

Theorem 1.17. *[6] Let G be a self-similar, regular branch group. Then the branch kernel is abelian and the rigid kernel has finite exponent.*

For any branch group G, the rigid kernel is

$$\ker(\widetilde{G} \to \overline{G}) = \varprojlim_{m\geq 1} Stab_G(m)/Rist_G(m)$$

where the maps $\rho_{m,m+k} : Stab_G(m+k)/Rist_G(m+k) \to Stab_G(m)/Rist_G(m)$ come from the natural inclusions $Stab_G(m+k) \hookrightarrow Stab_G(m)$ and $Rist_G(m+k) \hookrightarrow Rist_G(m)$. This is because, by definition, $\widetilde{G}$ is the subgroup of $\prod_{m\geq 1} G/Rist_G(m)$ consisting of sequences $(g_m Rist_G(m))_{m\geq 1}$ where

$$g_{m+1} Rist_G(m) = g_m Rist_G(m)$$

for all m. Likewise, by definition $\overline{G}$ is the subgroup of $\prod_{m\geq 1} G/Stab_G(m)$ consisting of sequences $(h_m Stab_G(m))_{m\geq 1}$ where

$$h_{m+1}Stab_G(m) = h_m Stab_G(m)$$

for all m. Thus the kernel of the map $\widetilde{G} \to \overline{G}$ is precisely those sequences $(g_m Rist_G(m))_{m\geq 1}$ where for all m, $g_m \in Stab_G(m)$, i.e. $\varprojlim_{m\geq 1} Stab_G(m)/Rist_G(m)$. Note that the maps $\rho_{m,m+k}$ are far from being surjective.

By similar reasoning, the branch kernel is given by

$$\ker(\widehat{G} \to \widetilde{G}) = \varprojlim_{\substack{m\geq 1\\ N\in\mathcal{N}}} Rist_G(m)N/N.$$

Fortunately, there are some tools available for studying the branch kernel. The following theorem can be derived from the proof of Theorem 4 in [13]. Since this is not a direct citation of a theorem, we include a proof.

Theorem 1.18. *Let G be a branch group. If H is a finite index subgroup of G then H contains $rist_G(m)'$ for some m.*

Proof. Let G be a branch group acting on a tree $\mathcal{T}_{\bar{n}}$. Then it suffices to show that for any $1 \neq g \in G$ that $\langle\langle g\rangle\rangle$ contains $Rist_G(k)'$ for some k.

Take g as above. There exists an m such that $g \in Stab_G(m)\backslash Stab_G(m+1)$. Choose a vertex u with $|u| = m$ and such that the state of g at u decomposes as $g_u = (g_1, \ldots, g_{n_{m+1}})\theta$ where θ is a non-trivial permutation. Write $f = (g_1, \ldots, g_{n_{m+1}})_1$ so that $g_u = f\theta$.

Choose x, y in $X_{n_{m+1}}$ such that $x^\theta = y \neq x$. Let $h = ux * h_{ux} = u * h_u$ be an element of $Rist_G(ux)$. Now

$$\begin{aligned}[g,h] &= (1,\ldots,1,[g_u,h_u],1,\ldots,1)_m\\ &= (1,\ldots,1,\theta^{-1}f^{-1}h_u^{-1}f\theta h_u,1,\ldots,1)_m\\ &= (1,\ldots,1,h_{ux}^{-g_x},1,\ldots,1,h_{ux},1,\ldots,1)_{m+1}\end{aligned}$$

where $h_{ux}^{-g_x}$ is in position uy and h_{ux} is in position ux. Take an arbitrary $l \in Rist_G(ux)$ then

$$[[g,h],l] = (1,\ldots,1,[h_{ux},l_{ux}],1\ldots1)_{m+1}.$$

Since l and h were arbitrary elements of $Rist_G(ux)$ we get that $\langle\langle g\rangle\rangle$ contains $Rist_G(ux)'$. Since G acts spherically transitively, we get that $\langle\langle g\rangle\rangle$ contains $Rist_G(m+1)'$. □

A direct consequence of this theorem is a criterion for checking if a branch group is just infinite.

Corollary 1.19. *[13] A branch group G is just infinite if and only if $rist_G(m)'$ has finite index in G for all m.*

The following proposition is also easy to see.

Proposition 1.20. *[6] Let G be a branch group and let H be a subgroup of finite index in G. Then H contains $Rist_G(m)^e$ for some $m, e \geq 0$.*

Now the product $Rist_G(m)^e Rist_G(m)'$ gives a finite index subgroup. A consequence of Proposition 1.20 and 1.18 is that for a branch group G, the sets

$$\{Rist_G(m)^e Rist_G(m)' \mid m, e \geq 0\}$$

and

$$\mathcal{N} = \{N \trianglelefteq G \mid |G : N| < \infty\}$$

are cofinal subsets of the collection of all finite index normal subgroups.

Proposition 1.21. *For a branch group G, the branch kernel is given by*

$$\ker(\widehat{G} \to \widetilde{G}) = \varprojlim_{\substack{m\geq 1\\ e\geq 1}} Rist_G(m)/[Rist_G(m)^e Rist_G(m)'].$$

In the case where G is just infinite, it is not necessary to include $Rist_G(m)^e$ to obtain a finite index subgroup and so in this special case the kernel can be simplified.

Proposition 1.22. *For a just infinite branch group, the branch kernel is given by*

$$\ker(\widehat{G} \to \widetilde{G}) = \varprojlim_{m\geq 1} Rist_G(m)/Rist_G(m)'.$$

Chapter 2 Towers of Hanoi

The Hanoi towers group G_3 was first introduced by Grigorchuk and Šunić in [14]. The action of G_3 on the first m levels of the tree models the "Towers of Hanoi" game with m disks, hence the name.

In [6], the authors compute the rigid, branch, and congruence kernels for the Hanoi Towers group. At the time, it was the only branch group that had been shown to have a non-trivial rigid kernel. Knowing a group has a particular property and understanding why are both equally important tasks, especially when the group is the first example having said property. For this reason, in Section 2.2 we provide a simplified, constructive proof that the rigid kernel for the Hanoi towers group is the Klein 4 group, proving various properties of the group along the way. The results in this section were published in [22].

For completeness, in Section 2.3 we discuss the remaining kernels.

2.1 The game and the group

We start by describing the game.

The "Towers of Hanoi" game for three pegs and m disks works as follows. It begins with 3 pegs and m disks each of varying size organized from largest to smallest on the first peg. Figure 2.1 shows this initial game state for $m = 6$. The goal of the game is to move each of the disks from the first peg to the third peg through a series of moves. Each move consists of taking the top disk from one peg and placing it atop another peg with the restriction that at no point can a larger disk be on top of a smaller disk.

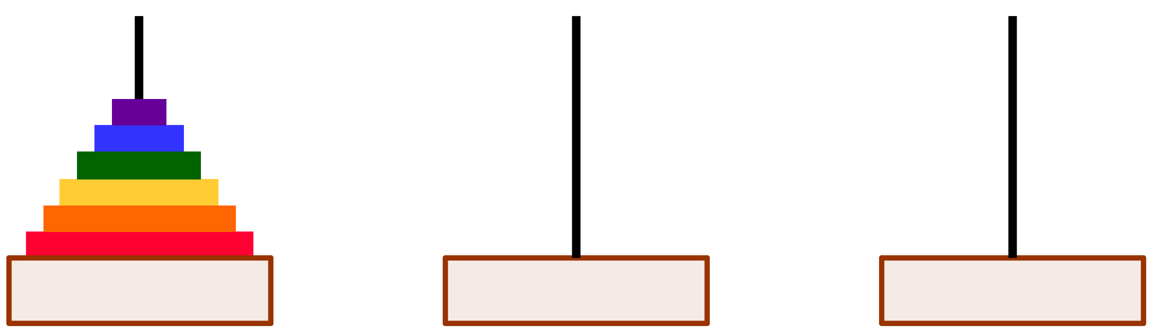

Figure 2.1: The beginning game state for the "Towers of Hanoi".

The restriction on the moves in the game limits a player's options to three possibilities. The first move, which will be called move a_1, transfers the smallest disk on pegs 2 and 3 between them. Likewise, move a_2 transfers the smallest disk on pegs 1 and 3 between them and move a_3 transfers the smallest disk on pegs 1 and 2 between them.

Any sequence of moves yields a game state which consists of the disks distributed across the three pegs such that on each peg, starting at the bottom and working up, the disks decrease in size. Thus, every game state in the m-disk game can be encoded as a sequence of m integers between 1 and 3 in the following way: the first integer indicates the location of the smallest disk, the second integer indicates the location of the next smallest disk and so forth until the final integer indicates the location of the largest disk. For example, the Figure 2.2 shows a possible game state for the 6-disk game corresponding to the sequence $(2, 1, 3, 2, 2, 1)$.

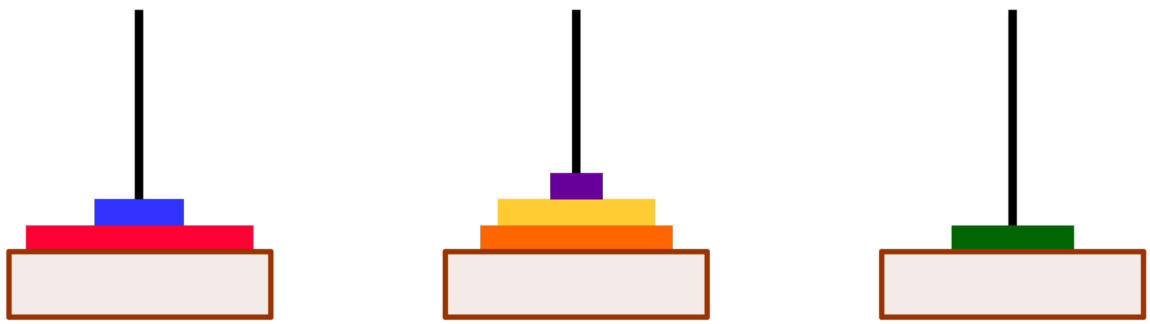

Figure 2.2: The game state corresponding to $(2, 1, 3, 2, 2, 1)$.

Recall that integer sequences of length m where the integers are between 1 and 3 can also be thought of as a vertex on the mth level in a rooted ternary tree as described in Section 1.3 and as seen in Figure 2.3.

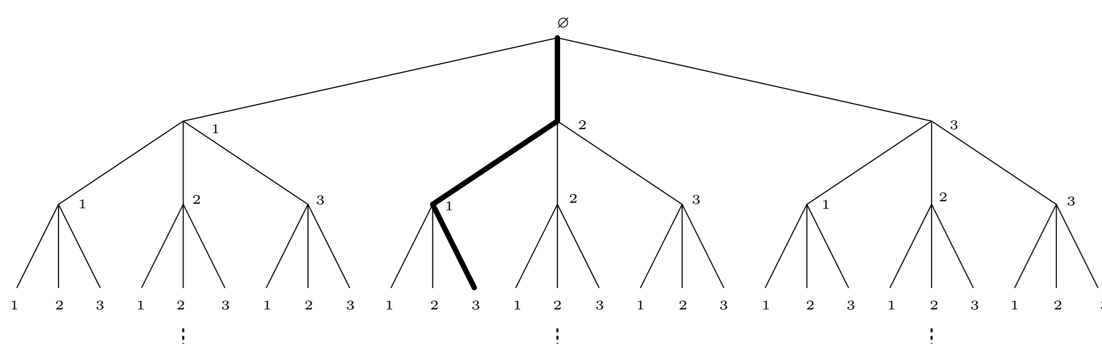

Figure 2.3: The rooted ternary tree

Since any move in the game takes one game state to another game state, i.e. takes one vertex on the mth level in the tree to another vertex on the mth level, each move can be thought of as an automorphism of the rooted ternary tree. Move a_1 should search for the first time a 2 or 3 appears in the path, and then switch it. Moves a_2 and a_3 should act similarly but instead with the numbers 1 and 3 and the numbers 1 and 2 respectively. For example, move a_2 takes the sequence $(2,1,3,2,2,1)$ to $(2,3,3,2,2,1)$.

In the same way we can define elements a_1, a_2 and a_3 acting on the whole ternary tree $\mathcal{T}$. They are as follows:

$$a_1 = (a_1,1,1)\sigma_1 \quad a_2 = (1,a_2,1)\sigma_2 \quad a_3 = (1,1,a_3)\sigma_3$$

where we are using the isomorphism $\mathrm{Aut}(\mathcal{T}) \cong \mathrm{Aut}(\mathcal{T}) \wr S_3$ and where σ_i is the transposition which fixes i for $1 \le i \le 3$.

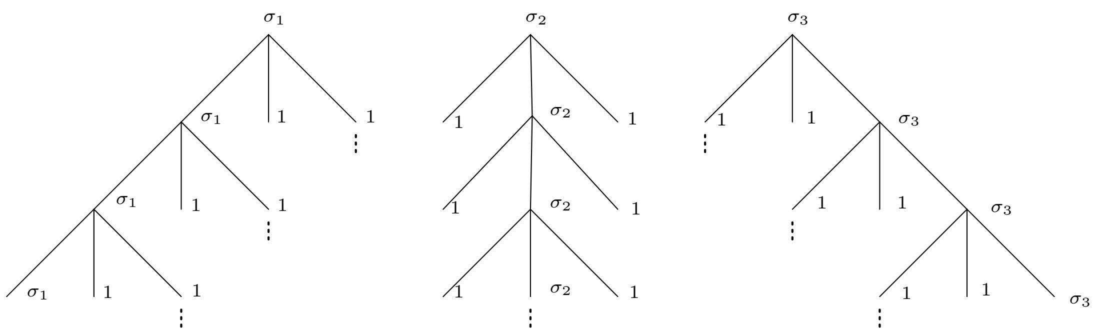

Figure 2.4: The generators a_1, a_2, and a_3 of the Hanoi towers group

Figure 2.4 shows the labeling of the vertices by elements in S_3 for a_1, a_2, and a_3 respectively. Then the Hanoi towers group is $G_3 = \langle a_1, a_2, a_3\rangle$. In [6], a full presentation for G_3 is obtained. It is:

$$G_3 = \langle a_1, a_2, a_3 \mid a_1^2, a_2^2, a_3^2, \tau^n(w_1), \tau^n(w_2), \tau^n(w_3), \tau^n(w_4) \text{ for all } n \ge 0\rangle \tag{2.1}$$

where τ is an endomorphism of G_3 defined by the substitution

$$a_1 \mapsto a_1 \qquad a_1 \mapsto a_2^{a_3} \qquad a_3 \mapsto a_3^{a_2}$$

and where

$$w_1 = [a_2,a_1][a_2,a_3][a_3,a_1][a_1,a_3]^{a_2}[a_1,a_2]^{a_3}[a_3,a_2]$$

$$w_2 = [a_2,a_3]^{a_1}[a_3,a_2][a_2,a_1][a_3,a_1][a_1,a_2][a_1,a_3]^{a_2}$$

$$w_3 = [a_3,a_2][a_1,a_2][a_2,a_3]^{a_1}[a_3,a_2]^2[a_2,a_1][a_2,a_3]^{a_1}[a_2,a_3]^{a_1}$$

$$w_4 = [a_2,a_3]^{a_1}[a_1,a_2]^{a_3}[a_2,a_1]^2[a_1,a_3][a_1,a_2]^{a_3}[a_3,a_1][a_3,a_2].$$

2.2 Properties of the Hanoi towers group and computation of the rigid kernel

In this section we compute the rigid kernel for G_3:

$$\ker(\widetilde{G}_3 \to \overline{G}_3) = \varprojlim_{m \geq 1} Stab_{G_3}(m)/Rist_{G_3}(m).$$

Since the maps $\rho_{n,n+k} : Stab_{G_3}(m+k)/Rist_{G_3}(m+k) \to Stab_{G_3}(m)/Rist_{G_3}(m)$ are far from being surjective most of our work in computing the rigid kernel for G_3 will be in determining the image $\rho_{m,m+k}(Stab_{G_3}(m+k)/Rist_{G_3}(m+k))$ for all m and k.

First we observe that since each generator of G_3 has order 2, any element in G_3 can be expressed as a word in a_1, a_2, and a_3 using only the positive alphabet. Further, since each relator in presentation 2.1 can be written as a product of commutators $G_3/G_3' \cong (C_2)^3$ where C_2 is a cyclic group of order 2. Thus a word in a_1, a_2, and a_3 is in G_3' if and only if the sum of the exponents on each letter is congruent to 0 modulo 2.

Using the Reidemeister-Schreier method, we obtain a generating set for $Stab_{G_3}(1)$:

$$\alpha = a_1 a_3 a_1 a_2 = (a_1, a_3 a_2, a_1)_1$$

$$\beta = a_1 a_2 a_1 a_3 = (a_1, a_1, a_2 a_3)_1$$

$$\delta = a_2 a_3 a_2 a_1 = (a_3 a_1, a_2, a_2)_1$$

$$\gamma = a_2 a_1 a_2 a_3 = (a_2, a_2, a_1 a_3)_1.$$

The details of this can be found in Appendix A.

Recall that a self-similar group G is self-replicating if $\pi_v(Stab_G(v)) = G$. If G is both self-replicating and acts transitively on the first level of the tree, then G is spherically transitive. As $G_3/Stab_{G_3}(1) = S_3$, G_3 clearly acts transitively on the first level of the ternary tree. Thus to show it is level transitive, it is sufficient to show it is self-replicating.

Lemma 2.1. *G_3 is self-replicating.*

Proof. From the generators obtained for $Stab_{G_3}(1)$ above we see $\pi_v(Stab_{G_3}(v)) = G_3$ for any vertex v of level 1. Now suppose for any vertex v of level m, $\pi_v(Stab_{G_3}(v)) = G_3$ and let w be an immediate descendant of v. Then let p, q, r, and s be the elements in $Stab_{G_3}(v)$ that act as α, β, δ, and γ on the subtree rooted at v. Then, p, q, r, and s are in $Stab_{G_3}(w)$ and p_w, q_w, r_w, and s_w generate G_3. Thus, $\pi_w(Stab_{G_3}(w)) = G_3$. □

An important observation that will be used frequently is that for a group G if $X^m * H \leq G$, then

$$Stab_G(m+k) \cap X^m * H = X^m * Stab_H(k). \tag{2.2}$$

This is because $X^m * H$ describes a disjoint action on each subtree rooted at the mth level, and so on each of these subtrees $Stab_G(m+k) \cap X^m * H$ describes the collection of elements that are contained in H and stabilize the kth level.

Lemma 2.2. *G_3 is a self-similar, regular branch group with branching subgroup G_3'.*

Proof. The definition of the generators of G_3 easily implies that G_3 is self-similar. We will show by induction that $X^m * G_3' \leq G_3'$. For $m = 1$, observe that

$$(a_1a_3a_2a_3)^2 = (a_1a_2a_1a_2, 1, 1)_1 = ([a_1, a_2], 1, 1)_1$$

$$(a_1a_2a_3a_2)^2 = (a_1a_3a_1a_3, 1, 1)_1 = ([a_1, a_3], 1, 1)_1$$

$$a_3(a_2a_1a_3a_1)^2a_3 = (a_2a_3a_2a_3, 1, 1)_1 = ([a_2, a_3], 1, 1)_1$$

and $(a_1a_3a_2a_3)^2$, $(a_1a_2a_3a_2)^2$, and $a_3(a_2a_1a_3a_1)^2a_3$ are all in G_3' since G_3/G_3' is an elementary abelian 2-group.

From the description of the generators for $Stab_{G_3}(1)$, we see that for all $g \in G_3$ there is an element $\tilde{g} \in Stab_{G_3}(1)$ whose state in the first coordinate is g. Conjugating $(a_1a_3a_2a_3)^2$ by $\tilde{g}$ produces the element $([a_1, a_2]^g, 1, 1)_1$. Likewise, we can obtain the element that has any conjugate of $[a_1, a_3]$ or $[a_2, a_3]$ in the first coordinate and 1 in the second and third coordinates. As G_3 is transitive on all levels of $\mathcal{T}$, we obtain $X * G_3' \leq G_3'$.

Now assume for some $m \geq 1$, that $X^m * G_3' \leq G_3'$. By the base case, each copy of G_3' contains a copy of $X * G_3'$. Therefore,

$$X^m * (X * G_3') \leq X^m * G_3' \leq G_3'.$$

But $X^m * (X * G_3') = X^{m+1} * G_3'$. □

Lemma 2.3. *For all $m \geq 1$, $Rist_{G_3}(m) = X^m * G_3'$.*

Proof. The proof is by induction on the level. By Lemma 2.2,

$$X * G_3' \leq Rist_{G_3}(1) \leq Stab_{G_3}(1) \leq X * G_3.$$

Note that

$$(X * G_3)/(X * G_3') \cong (G_3/G_3')^3 \cong [(\mathbb{Z}/2\mathbb{Z})^3]^3 \cong (\mathbb{Z}/2\mathbb{Z})^9.$$

Consider H, the rigid stabilizer of the first vertex of level 1. The image H in $(\mathbb{Z}/2\mathbb{Z})^9$ is contained in the subspace W consisting of vectors which have 0 in the ith coordinate for $i \geq 4$. On the other hand, the image U of $Stab_{G_3}(1)$ in $(\mathbb{Z}/2\mathbb{Z})^9$ is spanned by the images of α, β, δ, and γ which are

$$\widetilde{\alpha} = (1, 0, 0, 0, 1, 1, 1, 0, 0)$$

$$\widetilde{\beta} = (1,0,0,1,0,0,0,1,1)$$
$$\widetilde{\delta} = (1,0,1,0,1,0,0,1,0)$$
$$\widetilde{\gamma} = (0,1,0,0,1,0,1,0,1).$$

It is a simple exercise to see that $W \cap U = \{0\}$. It follows that $H \leq X * G_3'$ and thus $Rist_{G_3}(1) = X * G_3'$.

Now assume for some $m \geq 1$ that $Rist_{G_3}(m) = X^m * G_3'$. Then, again, by Lemma 2.2,

$$\begin{aligned} X^{m+1} * G_3' &\leq Rist_{G_3}(m+1) \\ &= Rist_{G_3}(m+1) \cap X^m * G_3' \\ &= X^m * Rist_{G_3'}(1) \\ &\leq X^m * Rist_{G_3}(1) \\ &= X^{m+1} * G_3' \end{aligned}$$

giving $X^{m+1} * G_3' = Rist_{G_3}(m+1)$. □

Corollary 2.4. *For all m, $Rist_{G_3}(m)Stab_{G_3}(m+1)/Stab_{G_3}(m+1) = (A_3)^{3^m}$ where A_3 is the alternating group on 3 letters.*

Proof. The projection $G_3 \to G_3/Stab_{G_3}(1) \cong S_3$ takes G_3' onto A_3, hence $G_3'/Stab_{G_3'}(1) \cong A_3$. Further,

$$\begin{aligned} &Rist_{G_3}(m)Stab_{G_3}(m+1)/Stab_{G_3}(m+1) \\ &\cong Rist_{G_3}(m)/[Rist_{G_3}(m) \cap Stab_{G_3}(m+1) \\ &= X^m * G_3'/[(X^m * G_3') \cap Stab_{G_3}(m+1)] \\ &\cong X^m * G_3'/X^m * Stab_{G_3'}(1) \\ &\cong (G_3'/Stab_{G_3'}(1))^{3^m} \cong (A_3)^{3^m}. \end{aligned}$$

□

Corollary 2.5. *The rigid kernel for G_3 is an elementary abelian 2-group.*

Proof. Since $Stab_{G_3}(m) \leq X^m * G_3$, we have

$$Stab_{G_3}(m)/Rist_{G_3}(m) = Stab_{G_3}(m)/X^m * G_3'$$

is a subspace of

$$X^m * G_3/X^m * G_3' \cong (G_3/G_3')^{3^m}$$

which is an elementary abelian 2-group. An inverse limit of elementary abelian 2-groups is an elementary abelian 2-group. □

Corollary 2.6. *$|Stab_{G_3}(1)/Rist_{G_3}(1)| = 16$ and $|Stab_{G'_3}(1)/Rist_{G'_3}(1)| = 4$.*

Proof. We have seen in the proof of Lemma 2.3 that

$$Stab_{G_3}(1)/Rist_{G_3}(1) = Stab_{G_3}(1)/X * G'_3 = U$$

is a four dimensional vector space over $\mathbb{F}_2$ (the images of α, β, δ, and γ form a basis). Hence U has 16 elements.

Now, by Lemmas 2.2 and 2.3, we see that $Rist_{G_3}(m) = Rist_{G'_3}(m) = X^m * G'_3$. This gives

$$Stab_{G'_3}(1)/Rist_{G'_3}(1)$$

$$= (Stab_{G_3}(1) \cap G'_3)/Rist_{G_3}(1)$$

$$= U \cap (G'_3/Rist_{G_3}(1)).$$

Moreover, since a word in a_1, a_2, and a_3 is in G'_3 if and only if each generator appears in it an even number of times, a word in α, β, δ, and γ is in G'_3 if and only if the number of appearances of α and β have the same parity and the number of appearances of δ and γ have the same parity. It follows that $U \cap (G'_3/Rist_{G_3}(1))$ is the two dimensional subspace spanned by $\widetilde{\alpha} + \widetilde{\beta}$ and $\widetilde{\delta} + \widetilde{\gamma}$. □

As G_3 is self-replicating, if $g \in Stab_{G_3}(u)$, then g_u must also be an element of G_3. Corollary 2.6 and the following lemma serve to elucidate the action of G_3 on the top levels of $\mathcal{T}$.

Lemma 2.7.

1. *$G_3/Stab_{G_3}(1) \cong S_3$, the symmetric group on three letters.*
2. *$Stab_{G_3}(1)/Stab_{G_3}(2)$ considered as a subgroup of $(S_3)^3$ is the kernel of the homomorphism $\phi : (S_3)^3 \to C_2$ where ϕ sums the signs of the permutation in each coordinate. This quotient has order $2^2 \cdot 3^3$.*

Proof. 1. We have already observed that $G_3/Stab_{G_3}(1) \cong S_3$.

2. The images of α, β, δ, and γ in $Stab_{G_3}(1)/Stab_{G_3}(2)$ are

$$\overline{\alpha} = (\sigma_1, (1,2,3), \sigma_1)$$

$$\overline{\beta} = (\sigma_1, \sigma_1, (1,3,2))$$

$$\overline{\delta} = ((1,2,3), \sigma_2, \sigma_2)$$

$$\overline{\gamma} = (\sigma_2, \sigma_2, (1,3,2)).$$

Thus $\overline{\alpha}, \overline{\beta}, \overline{\delta}$, and $\overline{\gamma}$ are in $\ker(\phi)$. Further $\overline{\delta}^2 = ((1,3,2),1,1)$ and, by spherical transitivity, this implies that $(A_3)^3 \leq Stab_{G_3}(1)/Stab_{G_3}(2)$. Also, $\overline{\alpha\beta} = (1,\sigma_2,\sigma_2)$ and $\overline{\delta\gamma} = (\sigma_1,1,\sigma_1)$. Collectively, these elements generate $\ker(\phi)$.

□

Now we apply our knowledge of the permutations appearing on the top levels of the tree to gain an understanding of action on subtrees rooted at the lower levels.

Lemma 2.8. *For $m \geq 1$, we have isomorphisms*

$$Stab_{G_3'}(m)/Stab_{G_3'}(m+1)$$

$$\cong Stab_{G_3}(m)/Stab_{G_3}(m+1)$$

$$\cong X^{m-1} * Stab_{G_3}(1)/X^{m-1} * Stab_{G_3}(2).$$

In particular, all three groups have order $2^{2\cdot 3^{m-1}} \cdot 3^{3^m}$.

Proof. Since

$$Stab_{G_3'}(m)/Stab_{G_3'}(m+1)$$

$$= Stab_{G_3'}(m)/(Stab_{G_3'}(m) \cap Stab_{G_3}(m+1)),$$

the group $Stab_{G_3'}(m)/Stab_{G_3'}(m+1)$ can be considered as a subgroup of

$$Stab_{G_3}(m)/Stab_{G_3}(m+1).$$

By self-similarity, $Stab_{G_3}(m)/Stab_{G_3}(m+1)$ can be considered as a subgroup of

$$(X^{m-1} * Stab_{G_3}(1))/(X^{m-1} * Stab_{G_3}(2)),$$

a group of order $2^{2\cdot 3^{m-1}} \cdot 3^{3^m}$. Therefore it suffices to prove that

$$|Stab_{G_3'}(m)/Stab_{G_3'}(m+1)| \geq 2^{2\cdot 3^{m-1}} \cdot 3^{3^m}.$$

Observe that $G_3'/Stab_{G_3'}(1) \cong A_3$, generated by the image of $[a_1,a_2] = (a_1a_2,a_1,a_2)(1,2,3)$ and recall that $Rist_{G_3}(m) = X^m * G_3' \leq G_3'$. It follows that $Stab_{G_3'}(m)/Stab_{G_3'}(m+1)$ contains $(X^m * G_3')Stab_{G_3'}(m+1)/Stab_{G_3'}(m+1)$. Note that

$$(X^m * G_3')Stab_{G_3'}(m+1)/Stab_{G_3'}(m+1)$$

$$\cong X^m * G_3'/(Stab_{G_3'}(m+1) \cap X^m * G_3')$$

$$= X^m * G_3'/X^m * Stab_{G_3'}(1)$$

$$\cong (G_3'/Stab_{G_3'}(1))^{3^m}$$

$$\cong (A_3)^{3^m}.$$

Therefore, $Stab_{G_3'}(m)/Stab_{G_3'}(m+1)$ has a subgroup of order 3^{3^m}.

Now, $Stab_{G_3'}(m)/Stab_{G_3'}(m+1)$ also contains a subgroup isomorphic to

$$(X^{m-1} * G_3' \cap Stab_{G_3'}(m))/(X^{m-1} * G_3' \cap Stab_{G_3'}(m+1)).$$

Moreover, by 2.2 this subgroup is isomorphic to $(Stab_{G_3'}(1)/Stab_{G_3'}(2))^{3^{m-1}}$ which has order $2^{2 \cdot 3^{m-1}}$ by Corollary 2.6. □

Now, we have all the tools needed to prove the main theorem.

Theorem 2.9. *The rigid kernel* $\ker(\widetilde{G}_3 \to \overline{G}_3)$ *is the Klein 4 group.*

Proof. By Corollary 2.5, the rigid kernel is an elementary abelian 2-group, so we only need to show that it has order 4.

For notational simplicity, for all $m \geq 1$, define ${}_mG_3 = Stab_{G_3}(m)/Rist_{G_3}(m)$. Further, under the natural map from ${}_{m+k}G_3$ to ${}_mG_3$, let ${}_{m,m+k}H$ be the image of ${}_{m+k}G_3$ in ${}_mG_3$, let ${}_{m,m+k}K$ be the kernel of this map, and let ${}_{m,m+k}Q$ be the cokernel of this map (note that ${}_{m,m+k}H \trianglelefteq {}_mG_3$).

Recall that the rigid kernel is $\varprojlim_{m\geq 1} {}_mG_3$. We will show that for all m, ${}_{m,m+1}H = {}_{m,m+2}H$ and that both have order 4. This implies that for each m, the maps ${}_{m+1,m+2}H \to {}_{m,m+1}H$ are isomorphisms and hence $\varprojlim_{m\geq 1} {}_mG_3 = \varprojlim_{m\geq 1} {}_{m,m+1}H$ also has order 4, completing the proof.

The first step in doing this is to determine ${}_{m,m+1}H$. We have the exact sequence

$$1 \to {}_{m,m+1}K \to {}_{m+1}G_3 \to {}_mG_3 \to {}_{m,m+1}Q \to 1. \tag{2.3}$$

Now

$$\begin{aligned} &{}_{m,m+1}K \\ &= (Stab_{G_3}(m+1) \cap Rist_{G_3}(m))/Rist_{G_3}(m+1) \\ &= (Stab_{G_3}(m+1) \cap X^m * G_3')/X^{m+1} * G_3' \\ &= X^m * Stab_{G_3'}(1)/X^{m+1} * G_3' \\ &\cong (Stab_{G_3'}(1)/Rist_{G_3'}(1))^{3^m}, \end{aligned}$$

hence $|{}_{m,m+1}K| = 2^{2 \cdot 3^m}$ from Corollary 2.6.

Also, ${}_{n,n+1}Q = Stab_{G_3}(m)/Rist_{G_3}(m)Stab_{G_3}(m+1)$ has $2^{2 \cdot 3^{m-1}}$ elements by Lemma 2.8 and Corollary 2.4.

Since the sequence 2.3 is exact,

$$\frac{|{}_{m+1}G_3|}{|{}_mG_3|} = \frac{|{}_{m,m+1}K|}{|{}_{m,m+1}Q|} = 2^{4 \cdot 3^{m-1}}.$$

Further, by Corollary 2.6, $|_1G_3| = 16$. Collectively, we obtain

$$|_mG_3| = 2^4 \prod_{i=2}^{m} 2^{4\cdot 3^{i-2}} = 2^{2\cdot(3^{m-1}+1)}$$

and the size of $_{m,m+1}H$ is

$$\frac{|_{m+1}G_3|}{|_{m,m+1}K|} = \frac{2^{2(3^m+1)}}{2^{2\cdot 3^m}} = 4.$$

Now it remains to show that $_{m,m+2}Q = {}_{m,m+1}Q$ as this would imply $_{m,m+2}H = {}_{m,m+1}H$ and moreover that $_{m+1,m+2}H$ maps isomorphically to $_{m,m+1}H$ for all m.

Now

$$_{m,m+i}Q = Stab_{G_3}(m)/(X^m * G_3{}')Stab_{G_3}(m+i).$$

Thus showing $_{m,m+1}Q = {}_{m,m+2}Q$ is the same as showing

$$(X^m * G_3')Stab_{G_3}(m+1) = (X^m * G_3')Stab_{G_3}(m+2).$$

By Lemma 2.8,

$$\begin{aligned} &Stab_{G_3}(m+1)/Stab_{G_3}(m+2) \\ &\cong X^m * Stab_{G_3}(1)/X^m * Stab_{G_3}(2) \\ &\cong X^m * Stab_{G_3'}(1)/X^m * Stab_{G_3'}(2). \end{aligned}$$

Hence, $Stab_{G_3}(m+1) = Stab_{G_3}(m+2)(X^m * Stab_{G_3'}(1))$ and we obtain

$$\begin{aligned} &(X^m * G_3')Stab_{G_3}(m+1) \\ &= (X^m * G_3')(X^m * Stab_{G_3'}(1))Stab_{G_3}(m+2) \\ &= (X^m * G_3')Stab_{G_3}(m+2). \end{aligned}$$

□

2.3 Branch kernel for the Hanoi towers group

Bartholdi, Siegenthaler, and Zalesskii additionally computed the branch and congruence kernel for the Hanoi towers group. For completeness, we include these here although we do not include a proof.

Theorem 2.10. *[6] The branch kernel of G_3 is isomorphic to $\widehat{Z}^3[[X^\omega]]$. The congruence kernel of G_3 is an extension of $\widehat{Z}^3[[X^\omega]]$ by the Klein 4 group where the action of the Klein 4 group is diagonal. Each non-trivial element of the Klein 4 group acts as a half-turn along a coordinate axis on $\widehat{Z}^3$.*

Here $\widehat{Z}^3[[X^\omega]]$ is used to denote the free profinite $\widehat{Z}^3$ module on the profinite space

$$X^\omega = \varprojlim_{m\geq 1} X^m.$$

Chapter 3 Generalized Groups

In this chapter, we introduce and study a natural generalization of the Hanoi towers group acting on an n-ary tree. The groups in the generalization turn out to be surprisingly different from the Hanoi towers group.

We start by introducing the groups in Section 3.1. Then we give an algorithm for solving the word problem in Section 3.2 which allows us to compute the abelianization. In Section 3.3, we determine the rigid stabilizers and level stabilizers for these groups and use them to find the rigid, branch, and congruence kernels. This is then applied to show that the groups in the generalization are just infinite if and only if they are not the Hanoi towers group in Section 3.4. And finally, in Section 3.5, we compute the Hausdorff dimension.

We remark that the group G_4 was studied briefly in [21], but a subtle overgeneralization in the hypotheses of earlier theorems led to some incorrect conclusions.

3.1 The groups

Let $n \geq 3$ and let $X = \{1, 2, \dots, n\}$. Let $\sigma_i = (1, 2, \dots, i-1, i+1, \dots, n-1, n)$, a permutation in S_n. Let a_i be the automorphism of the n-ary tree defined recursively as follows:

$$a_i = (1, \dots, 1, a_i, 1, \dots, 1)\sigma_i$$

where on the right side of the equation a_i appears in the i-th coordinate.

Definition 3.1. *The group G_n is the group generated by $\{a_1, \dots, a_n\}$.*

Our primary focus in this chapter will be on $n \geq 4$, but we will recall facts about G_3, the Hanoi towers group, as they are necessary.

Lemma 3.2. *For $n \geq 3$, $\langle \sigma_i \mid 1 \leq i \leq n \rangle$ is the alternating group on n letters, A_n, when n is even and the symmetric group on n letters, S_n, when n is odd.*

Proof. For all i, when n is even $\sigma_i \in A_n$ and when n is odd $\sigma_i \notin A_n$. Further, $\sigma_{i+1}^{-1}\sigma_i = (i, i+1, i+2)$ for $1 \leq i \leq n-2$. Since $\{(i, i+1, i+2) \mid 1 \leq i \leq n-2\}$ is a generating set of A_n, the result follows. $\square$

Recall that if a group is both self-replicating and acts transitively on the first level of the tree, then it is spherically transitive.

Lemma 3.3. *For all n, G_n is self-replicating.*

Proof. If a vertex v is a descendant of a vertex u (i.e. $v = uw$ for some $w \in X^*$), then

$$\pi_v(Stab_G(v)) = \pi_w(\pi_u(Stab_G(v))).$$

Thus G is self-replicating if and only if $\pi_u(Stab_G(u)) = G$ for every vertex u of level 1. Suppose u is in the i-th coordinate. Then for each a_j and a_k where $k \neq i$, there exists a number l such that $j^{\sigma_k^l} = i$. Moreover, $\sigma_j^{\sigma_k^l}$ fixes i. Therefore, $a_j^{a_k^l}$ is in $Stab_{G_n}(u)$ and $\pi_u(a_j^{a_k^l}) = a_j$. □

Corollary 3.4. *For all n, G_n is spherically transitive.*

3.2 Word problem and abelianization

A group G has a **solvable word problem** if it is finitely generated and there exists and algorithm in finite time for determining when two words in the generating set represent the same group element. Two words w_1 and w_2 represent the same element of the group if and only if $w_1 w_2^{-1} = 1$. Therefore, it is equivalent to finding an algorithm which determines whether or not a given word represents the identity element.

We remark that G_n is an example of an automaton group and as such there exists an algorithm in exponential time that solves the word problem [25]. Here we outline an alternative algorithm for G_n which also allows for the computation of the abelianization.

Let F_n be a free group with basis $\{s_1, \dots, s_n\}$. For a freely reduced word $w(s_1, \dots, s_n) = s_{i_1}^{r_1} s_{i_2}^{r_2} \cdots s_{i_k}^{r_k}$, define the length of w to be $|w| = k$. Let $\gamma : F_n \hookrightarrow F_n \wr S_n$ be the map defined by $\gamma(s_i) = (1, \dots, 1, s_i, 1, \dots 1)\sigma_i$ where s_i is in the i-th coordinate and $\sigma_i = (1, \dots i-1, i+1, \dots, n)$ as before. In other words, γ mimics the recursive definition of a_i.

Proposition 3.5. *Let $w(s_1, \dots, s_n)$ be an element of F_n and suppose $\gamma(w) = (w_1, \dots, w_n)\theta$. Then for all j, $|w_j| \leq \frac{|w|+1}{2}$.*

Proof. If w is of length 1, then w is of the form s_i^r so $\gamma(w) = (1, \dots, 1, s_i^r, 1, \dots, 1)\sigma_i^r$ and the claim is true.

Likewise if $w = s_{i_1}^{r_1} s_{i_2}^{r_2}$ where $i_1 \neq i_2$ then $\sigma_{i_1}^{r_1}$ is a permutation of $\{1, \dots, n\} \backslash \{i_1\}$. In particular, $\gamma(w)$ is of the form

$$(1, \dots, 1, s_{i_1}^{r_1}, 1, \dots, 1, s_{i_2}^{r_2}, 1, \dots, 1)\sigma_{i_1}^{r_1}\sigma_{i_2}^{r_2}$$

where $s_{i_2}^{r_2}$ is in the $i_2^{\sigma_{i_1}^{r_1}}$ coordinate and $i_2^{\sigma_{i_1}^{r_1}} \neq i_1$. Again the claim holds.

Now suppose $w = s_{i_1}^{r_1} s_{i_2}^{r_2} \cdots s_{i_k}^{r_k}$ has length k for some $k \geq 3$ and $\gamma(w) = (w_1, \ldots, w_n)\theta$. Then for $m = \lceil \frac{k}{2} \rceil$, w can be written as $u_1 \cdots u_m$ where $|u_i| \leq 2$ for each i. In this case,

$$\gamma(u_i) = (u_{i_1}, u_{i_2}, \ldots, u_{i_n})\theta_i$$

for some $\theta_i \in S_n$ and where for all i between 1 and m and all j between 1 and n, $|u_{i_j}|$ is either 0 or 1. Therefore each w_j is a product of m words of length 0 or 1 and $|w_j| \leq \lceil \frac{k}{2} \rceil \leq \frac{k+1}{2}$. $\square$

Now let $1 \to R_n \to F_n \overset{\phi_0}{\to} G_n \to 1$ be a presentation for G_n where $\phi_0(s_i) = a_i$. Since γ mimics the recursive definition of the generators of G_n, the following diagram commutes:

$$\begin{array}{ccc} F_n & \xrightarrow{\gamma} & Im(\gamma) \\ & \underset{\phi_0}{\searrow} & \downarrow \phi_1 \\ & & G_n \end{array}$$

where $\phi_1((1, \ldots, 1, s_i, 1, \ldots 1)\sigma_i) = a_i$.

This fact along with Proposition 3.5 provide tools for solving the word problem. Indeed, let $w(s_1, \ldots, s_n)$ be in F_n. If $|w| = 1$, then $w(a_1, \ldots, a_n)$ is trivial if and only if $w(s_1, \ldots, s_n) = s_i^{r(n-1)}$ for some i and r. If $|w| \geq 2$, then we can apply γ to w to obtain $\gamma(w) = (w_1, \ldots, w_n)\theta$ where $|w_j| < |w|$. If θ is a non-trivial permutation then $w(a_1, \ldots, a_n) \neq 1$ and we are done. Similarly, if θ is trivial and each w_j has length 0 or 1, then $w(a_1, \ldots, a_n) = 1$ if and only if each $w_j(s_1, \ldots, s_n)$ is of the form $s_{i_j}^{r_j(n-1)}$. The remaining possibility is that θ is the trivial permutation and for some w_j, the length of w_j is at least 2. In this case repeat the above process to the each w_j until either we find a non-trivial permutation or each obtained word has length at most 1 and is of the form $s_i^{r(n-1)}$.

As a result of the word problem algorithm, the abelianization of G_n is straightforward to compute. First, observe that the generators of G_n have order $(n-1)$ and so G_n/G_n' is a quotient of $(\mathbb{Z}/(n-1)\mathbb{Z})^n$. Now for a word $w(s_1, \ldots, s_n)$, let ϵ_{s_i} be the sum of the exponents on the s_i terms in w. Consider now $\gamma(w) = (w_1, \ldots, w_n)\theta$. By the way γ is defined

$$\epsilon_{s_i}(w(s_1, \ldots, s_n)) = \sum_{j=1}^{n} \epsilon_{s_i}(w_j(s_1, \ldots, s_n))$$

The algorithm states that if a word $w(s_1, \ldots, s_n)$ produces a trivial word in G_n, then after some number of iterations, the sum of the exponents of the s_i's over all the states on a given level is equal to 0 modulo $n-1$. But this is the same as $\epsilon_{s_i}(w)$. In other words, if $w(a_1, \ldots, a_n) = 1$ then $\epsilon_{s_i}(w(s_1, \ldots, s_n)) \equiv 0 \bmod (n-1)$ for all i. Thus $R_n \leq \langle F_n', s_1^{n-1}, \ldots, s_n^{n-1} \rangle$ and G_n surjects onto $(\mathbb{Z}/(n-1)\mathbb{Z})^n$.

Proposition 3.6. *The abelianization of G_n is $G_n/G_n' \cong (\mathbb{Z}/(n-1)\mathbb{Z})^n$.*

A similar property to what is described in Proposition 3.5 is frequently studied in the setting of self-similar groups.

Definition 3.7. *A self similar group G is called contracting if there exists a finite set $\mathrm{N} \subset G$ such that for every $g \in G$, there exists $k \in \mathbb{N}$ such that $g_v \in \mathrm{N}$ for all words $v \in X^*$ of length greater than or equal to k. The minimal set N with this property is called the nucleus of the self-similar action.*

Since the generators of G_n have finite order every element of G_n can be expressed as a positive word. Now the next result follows immediately from Proposition 3.5.

Corollary 3.8. *G_n is contracting with nucleus*

$$\mathrm{N} = \{1, a_i^j \mid 1 \leq i \leq n, 1 \leq j \leq n-2\}.$$

The abelianization also allows us to put some functions on G_n which will be of use to us later.

Definition 3.9. *Let g be an element of G_n. Let $w(s_1, \ldots, s_n) = s_{i_1}^{r_1} s_{i_2}^{r_2} \cdots s_{i_k}^{r_k}$ be a word in $s_1, s_2, \ldots, s_n$ such that $w(a_1, \ldots, a_n) = g$, then*

$$\epsilon(g) = (\sum_{j=1}^{k} r_i) \bmod (n-1).$$

Lemma 3.10. *$\epsilon : G_n \to \mathbb{Z}/(n-1)\mathbb{Z}$ is a well defined, surjective homomorphism.*

Proof. Since $G_n/G_n' \cong (\mathbb{Z}/(n-1)\mathbb{Z})^n$, ϵ is the composition of the abelianization map $[Ab] : G_n \to (\mathbb{Z}/(n-1)\mathbb{Z})^n$ with the map $\psi : (\mathbb{Z}/(n-1)\mathbb{Z})^n \to \mathbb{Z}/(n-1)\mathbb{Z}$ defined by $\psi : (b_1, b_2, \ldots b_n) \mapsto \sum_{i=1}^n b_i$. Clearly, this map is well defined and as both $[Ab]$ and ψ are surjective, ϵ is surjective. □

Definition 3.11. *Let $g = (g_1, \ldots, g_n)\sigma \in G_n$ where $g_i \in G_n$ for all i. Define*

$$\epsilon_1(g) = \sum_{i=0}^{n} \epsilon(g_i) \bmod (n-1).$$

Lemma 3.12. *For an element $g \in G_n$, $\epsilon(g) = \epsilon_1(g)$.*

Proof. This follows from the discussion preceding Proposition 3.6. □

3.3 The congruence subgroup problem

The first step in computing the kernels for the groups G_n, $n \geq 4$, is to understand their rigid stabilizers and level stabilizers.

We make the following observation.

Remark 3.13. *For any vertex v, conjugating any element $h \in Rist_{Aut(\mathcal{T})}(v)$ by an automorphism g of $\mathcal{T}$ works as follows:*

Let $m = |v|$ and suppose

$$h = (1, \ldots, 1, h_v, 1, \ldots, 1)_m$$

where h_v is in the v-th coordinate. Let g decompose as

$$(g_1, \ldots, g_{n^m})\sigma$$

where σ is in the m-fold iterated wreath product of S_n. Suppose σ sends the vertex v to the vertex u. Then

$$h^g = (1, \ldots, 1, h_v^{g_u}, 1, \ldots, 1)_m$$

where $h_v^{g_u}$ is in the u-th coordinate.

For spherically transitive, self-replicating groups, this significantly reduces the calculations for rigid stabilizers as illustrated by Proposition 3.14.

Proposition 3.14. *Suppose G is a level transitive, self-replicating group. If $v * g \in G$, then*

$$u * \langle\langle g \rangle\rangle \leq Rist_G(u)$$

for all u such that $|u| = |v|$.

Proof. Suppose $v * g \in G$ and that G is a level transitive, self-replicating group. Let g^h be a conjugate of g in G. Since G is level transitive, for any vertex u on the same level as v there exists $\tilde{h}_1 \in G$ such that $\tilde{h}_1$ takes v to u. Then by Remark 3.13, $(v * g)^{\tilde{h}_1} = u * g^{h_1}$ for some $h_1 \in G$. Since G is self-replicating there exists $\tilde{h} \in Stab_G(u)$ such that the state of $\tilde{h}$ at u is $h_1^{-1}h$. Then $(v * g)^{\tilde{h}_1 \tilde{h}} = u * g^h$. □

Additionally, for the groups G_n there is another simplification that comes from the symmetry of the generators.

Remark 3.15. *Let ω be the permutation $(1, 2, \cdots, n)$ and let λ be the automorphism of n-ary tree defined recursively by*

$$\lambda = (\lambda, \lambda, \ldots, \lambda)\omega.$$

Then conjugation by λ is an automorphism of the group G_n which takes $a_n \mapsto a_1$ and $a_i \mapsto a_{i+1}$ for $1 \leq i \leq n-1$. Further, if

$$g = (g_1, g_2, \ldots, g_n)\sigma,$$

then

$$g^\lambda = (g_n^\lambda, g_1^\lambda, \ldots, g_{n-1}^\lambda)\sigma^\omega.$$

Theorem 3.16. *For all n, G_n is a regular branch group with branching subgroup G_n'.*

Proof. By Lemma 3.3 and Corollary 3.4, G_n is level transitive and self-replicating. Therefore, by Proposition 3.14, it suffices to find $v * g$ for each g in some normal generating set for G_n' and for some $v \in X$. And finally, by Remark 3.15, it suffices to find a conjugate of $v * [a_1, a_i]$ for each i between 1 and $1 + \lfloor \frac{n}{2} \rfloor$ and for some $v \in X$.

The case when $n = 3$ is dealt with in Chapter 2.

When $n = 4$, we have the following elements:

$$[a_3^{-a_1}, a_3^{-a_2}](a_2^{-1}a_1)^3 = (1, 1, [a_1, a_2]^{a_2}, 1)_1$$

$$[a_2^{a_1^{-1}}, a_2^{a_3}](a_1 a_3)^{-3} = (1, [a_1, a_3]^{-a_3^{-1}}, 1, 1)_1$$

When $n = 5$, we have the following elements:

$$[(a_1 a_4^{-1})^2, (a_2 a_4^{-1})^2] = ([a_1, a_2], 1, 1, 1, 1)_1$$

$$[(a_3^{-1} a_1)^2, (a_3 a_1^{-1})^2] = (1, [a_1, a_3], 1, 1, 1)_1$$

When $n = 6$, we have he following elements:

$$[(a_1 a_4^{-1})^2, (a_2 a_4^{-1})^2] = ([a_1, a_2], 1, 1, 1, 1, 1)_1$$

$$[(a_3^{-1} a_1)^2, (a_3 a_1^{-1})^2] = (1, [a_1, a_3], 1, 1, 1, 1)_1$$

$$[(a_6^{-1} a_1 a_2 a_1^{-1})^{a_3}, (a_4 a_5^{-1} a_4^{-1} a_3)] = (1, 1, 1, [a_1, a_4], 1, 1)_1$$

For the remaining n, fix i, $1 \le i \le 1 + \lfloor \frac{n}{2} \rfloor$ and let $j = i + 2 \ge 4$. Then

$$[(a_1 a_j^{-1})^2, ((a_i a_j^{-1})^2)^{a_j^{-(i-2)}}] = ([a_1, a_i], 1, \ldots, 1)_1.$$

Since G_n' has finite index in G_n and we obtain the result. □

Remark 3.17. *Note that G_n' is not the maximal branching subgroup for $n \ge 4$. The maximal branching subgroup for G_n, which depends on the size of n and whether n is even or odd, will be computed in Thoerems 3.23, 3.26, and 3.28.*

Definition 3.18. *Let I_n be the collection of elements of the form*

$$(1, \ldots, 1, g, 1, \ldots, 1, g^{-1}, 1, \ldots, 1)_1$$

where g ranges over all elements of G_n and the coordinates in which g and g^{-1} appear ranges over the set $\{1, \ldots, n\}$.

Proposition 3.19. *When $n \ge 4$, I_n is contained in G_n'.*

Proof. First, we observe that if $g = (g_1, \ldots, g_n)_1$ is an element in $Stab_{G_n}(1)$ and $h = (h_1, \ldots, h_n)\sigma$ is an element of G_n, then $g^h = (g_{1^\sigma}^{h_1}, \ldots, g_{n^\sigma}^{h_n})_1$ which is equivalent to

$$(g_{1^\sigma}, \ldots, g_{n^\sigma})_1 \bmod G_n'$$

by Theorem 3.16.

Consider the element

$$[a_1^{a_2}, a_3] = (1, a_2^{-1}, [a_1, a_3], a_2^2, a_2^{-1}, 1, \ldots, 1)_1 \equiv (1, a_2^{-1}, 1, a_2^2, a_2^{-1}, 1, \ldots, 1)_1 \bmod G_n'$$

where the equivalence is again by Theorem 3.16. Letting $\delta = (1, a_2^{-1}, 1, a_2^2, a_2^{-1}, 1, \ldots, 1)_1$, we see that

$$\delta\delta^{-a_1 a_3^{-1}} = (1, a_2^{-1}, a_2^{a_3^{-1}}, 1, \ldots, 1)_1 \equiv (1, a_2^{-1}, a_2, 1, \ldots, 1)_1 \bmod G_n'.$$

Since G_n acts as either A_n or S_n on the first level, by our first observation all elements of the form $(1, \ldots, 1, a_2, 1, \ldots, 1, a_2^{-1}, 1, \ldots, 1)_1$ with the a_2 and a_2^{-1} in any coordinate are contained in G_n'. Similarly, by Remark 3.13 all elements of the form $(1, \ldots, 1, a_i, 1, \ldots, 1, a_i^{-1}, 1, \ldots, 1)_1$ for $1 \leq i \leq n$ are likewise in G_n'.

Finally suppose $g = a_{i_1}^{m_{i_1}} \cdots a_{i_k}^{m_{i_k}}$. Then,

$$(1, \ldots, 1, a_{i_1}, 1, \ldots, 1, a_{i_1}^{-1}, 1, \ldots, 1)_1^{m_{i_1}} \cdots (1, \ldots, 1, a_{i_k}, 1, \ldots, 1, a_{i_k}^{-1}, 1, \ldots, 1)_1^{m_{i_k}}$$
$$= (1, \ldots, 1, g, 1, \ldots, 1, a_{i_1}^{-m_{i_1}} \cdots a_{i_k}^{-m_{i_k}}, 1, \ldots, 1)_1$$

and

$$(1, \ldots, 1, g, 1, \ldots, 1, a_{i_1}^{-m_{i_1}} \cdots a_{i_k}^{-m_{i_k}}, 1, \ldots, 1)_1 \equiv (1, \ldots, 1, g, 1, \ldots, 1, g^{-1}, 1, \ldots, 1)_1 \bmod G_n'.$$

□

Remark 3.20. *Proposition 3.19 is explicitly not true when $n = 3$ which can be seen from the generators for $Stab_{G_3}(1)$ found in Section 2.2. This significantly contributes to the change in the rigid kernels for G_n starting at $n = 4$ described in Theorem 3.30.*

Corollary 3.21. *For $n \geq 4$, $(g_1, \ldots, g_n)_1$ is in $Stab_{G_n}(1)$ if and only if*

$$(1, \ldots, 1, g_{1^\theta} \cdots g_{n^\theta}, 1, \ldots 1)_1$$

is in $Rist_{G_n}(1)$ for every permutation θ of $\{1, \ldots, n\}$. In particular, for every θ

$$(g_1, \ldots, g_n)_1 \equiv (1, \ldots, 1, g_{1^\theta} \cdots g_{n^\theta}, 1, \ldots 1)_1 \bmod \langle I_n, X * G_n' \rangle.$$

3.3.1 Rigid kernels

Since by Lemma 3.2,

$$\langle a_1(\varnothing), a_2(\varnothing), \dots, a_n(\varnothing)\rangle = \begin{cases} S_n & n \text{ is odd} \\ A_n & n \text{ is even} \end{cases}$$

and since the normal subgroup structure of the alternating and symmetric groups changes starting at $n = 5$, we will split the next computations into three settings, when $n = 4$, when $n \geq 5$ is odd, and when $n \geq 5$ is even.

Proposition 3.22. *$Stab_{G_4}(1) = \langle a_1a_3a_4^{-1}, a_2a_1a_3a_1^{-1}, a_1a_3^{-1}a_4a_3, X * G_4', I_4\rangle$.*

Proof. This was done was done using the computer algebra system GAP (Groups, Algorithms, and Programming) [10] and the GAP package AutomGrp [17] which applies the Reidemeister-Schreier method to obtain a list of generators. We then eliminate the redundant ones. See Appendix B for more details. □

Let $K_4 = \langle a_1a_3a_4^{-1}, a_2a_1a_3a_1^{-1}, a_1a_3^{-1}a_4a_3, G_4'\rangle$, a subgroup of index 3 in G_4.

Theorem 3.23. *$K_4 = \langle\langle a_1a_2, a_2a_3, a_3a_4, a_4a_1\rangle\rangle$ and K_4 is the maximal branching branching subgroup of G_4. In particular, $X^m * K_4 = Rist_{G_4}(m)$ for all m. Consequently,*

$$Stab_{G_4}(m+1) = X^m * Stab_{G_4}(1) \leq Rist_{G_4}(m)$$

for all m and G_4 has trivial rigid kernel.

Proof. Since $G_4' \leq K_4$, K_4 is a normal subgroup. Moreover, $a_2a_1a_3a_1^{-1} \equiv a_2a_3 \bmod G_4'$ and similarly $a_1a_3^{-1}a_4a_3 \equiv a_4a_1 \bmod G_4'$. Further, a_3a_4 and a_1a_2 can be written as a product of the generators of K_4 and their conjugates. Let $\widetilde{K}_4 = \langle\langle a_1a_2, a_2a_3, a_3a_4, a_4a_1\rangle\rangle$. Now clearly $\widetilde{K}_4 \leq K_4$ and $G_4/K_4 \cong \mathbb{Z}/3\mathbb{Z}$ by Proposition 3.6, hence to check that $\widetilde{K}_4 = K_4$ it suffices to show that $G_4/\widetilde{K}_4$ has order at most three. This is immediate from the fact that

$$a_1 \equiv a_2^{-1} \equiv a_3 \equiv a_4^{-1} \bmod \widetilde{K}_4$$

and that each a_i has order 3.

Now to show that K_4 is a branching subgroup, by self-similarity it is only necessary to show that $K_4 \geq X * K_4 = Rist_{G_4}(1)$. First observe that K_4 has by index 3 in G_4 by Proposition 3.6 and therefore is a maximal subgroup. Now consider the elements

$$a_1a_3a_4^{-1} = (a_1, a_3, 1, a_4^{-1})_1,$$

$$a_2a_1a_3a_1^{-1} = (a_1^{-1}, a_2a_3, 1, a_1)_1,$$

$$a_1a_3^{-1}a_4a_3 = (a_1a_4, a_3^{-1}, a_3)_1.$$

Thus by Corollary 3.21, $(a_1a_3a_4^{-1},1,1)_1$, $(a_2a_1a_3a_1^{-1})_1$, and $(a_1a_3^{-1}a_4a_3,1,1)_1$ are in K_4 (since only elements in the commutator subgroup are required to shift the coordinates).

To show that K_4 is the maximal branching subgroup, observe that the generators obtained in Proposition 3.22 for $Stab_{G_4}(1)$ generate a subgroup of index 3 in $X * G_4$ and so in particular, for each vertex v on the first level $Rist_{G_4}(v)$ must be a proper subgroup of $v * G_4$.

By Theorem 3.16, Proposition 3.19, and Proposition 3.22, $Stab_{G_4}(1) \leq K_4$ and the rest follows from self-similarity. □

Now we move to odd $n \geq 5$.

Proposition 3.24. *For odd $n \geq 5$, if g is in $Stab_{G_n}(1)$, then $\epsilon(g) \equiv 0 \bmod 2$. Conversely, if $g_1,\dots,g_n$ is such that $\sum_{i=1}^{n} \epsilon(g_i) \equiv 0 \bmod 2$, then $(g_1,\dots,g_n)_1 \in Stab_{G_n}(1)$.*

Proof. First observe that since the $a_i(\varnothing)$ is an element in $S_n \backslash A_n$ for all i, if a word in $a_1,\dots,a_n$ produces an element g in $Stab_{G_n}(1)$, then it necessarily has even exponent sum. In particular, $\epsilon(g) \equiv 0 \bmod 2$.

Recall that I_n is the set of all elements of the form $(g,1,\dots,g^{-1},1,\dots,1)_1$ and that $I_n \subseteq Stab_{G_n}(1)$. Define $H_n = \langle I_n, X * G'_n \rangle \trianglelefteq G_n$. Observe that $G_n/X * G'_n$ is isomorphic to a subgroup of

$$(\mathbb{Z}/(n-1)\mathbb{Z})^n \wr S_n = \left((\mathbb{Z}/(n-1)\mathbb{Z})^n \times \cdots \times (\mathbb{Z}/(n-1)\mathbb{Z})^n\right) \rtimes S_n.$$

and hence G_n/H_n isomorphic to a subgroup of $(\mathbb{Z}/(n-1)\mathbb{Z})^n \times S_n$. We claim that in fact G_n/H_n is a subdirect product of $(\mathbb{Z}/(n-1)\mathbb{Z})^n \times S_n$. Indeed, H_n is contained in the kernel of ϵ_1, a surjective homomorphism onto $(\mathbb{Z}/(n-1)\mathbb{Z})^n$, and G_n surjects onto S_n.

Let $\pi_1 : G_n/H_n \twoheadrightarrow S_n$ and let $\pi_2 : G_n/H_n \twoheadrightarrow (\mathbb{Z}/(n-1)\mathbb{Z})^n$. By Goursat's Lemma, $(\mathbb{Z}/(n-1)\mathbb{Z})^n/\ker(\pi_1) \cong S_n/\ker(\pi_2)$. Since the only non-trivial abelian quotient of S_n has order 2, $(\mathbb{Z}/(n-1)\mathbb{Z})^n/\ker(\pi_1)$ is either trivial or order 2. But since a word in $a_1,\dots,a_n$ has trivial permutation only if it has even exponent sum, $\ker(\pi_1)$ is a proper subgroup of $(\mathbb{Z}/(n-1)\mathbb{Z})^n$. Therefore,

$$Stab_{G_n}(1) = \{(g_1,\dots,g_n)_1 \mid \sum_{i=1}^{n} \epsilon(g_i) \equiv 0 \bmod 2\}.$$

□

Definition 3.25. *For odd n, define $K_n = \{g \in G_n \mid \epsilon(g) \equiv 0 \bmod 2\} \leq G_n$.*

Theorem 3.26. *For odd $n \geq 5$, K_n is the maximal branching subgroup for G_n. In particular, $Rist_{G_n}(m) = X^m * K_n$ for all m. Consequently,*

$$Stab_{G_n}(m+1) = X^m * Stab_{G_n}(1) \leq Rist_{G_n}(m)$$

for all m and G_n has trivial rigid kernel.

Proof. Again, it suffices to show that $K_n \geq X * K_n = Rist_{G_n}(1)$. By Proposition 3.24, $(1,\dots,1,g,1,\dots,1)_1 \in G_n$ if and only if $\epsilon(g) \equiv 0 \bmod 2$ which is if and only if $g \in K_n$. Moreover, by Lemma 3.12 such a $(1,\dots,1,g,1,\dots,1)_1$ is in K_n. Now by Lemma 3.12 and Proposition 3.24, $Stab_{G_n}(1) \leq K_n$ and the rest follows from the above work. □

Now, we work with the remaining groups: G_n where $n \geq 5$ is even.

Definition 3.27. *A group $G \leq Aut(\mathcal{T})$ is called layered if G contains the direct product of $|X|$ copies of G each acting on one of the subtrees of $\mathcal{T}$ rooted at the first level, i.e.*

$$X * G \leq G.$$

Theorem 3.28. *For even $n \geq 5$, $Stab_{G_n}(m) = Rist_{G_n}(m) = X^m * G_n$. In particular, G_n is layered and consequently G_n has trivial rigid kernel.*

Proof. It suffices to show for $m = 1$. Let H_n be as in the proof of Proposition 3.24. By the same arguments presented there, for even $n \geq 5$, G_n/H_n isomorphic to a subgroup of $(\mathbb{Z}/(n-1)\mathbb{Z})^n \times A_n$ (since the root permutations generate A_n by Lemma 3.2). This time, G_n/H_n is a subdirect product of $(\mathbb{Z}/(n-1)\mathbb{Z})^n \times A_n$ as H_n is again contained in the kernel of ϵ_1 and G_n surjects onto A_n.

Let $\pi_1 : G_n/H_n \twoheadrightarrow A_n$ and let $\pi_2 : G_n/H_n \twoheadrightarrow (\mathbb{Z}/(n-1)\mathbb{Z})^n$. By Goursat's Lemma, $(\mathbb{Z}/(n-1)\mathbb{Z})^n/\ker(\pi_1) \cong A_n/\ker(\pi_2)$. Since the only non-trivial abelian quotient of A_n is the trivial group, $(\mathbb{Z}/(n-1)\mathbb{Z})^n/\ker(\pi_1)$ is trivial and so $Stab_{G_n}(1) = X * G_n$. Since $X * G_n$ is in fact a direct product, it is also $Rist_{G_n}(1)$. Moreover, as $X * G_n \leq G_n$, for any m we have $X^m * G_n \leq G_n$. Since the group is self similar, the result follows. □

Note that Theorem 3.28 tells us that for even $n \geq 5$, $G_n = G_n \wr A_n$. In particular, this implies the following corollary.

Corollary 3.29. *For even $n \geq 5$, $\overline{G}_n = (\cdots A_n \wr A_n) \wr A_n) \wr \cdots A_n)$, the infinitely iterated wreath product of A_n.*

3.3.2 Branch kernels

The combination of Theorems 3.23, 3.26, and 3.28 shows that, unlike when $n = 3$, when $n \geq 4$ the congruence kernel for G_n is the same as the branch kernel.

Recall that from Proposition 1.21, for a branch group G, the branch kernel is given by

$$\ker(\widehat{G} \to \widetilde{G}) = \varprojlim_{\substack{m\geq 1\\ e\geq 1}} Rist_G(m)/[Rist_G(m)^e Rist_G(m)'].$$

Theorem 3.30. *For $n \neq 3$, the branch kernel, and thus the congruence kernel, for G_n is the inverse limit*

$$\varprojlim_{m \geq 1} M_n^m$$

where M_n is a finite abelian group. When $n \geq 5$ is even, M_n is cyclic of order $n-1$ and when $n = 4$ or $n \geq 5$ is odd, M_n has exponent bounded between $(n-1)$ and $2(n-1)$.

Proof. For $n = 4$, $Rist_{G_4}(m)/Rist_{G_4}(m)' \cong (K_4)^{4^m}/(K_4')^{4^m} = (K_4/K_4')^{4^m}$. Now K_4 is a subgroup of index 3 containing G_4' and hence surjects onto a subgroup of index 3 in $G_4/G_4' = (\mathbb{Z}/3\mathbb{Z})^4$. The image of K_4 is then an abelian group of exponent 3 and so K_4/K_4' has exponent at least 3. It is easy to check that the normal generators of K_4 given by Theorem 3.23 have order 6. Since conjugating does not change the order of an element, K_4 has a generating set consisting of elements of order 6 and so K_4/K_4' has exponent at most 6. Now since K_4 has finite index in a finitely generated group, it is finitely generated. Therefore K_4/K_4' is a finite abelian group with exponent between 3 and 6.

For odd $n \geq 5$, $Rist_{G_n}(m)/Rist_{G_n}(m)' \cong (K_n)^{n^m}/(K_n')^{n^m} = (K_n/K_n')^{n^m}$. Now K_n is a subgroup of index 2 containing G_n' and as such surjects onto a subgroup of index 2 in $G_n/G_n' = (\mathbb{Z}/(n-1)\mathbb{Z})^n$. Since $n \geq 5$, the image of K_n is an abelian group of exponent $(n-1)$ and so K_n/K_n' has exponent at least $(n-1)$. Moreover, since n is odd, a generating set for K_n is $\{a_{n-1}a_1, a_na_2, a_ia_{i+2} \mid 1 \leq i \leq n-2\}$. It is easy to check that each of these elements has order $2(n-1)$. Thus K_n/K_n' has exponent at most $2(n-1)$.

For even $n \geq 5$,

$$Rist_{G_n}(m)/Rist_{G_n}(m)' \cong (G_n)^{n^m}/(G_n')^{n^m} = (G_n/G_n')^{n^m} = \left((\mathbb{Z}/(n-1)\mathbb{Z})^n\right)^{n^m}.$$

Now since for all $n \geq 4$, $Rist_{G_n}(m)/Rist_{G_n}(m)'$ has finite exponent, the collection $\{Rist_{G_n}(m)/Rist_{G_n}(m)'\}$ is cofinal with $\{Rist_{G_n}(m)/Rist_{G_n}(m)'Rist_{G_n}(m)^e\}$. Further, since

$$Rist_{G_n}(m)/Rist_{G_n}(m+1) = \begin{cases} (G_n/G_n')^{n^m} & \text{if } n \geq 5 \text{ is even} \\ (K_n/K_n')^{n^m} & \text{if } n = 4 \text{ or } n \geq 5 \text{ is odd} \end{cases}$$

we see that similarly $\{(G_n/G_n')^m\}$ and $\{(K_n/K_n')^m\}$ respectively also form cofinal sets. In particular, the branch kernel is

$$\varprojlim_{m \geq 1} M_n^m$$

where

$$M_n = \begin{cases} G_n/G_n' & \text{if } n \geq 5 \text{ is even} \\ K_n/K_n' & \text{if } n = 4 \text{ or } n \geq 5 \text{ is odd} \end{cases}$$

□

Remark 3.31. *Our techniques only put bounds on the exponent of K_n/K_n' for $n = 4$ and odd $n \geq 5$. It would be desirable to precisely understand this group.*

3.4 Just Infinite-ness

Recall that from Corollary 1.19, a branch group G is just infinite if and only if for each $m \geq 1$, the index of $Rist_G(m)'$ in $Rist_G(m)$ is finite.

In [6], it is shown the G_3/G_3'' is an infinite group and so G_3 is not just infinite. For $n \geq 4$, the proof of Theorem 3.30 shows $Rist_{G_n}(m)/Rist_{G_n}(m)'$ is finite. Thus we obtain the following result.

Theorem 3.32. *G_n is just infinite if and only if $n \neq 3$.*

3.5 Hausdorff Dimension

In this section, we define the Hausdorff dimension for a metric space and describe a standard metric on profinite group. We then use the work of the previous sections to compute the Hausdorff dimension of $\overline{G}_n$. For a more thorough discussion of Hausdorff dimension, the reader is directed to Chapter 2 of [8].

Let (X,d) be a metric space and let X' be a subset of X. A **δ-cover** of X' is a finite or countable collection of subsets $\{U_i\}_{i=1}^{\infty}$ such that for all i the diameter of U_i is at most δ and $X' \subseteq \cup_{i=1}^{\infty} U_i$.

For $r \geq 0$, define

$$\mathcal{H}_\delta^r(Y) = \inf\{\sum_{i=1}^{\infty} diam(U_i)^r \mid \{U_i\}_{i=1}^{\infty} \text{is a } \delta \text{ cover of Y}\}$$

Then the **r-dimensional Hausdorff measure** is

$$\mathcal{H}^r(Y) = \lim_{\delta \to 0} \mathcal{H}_\delta^r.$$

And finally the **Hausdorff dimension of** Y is

$$dim_{\mathcal{H}}(Y) = \sup\{r \geq 0 \mid \mathcal{H}^r(Y) = 0\}.$$

Recall now that a profinite group

$$H = \varprojlim_{m \geq 1} H_m$$

is metrizable via the metric given by $d(g,h) = \frac{1}{|H_i|}$ where $g = (g_m)_{m=1}^{\infty}$ and $h = (h_m)_{m=1}^{\infty}$ and i is the least value with $g_i h_i^{-1} \in H_i$.

Thus for a closed subgroup H of $Aut(\mathcal{T})$, we can compute the Hausdorff dimension. It was shown in [2] that the Hausdorff dimension of H is given by

$$dim_{\mathcal{H}}(H) = \liminf_{m \to \infty} \frac{\log |H/Stab_H(m)|}{\log |Aut(\mathcal{T})/Stab_{Aut(\mathcal{T})}(m)|}. \tag{3.1}$$

Abért and Virág showed that with probability 1 the closure of the subgroup generated by three random automorphisms of a binary tree has Hausdorff dimension 1 [1]. Shortly thereafter, Siegenthaler constructed the first explicit examples of topologically finitely generated groups of Hausdorff dimension 1 [20].

As a consequence of the work in previous sections, we show that $\overline{G}_n$ has Hausdorff dimension arbitrarily close to 1.

Theorem 3.33. *For $n \geq 3$, the Hausdorff dimension for $\overline{G}_n$ is*

$$dim_{\mathcal{H}}(\overline{G}_n) = \begin{cases} 1 - \frac{\log(48)}{\log(331776)} & \text{if } n = 4 \\ 1 - \frac{\log(2)}{\log(n!)} & \text{if } n \geq 5 \text{ is even} \\ 1 - \frac{\log(2)}{n\log(n!)} & \text{if } n \text{ is odd} \end{cases}$$

Proof. For $n = 4$, $|G_4/Stab_{G_4}(1)| = |A_4| = \frac{4!}{2}$ by Lemma 3.2. It can easily be checked from the generators of Proposition 3.22 that $Stab_{G_4}(1)/Stab_{G_4}(2)$ is an index 3 subgroup of $A_4 \times A_4 \times A_4 \times A_4$, so $|Stab_{G_4}(1)/Stab_{G_4}(2)| = \frac{4!^4}{3 \cdot 2^4}$. For $m \geq 2$, $|Stab_{G_4}(m-1)/Stab_{G_4}(m)| = |Stab_{G_4}(1)/Stab_{G_4}(2)|^{4^{m-2}}$ by Theorem 3.23. Hence equation 3.1 yields

$$\begin{aligned} dim_{\mathcal{H}}(\overline{G}_4) &= \liminf_{m\to\infty} \frac{\log\left(\frac{4!^{1+4+\cdots+4^{m-1}}}{2^{1+4+\cdots 4^{m-1}}3^{1+4+\cdots 4^{m-2}}}\right)}{\log(4!^{1+4+\cdots 4^{m-1}})} \\ &= \liminf_{m\to\infty} \frac{\log(4!^{\frac{4^m-1}{3}}) - \log(2^{\frac{4^m-1}{3}}) - \log(3^{\frac{4^{m-1}-1}{3}})}{\log(4!^{\frac{4^m-1}{3}})} \\ &= \liminf_{m\to\infty} 1 - \frac{\log(2)}{\log(4!)} - \frac{(4^{m-1}-1)\log(3)}{(4^m-1)\log(4!)} \\ &= 1 - \frac{\log(2)}{\log(4!)} - \frac{\log(3)}{4\log(4!)} \\ &= 1 - \frac{\log(48)}{\log(331776)}. \end{aligned}$$

For even $n \geq 5$, $G_n/Stab_{G_n}(1) = A_n$ and $Stab_{G_n}(m-1)/Stab_{G_n}(m) = (A_n)^{n^{m-1}}$ by Lemma 3.2 and Theorem 3.28. Therefore

$$\begin{aligned} dim_{\mathcal{H}}(\overline{G}_n) &= \frac{\log\left(\frac{n!^{1+n+\cdots n^{m-1}}}{2^{1+n+\cdots n^{m-1}}}\right)}{\log(n!^{1+n+\cdots n^{m-1}})} \\ &= \liminf_{m\to\infty} \frac{\log((n!)^{\frac{n^m-1}{n-1}}) - \log(2^{\frac{n^m-1}{n-1}})}{\log(n!^{\frac{n^m-1}{n-1}})} \\ &= 1 - \frac{\log(2)}{\log(n!)}. \end{aligned}$$

Finally, when n is odd $|G_n/Stab_{G_n}(1)| = |S_n| = n!$ by Lemma 3.2. Additionally,

$$|Stab_{G_n}(1)/Stab_{G_n}(2)| = \frac{n!^n}{2}$$

by Proposition 3.24. Moreover,

$$\begin{aligned}
&|Stab_{G_n}(m-1)/Stab_{G_n}(m)| \\
&= |Stab_{G_n}(1)/Stab_{G_n}(2)|^{4^{m-2}} \\
&= \frac{n!^{n^{m-1}}}{2^{n^{m-2}}}
\end{aligned}$$

by Theorem 3.28 and Lemma 2.8. Thus

$$\begin{aligned}
dim_{\mathcal{H}}(\overline{G}_n) &= \liminf_{m\to\infty} \frac{\log\left(\frac{n!^{1+n+\cdots n^{m-1}}}{2^{1+n+\cdots n^{m-2}}}\right)}{\log(n!^{1+n+\cdots n^{m-1}})} \\
&= \liminf_{m\to\infty} \frac{\log(n!^{\frac{n^m-1}{n-1}}) - \log(2^{\frac{n^{m-1}-1}{n-1}})}{\log(n!^{\frac{n^m-1}{n-1}})} \\
&= \liminf_{m\to\infty} 1 - \frac{(n^{m-1}-1)\log(2)}{(n^m-1)\log(n!)} \\
&= 1 - \frac{\log(2)}{n\log(n)}.
\end{aligned}$$

□

Corollary 3.34. *For all $\epsilon > 0$, there exists n such that $dim_{\mathcal{H}}(\overline{G}_n) > 1 - \epsilon$.*

Chapter 4 New examples of groups with non-trivial rigid kernel

In this final chapter, we present examples to show that triviality of the rigid kernel is not necessarily preserved when moving to subgroups of finite index, even if they are maximal. In doing so, we present infinitely many new examples of branch groups with non-trivial rigid kernel, adding to the only currently known example of the Hanoi towers group.

Theorem 4.1. *For $n \geq 4$, let $1 \neq d > 2$ be such that $d \mid (n-1)$ and let $H_{n,d}$ be the set of elements g of G_n with $\epsilon(g) \equiv 0 \bmod d$. The $H_{n,d}$ is a subgroup of index d in G_n and is a branch group with non-trivial rigid kernel.*

Proof. We will construct explicit elements that are in $Stab_{H_{n,d}}(m)$ but not in $Rist_{H_{n,d}}(k)$ for all $k \leq m$. For any n and d as in the theorem, $H_{n,d}$ is a subgroup of G_n of index d by Lemma 3.10. Let $\beta = a_1 a_2 \cdots a_n$.

If $n \geq 4$ is odd, then

$$\beta = (a_1 a_3 \cdots a_n, 1, \ldots, 1, a_2 a_4 \cdots a_{n-1})(1, n)$$

and

$$\beta^2 = (a_1 a_3 \cdots a_n a_2 a_4 \cdots a_{n-1}, 1, \ldots, 1, a_2 a_4 \cdots a_{n-1} a_1 a_3 \cdots a_n)_1.$$

Clearly, β^2 has exponent sum $2n$ and is not an element of $H_{n,d}$. But $H_{n,d}$ does contains G_n' and therefore also $X * G_n'$ and all elements of the form $(g, 1, \ldots, 1, g^{-1})_1$ for $g \in G_n$. Combining these elements we get that $\beta^2 \equiv (\beta^2, 1, \ldots, 1)_1 \bmod H_{n,d}$ and so likewise $(\beta^2, 1, \ldots, 1)_1$ is not contained in $H_{n,d}$. Inductively we get for any m,

$$\beta^2 \equiv (\beta^2, 1, \ldots, 1)_m \bmod H_{n,d}$$

and so $(\beta^2, 1, \ldots, 1)_m$ is not contained in $H_{n,d}$.

But again, since $H_{n,d}$ contains all elements of the form $(g, 1, \ldots, 1, g^{-1})_1$, the element $(\beta^2, 1, \ldots, 1, \beta^{-2})_1 \in Stab_{H_{n,d}}(1)$ and again inductively for all m, $(\beta^2, 1, \ldots, 1, \beta^{-2})_m \in Stab_{H_{n,d}}(m)$. But $(\beta^2, 1, \ldots, 1, \beta^{-2})_m \notin Rist_{H_{n,d}}(k)$ for any k, otherwise $(\beta^2, 1, \ldots, 1)_{m-k}$ would be in the group $H_{n,d}$, a contradiction.

Now if $n \geq 4$ is even, then

$$\beta = (a_1 a_3 \cdots a_{n-1}, 1, \ldots, 1, a_2 a_4 \cdots a_n)_1$$

and so by the same discussion above $\beta \notin H_{n,d}$ and for all m,

$$\beta \equiv (\beta, 1, \ldots, 1)_m \bmod H_{n,d}$$

so $(\beta, 1, \ldots, 1)_m$ is not an element of $H_{n,d}$ but $(\beta, 1, \ldots, 1, \beta^{-1})_m$ is. The same arguments show that $(\beta, 1, \ldots, 1, \beta^{-1})_m$ is not in $Rist_{H_{n,d}}(k)$ for any k. □

Chapter 5 Open questions

In this final chapter we discuss some open questions and avenues for additional research related to the above material.

The first comes from an alternative generalization of the Hanoi towers group. Instead of playing the game on three pegs, one could play the game on n-pegs ($n \geq 3$) and produce a group acting on the n-ary tree. For each i and j where $1 \leq i \leq j \leq n$ we define automorphisms α_{ij} to model the moves in the game as

$$\alpha_{ij} = (\alpha_{ij}, \ldots, \alpha_{ij}, 1, \alpha_{ij}, \ldots, \alpha_{ij} 1, \alpha_{ij}, \ldots, \alpha_{ij})(i, j)$$

where α_{ij} repeats in every coordinate excepts the i and j ones.

Then the n-ary Hanoi towers group Γ_n is defined to be $\langle \alpha_{ij} \mid 1 \leq i \leq j \leq n \rangle$. Very little is known about Γ_n for $n \geq 4$.

Question 5.1. *For $n \geq 4$, does Γ_n have the congruence subgroup property?*

Many of the tools available for G_3 are not available yet for Γ_n without further study. In particular, the following is still open.

Question 5.2. *For $n \geq 4$, is Γ_n a branch group?*

It is known that Γ_n is weakly branch. A group G is **weakly branch** if there is an embedding of G into $\mathrm{Aut}(\mathcal{T})$ such that G acts transitively on every level and for all $m \geq 1$, $Rist_G(m) \neq \{1\}$. This can be shown by observing that α_{ij}, α_{jk}, and α_{ik} generate a subgroup of Γ_n isomorphic to G_3. Part of the difficulty in studying Γ_n comes from the fact that Γ_n is not contracting for $n \geq 4$.

Another more general open question about the congruence subgroup property is the following.

Question 5.3. *Does there exist a branch group with the congruence subgroup property that is not just infinite?*

This question is still open because the standard technique for showing a branch group G has the congruence subgroup property is to use Theorem 1.18 and then to show that for all m, $Rist_G(m)'$ contains a level stabilizer. In fact, to the author's knowledge this is the only technique so far that has been succesfully applied to obtain a positive answer to the congruence subgroup problem. Unfortunately, this technique fails if the group is not just infinite.

Appendix A Generators of $Stab_{G_3}(1)$ via the Reidemeister-Schreier Method

The Reidemeister-Schreier method is a method for determining a presentation for a finite index subgroup of a group with a given presentation. Here we are only interested in finding a generating set for $Stab_{G_3}(1)$ and so we only discuss this part of the process. For a more detailed description and justification of the procedure the reader is directed to Chapter 2 Section 3 of [16].

Let G be a group with a symmetric generating set and let H be a finite index subgroup. We construct the **Schreier graph** S **for** H **in** G. This graph has vertices corresponding to cosets of H in G and two vertices are connected by an edge if one can get from one coset to the other via left multiplication by a generator. The edge is then labeled by the appropriate generator. Next we choose a maximal spanning tree T inside the coset graph.

For each edge $e \in S\backslash T$, let e_i be the unique path in T from the vertex H to one endpoint of e and let e_t be the unique path in T from the vertex H to other endpoint of e. Then the set $\{e_i e (e_t)^{-1} \mid e \in S\backslash T\}$ forms a generating set for H.

Recall that $G_3/Stab_{G_3}(1) \cong S_3$ and so from this we construct the Schreier graph for $Stab_{G_3}(1)$ in G_3 seen in Figure A.1. We choose the red edges to form the spanning tree T. Recalling that $a_i = a_i^{-1}$, we obtain the following list of generators.

$$a_1 a_3 a_1 a_2 \qquad a_1 a_2 a_1 a_3 \qquad a_2 a_3 a_2 a_1 \qquad a_2 a_1 a_2 a_3$$

These generators are precisely α, β, δ, and γ as defined in Section 2.2.

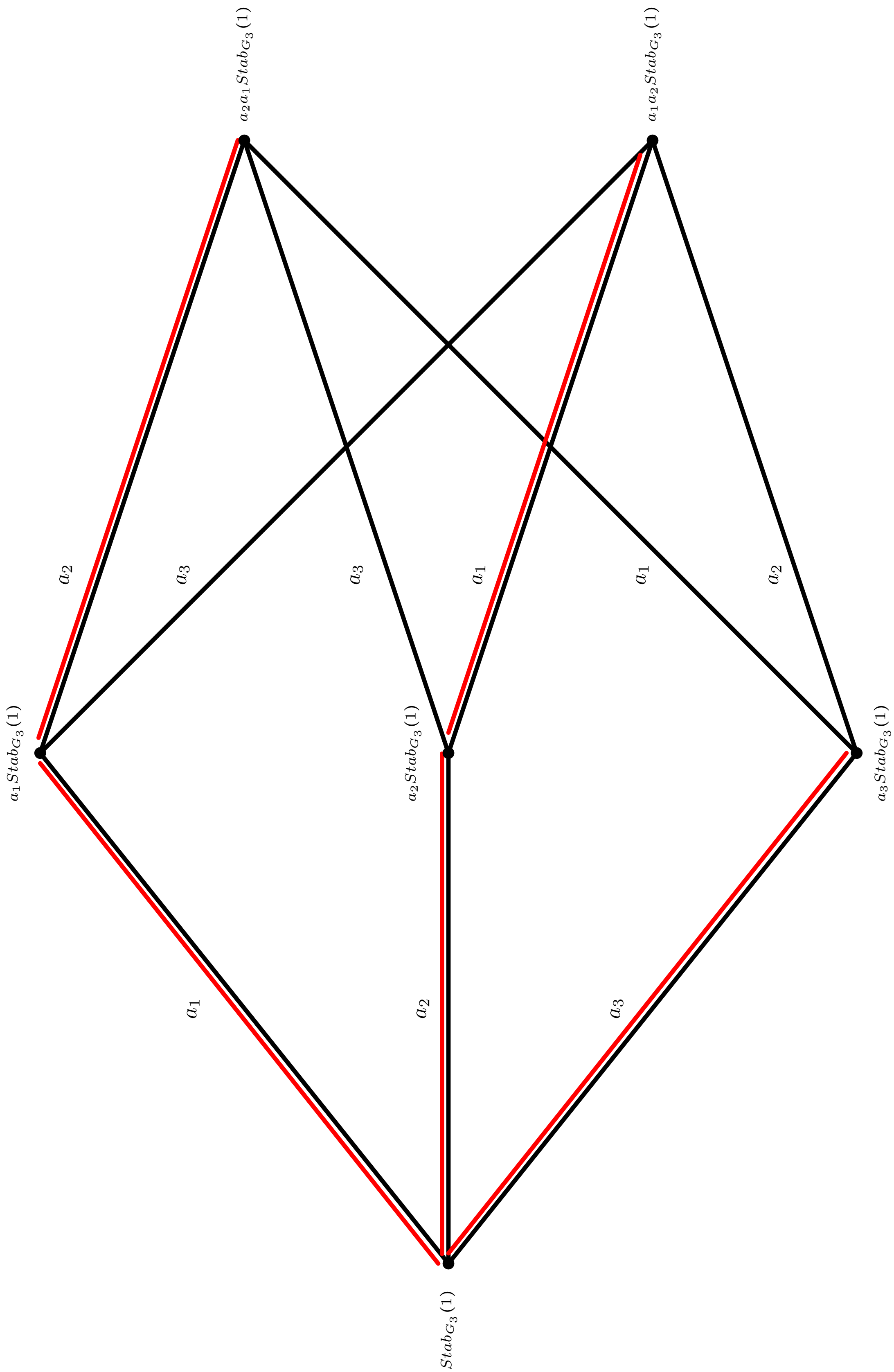

Figure A.1: The Schreier coset graph for $Stab_{G_3}(1)$ in G_3.

Appendix B Generators of $Stab_{G_4}(1)$ via GAP

Here we include the remaining details of the proof for Proposition 3.22.

Proposition 3.22. $Stab_{G_4}(1) = \langle a_1a_3a_4^{-1}, a_2a_1a_3a_1^{-1}, a_1a_3^{-1}a_4a_3, X * G_4', I_4 \rangle$.

Computing the Reidermeister-Schreier method by hand becomes unwieldy as the index of the subgroup grows. For the group G_4, the index of the stabilizer of the first level in G_4 is 12. Since G_4 has four generators, the Schreier graph for $Stab_{G_4}(1)$ in G_4 would have 12 vertices and 48 edges. Rather than drawing this, we using the computer algebra system GAP (Groups, Algorithms, and Programming) [10] and the GAP package AutomGrp [17] to produce the generators of $Stab_{G_4}(1)$. We include here the code necessarily to obtain Proposition 3.22 and an explanation of the reductions made.

```
gap> LoadPackage("AutomGrp");;
gap> G4:=AutomatonGroup("a1=(a1,1,1,1)(2,3,4), a2=(1,a2,1,1)(1,3,4),
     a3=(1,1,a3,1)(1,2,4), a4=(1,1,1,a4)(1,2,3)");;
gap> StabilizerOfLevel(G4,1);
   < a1^3, a1*a3*a4^-1, a2*a1*a3*a1^-1, a2^3, a2*a3*a2^-1*a1^-1,
     a2*a4*a1^-1, a3*a1*a2^-1, a3*a2*a4^-1*a1^-1, a3^3, a3*a4*a2^-1*a1^-1,
     a4*a1*a2^-1*a1^-1, a4*a2*a3^-1, a4*a3^2*a1^-1, a4^3, a1^-1*a2*a4,
     a1^-1*a3*a4^-1*a1^-1, a1^-1*a4*a3^-1, a2^-1*a1*a4^-1, a2^-1*a3*a1,
     a2^-1*a4*a3*a1^-1, a3^-1*a1*a4^-1*a1^-1, a3^-1*a2*a1^-1, a3^-1*a4*a2,
     a4^-1*a1*a3, a4^-1*a2*a3*a1^-1, a4^-1*a3*a2^-1, (a1*a2)^2, a1*a2^2*a3,
     a1*a2*a3*a4, a1*a2*a4*a1, (a1*a4)^2, a1*a4*a2*a4^-1, a1*a4*a3*a2,
     a1*a4^2*a2^-1, (a1*a3^-1)^2, a1*a3^-1*a2*a1, a1*a3^-1*a4*a3 >
```

Recall now that a_1, a_2, a_3, and a_4 each have order 3. This allows us to immediately remove a_1^3, a_2^3, a_3^3 and a_4^3 from the list GAP produced. This also gives us that $a_i^{-1} = a_i^2$ for $1 \le i \le 4$.

Let $\alpha = a_1a_3a_4^{-1}$, $\beta = a_2a_1a_3a_1^{-1}$, and $\gamma = a_1a_3^{-1}a_4a_3$. It now suffices to write each of the remaining generators as a product of α, β, and γ modulo $\langle I_4, X * G_4' \rangle$. We will use the following reduction obtained in Corollary 3.21: $(g_1, g_2, g_3, g_4)_1 \equiv (1, \ldots, 1, g_{1^\theta} g_{2^\theta} g_{3^\theta} g_{4^\theta}, 1, \ldots 1)_1 \bmod \langle I_4, X * G_4' \rangle$ for any permuation θ of $\{1, 2, 3, 4\}$.

$$a_2 a_3 a_2^{-1} a_1^{-1} \equiv \alpha\gamma \bmod \langle I_4, X * G_4' \rangle$$
$$a_2 a_4 a_1^{-1} \equiv \alpha^2\beta \bmod \langle I_4, X * G_4' \rangle$$
$$a_3 a_1 a_2^{-1} \equiv \alpha^2\beta^2\gamma^2 \bmod \langle I_4, X * G_4' \rangle$$
$$a_3 a_2 a_4^{-1} a_1^{-1} \equiv \beta\gamma^2 \bmod \langle I_4, X * G_4' \rangle$$
$$a_3 a_4 a_2^{-1} a_1^{-1} \equiv \alpha^2\beta^2 \bmod \langle I_4, X * G_4' \rangle$$
$$a_4 a_1 a_2^{-1} a_1^{-1} \equiv \alpha\beta^2\gamma^2 \bmod \langle I_4, X * G_4' \rangle$$
$$a_4 a_2 a_3^{-1} \equiv \alpha\beta\gamma^2 \bmod \langle I_4, X * G_4' \rangle$$
$$a_4 a_3^2 a_1^{-1} \equiv \alpha^2 \bmod \langle I_4, X * G_4' \rangle$$
$$a_1^{-1} a_2 a_4 = \alpha^2\beta \bmod \langle I_4, X * G_4' \rangle$$
$$a_1^{-1} a_3 a_4^{-1} a_1^{-1} \equiv \alpha \bmod \langle I_4, X * G_4' \rangle$$
$$a_1^{-1} a_4 a_3^{-1} \equiv \alpha^2 \bmod \langle I_4, X * G_4' \rangle$$
$$a_2^{-1} a_1 a_4^{-1} \equiv \alpha\beta^2 \bmod \langle I_4, X * G_4' \rangle$$
$$a_2^{-1} a_3 a_1 \equiv \alpha^2\beta^2\gamma^2 \bmod \langle I_4, X * G_4' \rangle$$
$$a_2^{-1} a_4 a_3 a_1^{-1} \equiv \alpha^2\beta^2 \bmod \langle I_4, X * G_4' \rangle$$
$$a_3^{-1} a_1 a_4^{-1} a_1^{-1} \equiv \alpha^2\gamma \bmod \langle I_4, X * G_4' \rangle$$
$$a_3^{-1} a_2 a_1^{-1} \equiv \alpha\beta\gamma \bmod \langle I_4, X * G_4' \rangle$$
$$a_3^{-1} a_4 a_2 \equiv \alpha\beta\gamma^2 \bmod \langle I_4, X * G_4' \rangle$$
$$a4^{-1} a_1 a_3 \equiv \alpha \bmod \langle I_4, X * G_4' \rangle$$
$$a4^{-1} a_2 a_3 a_1^{-1} \equiv \beta^2\gamma^2 \bmod \langle I_4, X * G_4' \rangle$$
$$a_4^{-1} a_3 a_2^{-1} \equiv \alpha^2\beta^2\gamma \bmod \langle I_4, X * G_4' \rangle$$
$$(a_1 a_2)^2 \equiv \alpha\beta^2\gamma \bmod \langle I_4, X * G_4' \rangle$$
$$a_1 a_2^2 a_3 \equiv \alpha^2\beta^2\gamma^2 \bmod \langle I_4, X * G_4' \rangle$$
$$a_1 a_2 a_3 a_4 \equiv \beta\gamma \bmod \langle I_4, X * G_4' \rangle$$
$$a_1 a_2 a_4 a_1 \equiv \alpha^2\beta \bmod \langle I_4, X * G_4' \rangle$$
$$(a_1 a_4)^2 \equiv \gamma^2 \bmod \langle I_4, X * G_4' \rangle$$

$$a_1 a_4 a_2 a_4^{-1} \equiv \alpha^2 \beta \gamma^2 \bmod \langle I_4, X * G_4' \rangle$$

$$a_1 a_4 a_3 a_2 \equiv \beta \gamma \bmod \langle I_4, X * G_4' \rangle$$

$$a_1 a_4^2 a_2^{-1} \equiv \alpha \beta^2 \bmod \langle I_4, X * G_4' \rangle$$

$$(a_1 a_3^{-1})^2 \equiv \alpha + \delta \bmod \langle I_4, X * G_4' \rangle$$

$$a_1 a_3^{-1} a_2 a_1 \equiv \alpha \beta \gamma \bmod \langle I_4, X * G_4' \rangle$$

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