1971

Concerning the condition that a disk in E^3/G be the image of a disk in E^3

Edythe Parker Woodruff

CONCERNING THE CONDITION THAT A DISK IN E^3/G BE THE IMAGE OF A DISK IN E^3

A Dissertation Presented

By

EDYTHE PARKER WOODRUFF

Submitted to the Graduate School of the
State University of New York at Binghamton

DOCTOR OF PHILOSOPHY

May 1971

MATHEMATICS

CONCERNING THE CONDITION THAT A DISK

IN E^3/G BE THE IMAGE OF A DISK IN E^3

A Dissertation

By

EDYTHE PARKER WOODRUFF

Approved as to style and content by:

Louis F. McAuley

Chairman of the Mathematics Department
and Chairman of the Committee

Walter E Koehn

Member of the Committee

Patricia Tulley McAuley

Member of the Committee

Harry W. Berkowitz

Member of the Committee

Dick Wick Hall

May 1971

ACKNOWLEDGMENTS

To Prof. Louis F. McAuley the author wishes to express her appreciation for direction, patient listening, and help. Without his encouragement this work might never have been completed.

For many useful discussions the author thanks Prof. Harry Berkowitz.

The author appreciates her family's understanding and patience thoughout graduate study. She also wishes to thank her husband Robert and children Andrew and Jeanne for aid in the final stages of typing and printing.

INTRODUCTION

A generalization of the concept of lifting of an n-cell is studied. In the standard definition a lift of an n-cell is another n-cell that maps homeomorphically onto the given n-cell under the projection mapping. The generalization is defined as follows: A set X' is said to be a P-lift of a set X if X' is homeomorphic to X and X' projects onto X. Notice that a point in X may be the image of a nondegenerate set in X'. Methods of upper semicontinuous decomposition are used throughout the thesis.

Conditions under which P-lifts exist are investigated. It is not always true that when X and Y each P-lift that $X \cup Y$ will P-lift. Many such examples are given. Additional conditions are imposed to make $X \cup Y$ P-lift. One such theorem states: Suppose that A and B are compact manifolds that each P-lift; $A \cup B$ is a disk $D \subset E^3/G$; $A \cap B \cap P(H) = \emptyset$; and $(A \cup B) \cap P(H)$ is 0-dimensional. Then $A \cup B$ P-lifts.

It is easy to construct a decomposition G of E^3 such that there is a disk $D \subset E^3/G$ that is not the image of any disk in E^3. This implies that the disk D does not P-lift. Consider, for example, the set D' one gets from rotating the following around the z-axis:
Cl $\{y \mid y = \sin 1/x,\ 0 < x \leq 1\}$. This set D' is not a

disk because it is not locally connected on the segment $s = \{z \mid 0 \leq z \leq 1\}$. If we use the decomposition of E^3 in which s is the only nondegenerate element, then the image of this set D' is a disk D, but D' does not contain any disk that projects onto D. This condition is not true for all disks homeomorphically close to D. There are disks that agree with D' except very close to s and do project onto disks. In the thesis hypotheses are studied that assure that even when a set does not P-lift, there is a homeomorphic copy close to it that does P-lift.

An example will be given of a disk D in a decomposition space such that for a particular $\epsilon > 0$ there is no disk D_ϵ that is ϵ-homeomorphic to D and P-lifts. The decomposition space has the following properties: (1) E^3/G is homeomorphic to E^3; (2) each $g \in H$ is a tame arc; (3) H is continuous and closed; (4) P(H) is a Cantor set; and (5) H is not countable and is not definable by 3-cells. The decomposition space is a modification of a toroidal decomposition. A construction called knitting is used. This resembles a single stitch of ordinary knitting that progresses through a countable number of bundles of nondegenerate elements. The proof of this example will constitute a major part of the thesis. It is conjectured that this example would

not exist if the space satisfied conditions (1) - (5) and the following new definition: A collection G of closed point sets in E^3 is said to be <u>equi-locally connected</u> provided that, if y is a point of an element g_o of G and ϵ is a positive number, there is a topological 3-cell B contained in the ϵ-neighborhood of y such that if g is an element of G, then $g \cap \text{Int } B$ is connected.

Another example closely related to the above one does not have the property of containing a disk such that no disk close to it P-lifts. This pair of examples is particularly interesting because their decomposition spaces are "equivalent" in the terminology of Armentrout, Lininger, and Meyer.

The author conjectures that another related example provides a negative answer to a question asked by Armentrout: Suppose G is a point-like decomposition of E^3. If S is a 2-sphere in E^3/G, does there exist a 2-sphere S' in E^3 such that P[S'] is a 2-sphere homeomorphically close to S?

TABLE OF CONTENTS

CHAPTER I 1

The Concept of a P-lift

CHAPTER II 10

Conditions under which the Union of Two Sets P-lifts

CHAPTER III 40

Decompositions in which there exists a P-liftable Disk Homeomorphically Close to any Given Disk

CHAPTER IV 82

A Decomposition Space Containing a Disk that is not Homeomorphically Close to a P-liftable Disk

CHAPTER I

THE CONCEPT OF A P-LIFT

The standard definition of lifting applies this term to a mapping.

<u>Definition</u>. Suppose that P is a continuous map of a space X onto a space Y and that f is a continuous map of a space Z into the space Y. There is said to be a <u>lifting of the map f</u> if there exists a map $g: Z \to X$ such that $f = Pg$.

$$\begin{tikzcd} & X \arrow[d, "P"] \\ Z \arrow[ur, dashed, "g"] \arrow[r, "f"] & Y \end{tikzcd}$$

McAuley [23] uses the term lifting in connection with a space:

<u>Definition</u>. Suppose that Z is an n-cell and that f is a homeomorphism of Z into Y. If there exists a lifting g of f, then g is also called a <u>lifting of the n-cell f(Z)</u>.

The emphasis here is on the n-cell $D = f(Z) \subset Y$. Although this is called a lifting of an n-cell, D, whenever it occurs, it implies that every homeomorphism $h: Z \xrightarrow{\text{onto}} D$ has a lifting.

$$\begin{tikzcd} & & X \arrow[d, "P"] \\ Z \arrow[urr, dashed] \arrow[r, "h"] & D \subset & Y \end{tikzcd}$$

Hence, it is a concept concerning mappings--i.e., members of this set of homeomorphisms. For any given h that lifts, let g_h be its lifting. The mapping P restricted to the space $D' = g_h(Z)$ must be a homeomorphism. Of course, $g_h(Z)$ and D are both n-cells.

$$\begin{tikzcd} & D' \arrow[d, "P|D'"] & \subset & X \arrow[d, "P"] \\ Z \arrow[ur, "g_h"] \arrow[r, "h"'] & D & \subset & Y \end{tikzcd}$$

Our interest in this thesis is in the question of the existence of part of the last diagram:

$$\begin{tikzcd} D' \arrow[d, "P|D'"] & \subset & X \arrow[d, "P"] \\ D & \subset & Y \end{tikzcd}$$

We will not require that $P|D'$ be a homeomorphism. Notice that, as in the case of lifting, D' is not necessarily unique. If the diagram exists with a particular D', then we have two cases:

(1) $P|D'$ is a homeomorphism. In this case, D' is a lift.

(2) $P|D'$ is not a homeomorphism. Now, D' is not a lift of D.

In the second case, we enlarge the diagram to the following one:

$$\begin{tikzcd}
 & D' \arrow[r, phantom, "\subset"] \arrow[d, "P|D'"] & X \arrow[d, "P"] \\
Z \arrow[ur, "g"] \arrow[r, "f"] \arrow[d, "\tilde{P}"'] & D \arrow[r, phantom, "\subset"] & Y \\
Z \arrow[ur, "h"'] & &
\end{tikzcd}$$

g and h are homeomorphisms. The other maps are not necessarily homeomorphisms. Now the question becomes: Given a homeomorphism $h: Z \to Y$, does there exist a homeomorphism $g: Z \to X$ such that $Pg(Z) = h(Z)$?

If the answer is yes, then the space D' exists and is $g(Z)$. Notice that $Pg = h$ for maps is a stronger statement; it means that for any $z \in Z$, $Pg(z) = h(z)$.

For the following definition we do not assume that D and D' are n-cells, but we do require that they are homeomorphic spaces.

<u>Definition</u>. If g exists in the above diagram, then we say that g is a <u>P-lift of the map h</u>, that D' is a <u>P-lift of the space D</u>, and that D <u>P-lifts</u>. The term is chosen to suggest the importance of the natural projection in the concept. In the diagram, D and D' are not connected by a homeomorphism as in McAuley's diagram, but rather by a map, which is the projection map P if $Y = X/G$. Also, notice that, given an arbitrary homeomorphism h, the map f that is in the diagrammatic position of a map that lifts is the composition of h and

$\tilde{P}$. Of course, we can not find f until we know that g exists.

We could have chosed to call this concept a pseudolifting. We prefer not to because in previous terminology a lifting is always a mapping. We wish to apply our new term also to the space D'. It seems natural to emphasize spaces over maps in this concept because $Pg(Z) = h(Z)$ is an identity on spaces.

Another alternate terminology for this concept is to say that D' is a projecting copy of D and that D has a projecting copy. To fully describe the concept, we need to say that D' is an epi-projecting copy. This is rather cumbersome. Without the epi-, the reader is apt to forget that we wish to require that D' project onto D.

Notation. Throughout this thesis, the prime notation will always indicate a P-lift, i.e., given a set $X \subset E^3/G$, X' will denote a P-lift of X.

For any P-lift X' of X and decomposition G of E^3, X' is naturally decomposed by G. We call this decomposition $G_{X'}$. $G_{X'} = \{g \cap X' \mid g \in G\}$. The corresponding set of nondegenerate elements is $H_{X'} = \{g \cap X' \mid g \in G$ and $g \cap X'$ is nondegenerate.$\}$

When a manifold is contained in a larger set, Bd

will denote the manifold boundary. Int will denote the manifold minus its boundary. If M is a manifold imbedded in a larger set S, then Fr M will denote the set theoretic boundary of M with respect to S; i.e., $\mathrm{Cl}\, M \cap \mathrm{Cl}\,(S - M)$.

If H is a set whose elements are point sets, then H^* will denote the union of all points that lie in elements of H.

The ϵ-neighborhood of a set S will be denoted by $N_\epsilon(S)$.

The symbol $d(x,y)$ will denote the Euclidean distance between the points x and y. For sets X and Y the symbol $d(X,Y)$ will denote the greatest lower bound of the set $\{d(x,y) \mid x \in X, y \in Y\}$.

For definitions concerning upper semi-continuous decompositions, see Steve Armentrout [2].

Remarks. It is easy to construct a decomposition G of E^3 such that there is a disk $D \subset E^3/G$ that is not the image of any disk in E^3. This implies that the disk D does not P-lift. Consider, for example, the set D' one gets from rotating the following around the z-axis: $\mathrm{Cl}\,\{y \mid y = \sin 1/x,\ 0 < x \leq 1\}$. This is not a disk because it is not locally connected on the segment $s = \{z \mid 0 \leq z \leq 1\}$. If we use the decomposition of E^3

in which s is the only nondegenerate element, then the image of this set D' is a disk D, but D' does not contain any disk that projects onto D. This condition is not true for all disks homeomorphically close to D. There are disks that agree with D' except very close to s and do project onto disks. In this thesis we will prove an example in which there is a disk D in a decomposition space E^3/G such that for a particular $\epsilon > 0$ there is no disk D_ϵ that is ϵ-homeomorphic to D and P-lifts. Also, theorems will be proven giving some conditions that guarantee that given a disk $D \subset E^3/G$ and an $\epsilon > 0$ there is a disk ϵ-homeomorphic to D that does P-lift. The following is such a theorem.

<u>Theorem 1</u>. Suppose E^3/G is homeomorphic to E^3 and that H is countable. Given a disk D in E^3/G and $\epsilon > 0$. Then there is a disk D^t_ϵ such that

(1) D^t_ϵ is ϵ-homeomorphic to D.

(2) D^t_ϵ is tame.

(3) D^t_ϵ P-lifts to $D^{t\,\prime}_\epsilon$.

(4) $D^{t\,\prime}_\epsilon$ is tame.

Proof. We will use Bing's Approximation Theorem [7], which states: For each 2-manifold M in a triangulated 3-manifold-with-boundary and each non-negative continuous function f defined on M, there is a 2-manifold M^* and a homeomorphism h of M onto M^* such

that $d(x, h(x)) \leq f(x)$, $(x \in M)$, and M' is locally polyhedral at $h(x)$ if $F(x) > 0$.

Since the decomposition space is E^3, the given disk D is contained in a polyhedral 3-ball. We use this 3-ball for the triangulated 3-manifold-with-boundary in Bing's hypothesis. For the continuous function we use $f(x) = \epsilon/2$ for all $x \in D$. Hence, there is a polyhedral disk D_p which is $\epsilon/2$-homeomorphic to D.

We now will show that in a neighborhood of D_p there is a set homeomorphic to $D_p \times [0,1]$. Since D_p is a polyhedral disk, it can be extended to a polyhedral sphere S. The sphere S has a piecewise linear collar on each side. By definition, a collar is a closed neighborhood N of S in $S \cup$ (either complementary domain of S) such that there is a homeomorphism $\theta : S \times [0,1] \xrightarrow{\text{onto}} N$ and $\theta(p \times \{0\}) = p$ for each $p \in S$. Hence, $\theta(D_p \times \{0\}) = D_p$. Notice that $D_p \times \{s_1\}$ and $D_p \times \{s_2\}$ are disjoint disks in $D_p \times [0,1]$ for any $s_1, s_2 \in [0,1]$ and $s_1 \neq s_2$. Hence, $\theta(D_p \times \{s_1\})$ and $\theta(D_p \times \{s_2\})$ are disjoint disks. Since $\theta(p \times \{0\}) = p$ for each $p \in D_p$, there is some δ such that $d(\theta(p \times \{s\}), p) < \epsilon/2$ for each $s \in [0,\delta]$. This gives us $\theta(D_p \times [0,\delta]) \subset N_{\epsilon/2}(D_p)$. For $s \in [0,\delta]$, $\theta(D_p \times \{s\}) \subset N_{\epsilon/2}(D_p)$. Let $D_s = \theta(D_p \times \{s\})$. Each D_s is $\epsilon/2$-homeomorphic to D_p. We now have

$C = \{ D_s \mid s \in [0,\delta]\}$, which is an uncountable collection of disjoint disks, each ϵ-homeomorphic to D_p.

For every $s \in [0,\delta)$ the disk D_s is tame because it is collarable. At most a countable number of members of C can contain points of the countable set P(H). Hence, we have an uncountable subcollection $C_1 \subset C$ of tame disks that contain no points of P(H). Observe that if a compact set contains no points of P(H), then P^{-1} is a homeomorphism on that set. This implies that each disk in C_1 P-lifts. It remains to be shown that the P-lift of each element of C_1 is tame.

<u>Theorem</u> (Armentrout [3]). Suppose that M is a 3-manifold with boundary and G is a cellular decomposition of M such that M/G is a 3-manifold with boundary. Suppose K is a 2-manifold with boundary in M such that K misses H. Then P(K) is tame in M/G if and only if K is tame in M.

We will apply this theorem to our example. For M/G use a 3-ball, B, containing C_1^* in its interior and such that its boundary misses P(H). For M use $P^{-1}(B)$. Let K be any member of the collection C_1. We still need to know that G is a cellular decomposition. This we get by citing the following:

<u>Theorem</u> (Kwun [21]). If E^3/G is homeomorphic to E^3 and H is countable, then each element of H is

cellular.

We can now use the conclusion of Armentrout's theorem. Every member of the collection C_1 P-lifts to a tame set. Hence, any member of C_1 satisfies the requirements for D_{ϵ}^{t}. $\square$

C H A P T E R   I I

CONDITIONS UNDER WHICH THE UNION OF TWO SETS P-LIFTS

<u>Theorem 2.1</u>. Suppose that A and B are compact manifolds (with boundary), $A \cup B$ is a disk in E^3/G, and $A \cap B \cap P(H) = \emptyset$. Suppose that A has a particular P-lift A' such that

(1) $P|A'$ is monotone,

(2) Any imbedding $j: A' \to E^2$ has the property that for any $g \in H_{A'}$, $j(g)$ does not separate any component of $j(A')$ nor E^2, and

(3) $H^* \cap \mathrm{Bd}\, A' = \emptyset$.

Also, suppose that B has a particular P-lift B' satisfying similar conditions. Then $A \cup B$ P-lifts.

Proof. We must show the existence of the following P-lift diagram, where $(A \cup B)''$ is a copy of $A \cup B$ and ϕ and h are homeomorphisms.

$$\begin{tikzcd}
 & (A \cup B)' \arrow[d, "P|(A\cup B)'"] & \subset & E^3 \arrow[d] \\
(A \cup B)'' \arrow[ur, "\phi"] \arrow[d, "\tilde{P}"'] & A \cup B & \subset & E^3/G \\
(A \cup B)'' \arrow[ur, "h"] & & &
\end{tikzcd}$$

The following diagrams do exist because A' and B' are P-lifts.

$$\begin{tikzcd}
 & A' \arrow[d, "P|A'"] & \subset & E^3 \arrow[d] \\
A'' \arrow[ur, "\phi_A"] \arrow[d, "\tilde{P}_A"'] & A' & \subset & E^3/G \\
A'' \arrow[ur, "h_A"'] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B' \arrow[d, "P|B'"] & \subset & E^3 \arrow[d] \\
B'' \arrow[ur, "\phi_B"] \arrow[d, "\tilde{P}_B"'] & B' & \subset & E^3/G \\
B'' \arrow[ur, "h_B"'] & & &
\end{tikzcd}$$

We define $(A \cup B)'$ to be $A' \cup B'$ for the particular A' and B' given in the hypothesis. Then, by the definition of P-lift, the natural projection takes $(A \cup B)'$ onto $A \cup B$.

We will define $(A \cup B)''$ in such a way that it is homeomorphic to $(A \cup B)'$. Then it will be necessary to show that $(A \cup B)''$ is homeomorphic to $A \cup B$.

We first wish to consider the set that maps onto $A \cap B$ in each of the two given P-lift diagrams above. Since A and B are compact, $A \cap B \cap P(H) = \emptyset$ implies that P^{-1} is a homeomorphism on $A \cap B$. Hence, $A \cap B$ lifts by McAuley's definition of lift. Since no other points in E^3 map onto $A \cap B$ by P, this lift is the only possible preimage of $A \cap B$. Therefore, it must be the P-lift of this set in each of the given diagrams. Therefore, the subset $A \cap B$ of A gives us the following restricted diagram, in which $(A \cap B)'_A$ and $(A \cap B)''_A$ are the appropriate subsets from the diagram for A.

$$\begin{tikzcd}
 & & (A\cap B)'_A \arrow[r, phantom, "\subset"] \arrow[d, "P|(A\cap B)'_A"] & A' \arrow[r, phantom, "\subset"] \arrow[d, "P|A'"] & E^3 \arrow[d, "P"] \\
A'' \arrow[r, phantom, "\supset"] \arrow[d, "\tilde{P}_A"] & (A\cap B)''_A \arrow[ur, "\phi|(A\cup B)''_A"] & A\cap B \arrow[r, phantom, "\subset"] & A \arrow[r, phantom, "\subset"] & E^3/G \\
A'' \arrow[r, phantom, "\supset"] & (A\cap B)''_A \arrow[ur, "h_A|(A\cup B)''_A"] & & &
\end{tikzcd}$$

There is a similar diagram for $A \cap B$ considered as a subset of B. By the above argument, we know that the subsets $(A \cap B)'_A$ of A' and $(A \cap B)'_B$ of B' are identical.

We are ready to define $(A \cup B)''$. In the disjoint union of the sets A'' and B'', identify points of $(A \cap B)''_A$ with points of $(A \cap B)''_B$ by $x_B = \phi_B^{-1}\,\phi_A\, x_A$ for $x_A \in (A \cap B)''_A$ and $x_B \in (A \cap B)''_B$. Give this set the obvious topology: A set U is open in the new space if $U \cap A''$ is open in A'' and $U \cap B''$ is open in B''. The set $(A \cup B)''$ is defined to be this new space. (It is the adjunction space $A'' \cup_{\phi_B^{-1}\phi_A} B''$.)

We must now show that $(A \cup B)''$ is homeomorphic to $A \cup B$. We know that the subsets A'', B'', and $(A \cap B)''$ are homeomorphic to A, B, and $A \cap B$, respectively, but these homeomorphisms do not necessarily agree on $(A \cap B)''$. The monotone hypothesis has not been used yet. Note the counterexamples (given after this set of theorems)

if this hypothesis is omitted.

We define $G_{A'}$ to be the set $\{ g \cap A' \mid g \in G\}$. By the definition of a decomposition space, $A'/G_{A'}$ is A. By the definition of P-lift, A' is homeomorphic to A. Hence, A' is homeomorphic to $A'/G_{A'}$.

Now we will prove that no element of $H_{A'}$ separates A'. Suppose there is a $g \in H_{A'}$ such that g does separate A'. Let $A_1' \cup A_2' = A' - g$, where A_1' and A_2' are separated sets. The sets A_1' and A_2' are closed in $A' - g$ by the definition of separated. Let $A_1 = P(A_1')$ and $A_2 = P(A_2')$. Then A_1 and A_2 are closed in $A - P(g)$, since the natural map is closed. Hence, if $A_1 \cup A_2$ is connected, then $A_1 \cap A_2 \neq \emptyset$. Let $p \in A_1 \cap A_2$. Then $P^{-1}(p) \cap A_1' \neq \emptyset$ and $P^{-1}(p) \cap A_2' \neq \emptyset$, and, hence, $P^{-1}(p)$ is an element of $H_{A'}$ that is not connected. This contradicts the hypothesis. Hence, it must be that A_1 and A_2 are separated sets such that $A_1 \cup A_2 = A - P(g)$. Now, since $A \cup B$ is a disk, $P(g)$ must have a neighborhood N in $A \cup B$ homeomorphic to the plane or to the closed upper half plane. The set $N - P(g) =$ $N \cap (A_1 \cup A_2 \cup B)$ is connected. Since N can have an arbitrarily small diameter, $P(g)$ must be a limit point of B. Since B is closed, this implies that $P(g) \in B$. Since g was chosen as an element of $H_{A'}$, $P(g) \in A \cap B$. This implies that $P(g) \cap A \cap B \neq \emptyset$, which contradicts

the hypothesis that $A \cap B \cap P(H) = \emptyset$. Hence, no element of $H_{A'}$ separates A'.

Since A is contained in a disk, it is embeddable in E^2. By the definition of P-lift, A' and A'' are homeomorphic to A. Hence, there is an imbedding $i_a: A'' \to E^2$. By our above work, for each $g \in G$, $i_a\phi^{-1}(g \cap A')$ does not separate $i_a(A'')$; by hypothesis $g \cap A'$ is connected, so its copy $i_a\phi^{-1}(g \cap A')$ is connected. Since $i_a\phi^{-1}$ is an imbedding of A' into E^2, no nondegenerate element intersects the boundary of $i_a\phi^{-1}(g \cap A') = i_a(A'')$. Hence, certainly no nondegenerate element separates E^2. Therefore, $i_a\phi^{-1}(g \cap A')$ does not separate E^2. We now define a decomposition of E^2 such that the elements are points of $E^2 - i_a(A'')$ and the sets $i_a\phi^{-1}(g \cap A')$. Since this decomposition is closely related to $G_{A''}$ of A'', let us call it $\hat{G}_{A''}$.

Analogous statements can be made for B' and B'', and finally a decomposition $\hat{G}_{B''}$ of E^2.

We now quote Moore's Theorem: [26]: If G is an upper semicontinuous decomposition of the Euclidean plane E^2 into compact continua such that no element of G separates E^2, then the decomposition space associated with G is homeomorphic to E^2. Our decompositions $\hat{G}_{A''}$ and $\hat{G}_{B''}$ of E^2 satisfy these hypotheses. Therefore, $E^2/\hat{G}_{A''}$ and $E^2/\hat{G}_{B''}$ are homeomorphic to E^2.

In [22] McAuley defines shrinkable in the following manner.

<u>Definition</u>. Suppose that G is a u.s.c. collection filling up a metric space M and that H is the collection of all non-degenerate elements of G. We say that H is <u>shrinkable</u> in M if and only if for each open covering U of H, $\epsilon > 0$, and a homeomorphism h of M onto M; there exists a homeomorphism f_ϵ of M onto M such that

(1) $f_\epsilon = h$ on $M - U^*$, and

(2) for each g in H

(a) diam $f_\epsilon(g) < \epsilon$, and

(b) there exists D in U such that $h(D) \supset h(g) \cup f_\epsilon(g)$.

The following recent result of Siebenmann [28] is stated here in the 2-dimensional case without boundary.

<u>Theorem.</u> Suppose that $P: X \to X/G$ where X and X/G are 2-manifolds without boundary and P is a CE map. Then G is shrinkable (McAuley).

Here, a <u>CE map</u> is a <u>cell-like map</u>, which means that it is a map $f: X \to Y$ such that for each point $y \in Y$, $f^{-1}(y)$ is a compactum that can be imbedded in some euclidean space as a cellular set.

We wish to use this theorem for our decompositions $E^2/\hat{G}_{A''}$ and $E^2/\hat{G}_{B''}$. In E^2, if a set S is a

compact continuum that doesn't separate E^2, then S is cellular. The proof of this well-known fact easily follows from an argument in Whyburn [29, p. 170]. Since both our decomposition spaces are homeomorphic to E^2, we satisfy Siebenmann's hypothesis that X and X/G are 2-manifolds. We now know that $E^2/\hat{G}_{A''}$ and $E^2/\hat{G}_{B''}$ are shrinkable.

We wish now to apply the following:

Theorem (McAuley [22]). Suppose that G is a u.s.c. decomposition of a complete metric space M. Furthermore, H (the collection of all non-degenerate elements of G) is shrinkable and M is locally compact at each point of H^*. Then the decomposition space is homeomorphic to M.

We will use shrinkability of each of $E^2/\hat{G}_{A''}$ and $E^2/\hat{G}_{B''}$ to get shrinkability of $(A \cup B)''/G_{(A \cup B)''}$, where $G_{(A \cup B)''}$ is the decomposition induced by the decomposition of $(A \cup B)'$. Let $\epsilon > 0$ be given. Let $\mathcal{U}$ be an open covering in $(A \cup B)''$ of $H_{(A \cup B)''}$. Let $\mathcal{V} = \{\text{a component of } [U - (A \cap B)'' - \text{Bd } A'' - \text{Bd } B''] \mid U \in \mathcal{U}\}$. Let $\mathcal{V}_{A''} = \{V \in \mathcal{V} \mid V \cap A'' \neq \emptyset\}$ and $\mathcal{V}_{B''} = \{V \in \mathcal{V} \mid V \cap B'' \neq \emptyset\}$. Notice that $\mathcal{V}^*_{A''} \cap \mathcal{V}^*_{B''} = \emptyset$, because elements of these covers are connected sets and $(A \cap B)''$ separates $(A \cup B)''$. $\mathcal{V}_{A''}$ is an open cover in A'' of $H_{A''}$. Let $i_a\mathcal{V}_{A''} = \{i_a(V) \mid V \in \mathcal{V}_{A''}\}$. This

is an open cover of $\dot{H}_{A''}$, the nondegenerate elements of $E^2/\dot{G}_{A''}$. Since this decomposition has been shown to be shrinkable, given $\delta > 0$, there is a homeomorphism $f_a: E^2 \to E^2$ such that

(1) f_a is the identity on $E^2 - i_a \mathcal{V}^*_{A''}$, and

(2) for each $g \in H_{A''}$

(a) diam $f_a(g) < \delta$, and

(b) there is a $i_a(V) \in i_a \mathcal{V}_{A''}$ such that

$f_a(i_a(V)) \supset f_a(g) \cup g$.

Define $f_{A''} = i_a^{-1} f_a \mid i_a A''$. Since we have condition (1) on f_a, we know that $f_{A''}$ is the identity on $A'' - \mathcal{V}^*_{A''}$. Since i_a is an imbedding, we can take it to be the identity on the set on which it acts. Hence, we can take δ above to be the given ϵ. We now have that

(2) for each $g \in H_{A''}$

(a) diam $f_{A''}(g) < \epsilon$, and

(b) there is a $V \in \mathcal{V}_{A''}$ such that $f_{A''}(V) \supset f_{A''}(g) \cup g$.

On B'' there is an analogous shrinking homeomorphism $f_{B''}$. We can now define a homeomorphism $f_{(A \cup B)}: (A \cup B)'' \to (A \cup B)''$ by

$$f_{(A \cup B)''} = \begin{cases} f_{A''} & \text{on } A'' \\ f_{B''} & \text{on } B'' \end{cases} .$$

The homeomorphism $f_{(A \cup B)''}$ has properties on $(A \cup B)''$ with covering $\mathcal{U}$ analogous to those stated above for $f_{A''}$

with covering $\mathcal{V}_{A''}$. Hence, $H_{(A \cup B)''}$ is shrinkable. We are now ready to apply McAuley's theorem. For M in it we use $(A \cup B)''$. The set M is then a complete metric space and is locally compact. Hence, $(A \cup B)''/G_{(A \cup B)''}$ is homeomorphic to $(A \cup B)''$, where $G_{(A \cup B)''}$ is the decomposition of $(A \cup B)''$ with nondegenerate elements $H_{(A \cup B)''}$.

Now by the homeomorphism ϕ we carry this over to the space $(A \cup B)'$ and find that $(A \cup B)'/G_{(A \cup B)'}$ is homeomorphic to $(A \cup B)'$. (Notice that we chose to work with the decomposition of $(A \cup B)''$ instead of $(A \cup B)'$, because $(A \cup B)' \subset E^3$ and the shrinking in an open cover of $H_{(A \cup B)'}$ may move nondegenerate elements out of $(A \cup B)'$.) By the definition of the decomposition $G_{(A \cup B)'}$, the set $A \cup B$ is homeomorphic to $(A \cup B)'/G_{(A \cup B)'}$. Hence, $A \cup B$ is homeomorphic to $(A \cup B)'$, which we wished to know for the P-lift diagram. Also, $A \cup B$ is homeomorphic to $(A \cup B)''$, giving us h in the diagram. $\square$

The conditions in the hypothesis of Theorem 2.1 on particular P-lifts A' and B' may make the theorem hard to apply. We will prove some lemmas to replace hypotheses about A' and B' with hypotheses about A, B, and the decomposition G.

<u>Lemma 2.1</u>. Suppose that A is a compact manifold contained in a disk $D \subset E^3/G$ and that A has a P-lift A'. Also, suppose that $A \cap P(H)$ is 0-dimensional. Then $P|A'$ is monotone.

Proof. Suppose that $P|A': A' \to A$ is not monotone. Then there is an element g of G such that $g \cap A'$ is not connected. Let s and t be components of $g \cap A'$ (not necessarily all of $g \cap A'$). For each positive integer n, consider the 1/n neighborhood of P(g). In $A \cap N_{1/n}(P(g))$ let N_n denote the component containing P(g). If, for every n, s and t lie in the same component of $P^{-1}(N_n)$, then s and t are connected. (We use here that A is compact implies that A' is compact, and hence, that in A' the limiting set of connected sets is connected.) Since this contradicts our assumption, s and t must lie in different components. Let U_s and U_t be the components of $P^{-1}(N_{n_o})$ containing s and t, respectively.

The set N_{n_o} is an open (with respect to the disk) subset of a disk. Hence, it is a 2-manifold. Hurewicz and Wallman [19, p. 48] prove that any n-manifold cannot be disconnected by a subset of dimension less than or equal to n - 2. Therefore, $N_{n_o} - P(H)$ is connected. The map P^{-1} is a homeomorphism on this set. Hence, $P^{-1}(N_{n_o} - P(H))$ is connected. Therefore, U_s or U_t

contains no point of $P^{-1}(N_{n_o} - P(H))$. Suppose U_s contains no point of this set, and therefore, that $U_s \subset H^*$.

We note that a compact manifold can have only a finite number of components. Hence, A' has a finite number of components. Let C be the component of A' containing U_s. If C is not also the component of A' containing U_t, then $P(A) = A$ has fewer components than A' and this contradicts the P-lift requirement that A be homeomorphic to A'. Hence, $U_t \subset C$ and U_s is not all of C. The set U_s is open in C and s is compact, so s is not all of U_s. The facts that U_s is connected and s is a component of $g \cap A'$ imply that $U_s - s \not\subset g \cap A'$. Therefore, $P(U_s - s) \not\subset P(g) = P(s)$. But $P(U_s - s) \cup P(s) = P(U_s)$. Hence, $P(U_s)$ is not degenerate. The set U_s is connected, so $P(U_s)$ is connected. Now, since $P(U_s) \subset P(H)$, the set $P(U_s)$ is 0-dimensional. We now have a contradiction of the fact that any connected 0-dimensional set is degenerate and this proves our lemma. $\square$

<u>Lemma 2.2</u>. Suppose that A and B are closed sets, $A \cup B$ is a disk in E^3/G, the set A P-lifts, and $A \cap B \cap P(H) = \emptyset$. Also, suppose that for a particular A' the map $P|A'$ is monotone and that each $g \in H_{A'}$ is a

dendron. Then, for any imbedding $j: A' \to E^2$ and any $g \in H_{A'}$, the image $j(g)$ does not separate E^2 or any component of $j(A')$.

Proof. First, we show that, for any $g \in H_{A'}$, the set g does not separate any component of A'. Suppose there is a g that does separate a component C of A'. Also, suppose that $P(g)$ does not separate $P(C)$. Since g separates C, there are disjoint sets S_1, and $S_2 \subset C - g$ such that S_1 and S_2 are each closed with respect to $C - g$. Since P is a closed map, $P(S_1)$ and $P(S_2)$ are each closed in $P(C) - P(g)$. Therefore, if they are not a separation of $P(C) - P(g)$, they must have a common point, say x. Now $P^{-1}(x)$ is not connected. This contradicts the hypothesis that $P|A'$ is monotone. Therefore, not both assumptions at the beginning of the paragraph are valid and we suppose that $P(g)$ does separate $P(C)$.

Now $A \cup B$ is a disk and one point can not separate a disk. Therefore, $P(g)$ does not separate $A \cup B$. Hence, $P(g)$ is the limit point of a sequence of points $x_1, x_2, \cdots, x_i, \cdots$ such that for each i, the point x_i is in $(A \cup B) - P(C)$. If, for infinitely many i, the point x_i is in B, then $P(g) \in B$. Since $P(g) \in A$ and g is nondegenerate, we have $P(g) \in A \cap B \cap P(H)$, which contradicts a hypothesis. Therefore, we can assume

that for every i, the point $x_i \in A - P(C)$. But $P(x_i) \in A' - C$ for every i contradicts the assumption that C is a component of A' and, hence, separated from $A' - C$. Therefore, g does not separate any component of A' and $j(g)$ does not separate any component of $j(A')$.

It remains to be shown that $j(g)$ does not separate E^2. The set $j(g)$ is a dendron or a point, because it is a connected subset of a dendron. Hence, it cannot separate the space, and we are through. $\square$

<u>Lemma 2.3</u>. Suppose that A is a compact manifold contained in a disk $D \subset E^3/G$, A has a P-lift, and $(\mathrm{Bd}\ A) \cap P(H) = \emptyset$. Furthermore, assume that $P|A'$ is monotone for every P-lift A'. Then, for any imbedding $j: A' \to E^2$ and any $g \in H_{A'}$, $j(g)$ does not separate any component of $j(A')$ or E^2. Then, also, $H^* \cap \mathrm{Bd}\ A' = \emptyset$.

Proof. Observe that each component of Bd A is a simple closed curve. This is shown as follows: Since A is a compact 2-manifold, its boundary is a compact 1-manifold. A compact 1-manifold is an arc or a simple closed curve. Since an arc can not be a boundary component of a compact 2-manifold, each component of Bd A must be a simple closed curve. Notice that $(\mathrm{Bd}\ A) \cap P(H) = \emptyset$ implies that $P^{-1}|(\mathrm{Bd}\ A)$ is a homeomorphism, so each component of Bd A lifts to a

simple closed curve.

We first show that $P^{-1}(\text{Bd } A) \subset \text{Bd } A'$ for any A'. Suppose not. Then let $x \in \text{Bd } A$ be a point such that $P^{-1}(x) \not\subset \text{Bd } A'$. Let J be the component of Bd A containing x. Let C be the component of A' containing $P^{-1}(J)$. Since $P^{-1}(J)$ has a point not in Bd A', the simple closed curve $P^{-1}(J)$ separates C into disjoint, open sets C_1 and C_2 that have $P^{-1}(x)$ in their boundaries. Now $P(C_1 \cup C_2) = P[C_1 \cup C_2 \cup P^{-1}(J) - P^{-1}(J)] =$ $P[C_1 \cup C_2 \cup P^{-1}(J)] - J$. The set $P[C_1 \cup C_2 \cup P^{-1}(J)]$ is connected. The simple closed curve J is in the boundary of this connected set, so it cannot separate it. Therefore, $P[C_1 \cup C_2 \cup P^{-1}(J)] - J$ is connected and, hence, $P(C_1 \cup C_2)$ is connected. Of course, $C_1 \cup C_2$ is not connected. Since P is a closed map and C_1 and C_2 are each closed in $C_1 \cup C_2$, the sets $P(C_1)$ and $P(C_2)$ must each be closed in $P(C_1 \cup C_2)$. Therefore, $P(C_1) \cup P(C_2)$ being connected implies that there exists a point $y \in P(C_1) \cap P(C_2)$. Then $P^{-1}(y) \cap C_1$ and $P^{-1}(y) \cap C_2$ must both be nonempty. This means that $P^{-1}(y)$ is not connected and contradicts the hypothesis that $P|A'$ is monotone. Therefore, $P^{-1}(\text{Bd } A) \supset \text{Bd } A'$.

Next we show that $P^{-1}(\text{Bd } A) \subset \text{Bd } A'$. Since A is a compact manifold, Bd A has a finite number of components. By the P-lift hypothesis, Bd A' must have the same

number of components. Since each component of Bd A is a simple closed curve that lifts into Bd A', these simple closed curves must be all of Bd A', and there can be no point in Bd A' - P^{-1}(Bd A). We now have that Bd A' = P^{-1}(Bd A).

Since $P^{-1}[(\text{Bd } A) \cap P(H)] \supset P^{-1}(\text{Bd } A) \cap H^* =$ Bd $A' \cap H^*$, the hypothesis that (Bd A) $\cap$ P(H) = $\emptyset$ implies that Bd $A' \cap H^* = \emptyset$, which is one of the conclusions of this lemma.

Now we will show that, for any imbedding $j: A' \to E^2$ and any $g \in H_{A'}$, j(g) does not separate any component of j(A'). Suppose that $j(g_o)$ does for some $g_o \in H_{A'}$. Then from Bd $A' \cap H^* = \emptyset$, it follows that $g_o \subset$ Int A'. Let K be the component of A' containing g_o. Because A has only finitely many components and is homeomorphic to A', P can not map two components of A' into one component of A. Therefore, P(K) is all of the component of A containing $P(g_o)$. Since g_o separates K, we can set $K - g_o = S \cup T$, where S and T are disjoint sets each closed in $K - g_o$. Since P is a closed map, either P(S) and P(T) have a point in common or they are separated. Suppose $y \in P(S) \cap P(T)$. Then $P^{-1}(y) \cap S \neq \emptyset$ and $P^{-1}(y) \cap T \neq \emptyset$, which contradicts the monotoneity of P. Hence, P(S) and P(T) are separated in P(K) - $P(g_o)$, and $P(S) \cup P(T) \cup P(g_o) = P(K)$. Since $P(g_o) \in$

Int A, we now have the open set Int P(K) separated by a point. Because Int P(K) is homeomorphic to an open subset of the plane, we have a contradiction, and hence, $j(g_o)$ does not separate any component of $j(A')$ for any $g_o \in H_{A'}$.

If, for some $g_o \in H_{A'}$, $j(g_o)$ separates E^2 but not any component of $j(A')$, then $j(g_o)$ is in Bd $j(A')$. This implies that $P(g_o) \in$ Bd A, which contradicts the hypothesis, thereby completing the proof of the lemma. $\square$

We can combine Theorem 2.1 with Lemmas 2.1, 2.2, and 2.3 in various ways. In Chapter III we will use:

Theorem 2.2. Suppose that A and B are compact manifolds that each P-lift; $A \cup B$ is a disk $D \subset E^3/G$; $A \cap B \cap P(H) = \emptyset$; $(A \cup B) \cap P(H)$ is 0-dimensional. Then $A \cup B$ P-lifts.

Examples showing the necessity of some of the hypotheses of theorems and lemmas in Chapter II.

Example (1). If "A is a closed set" is omitted from the hypotheses of Theorems 2.1 and 2.2, we have the following counterexample, which is shown in Figure 1.

Let $\{S_i\}$ be the sequence of nested topological 2-spheres:

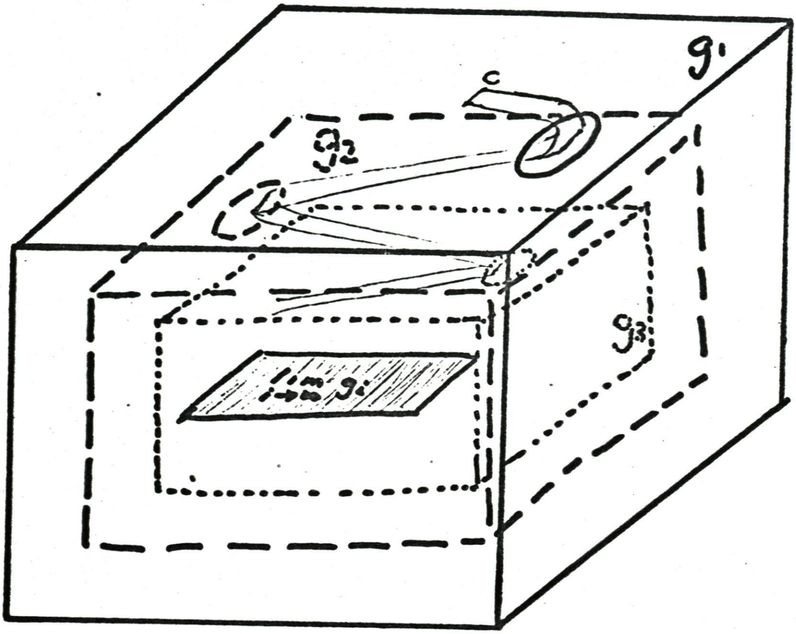


Figure 1

$S_i = \{(x,y,z) \mid |x| \leq 1/2 + 1/i,\ |y| \leq 1/2 + 1/i,\ |z| = 1/i\}$
$\cup \{(x,y,z) \mid |x| \leq 1/2 + 1/i, |y| = 1/2 + 1/i, |z| \leq 1/i\}$
$\cup \{(x,y,z) \mid |x| = 1/2 + 1/i, |y| \leq 1/2 + 1/i, |z| \leq 1/i\}$.
From the top face of each S_i remove the interior of a disk, D_i, with diameter 1/10. If i is odd, remove Int D_i from the left third of the top face of S_i. If i is even, remove Int D_i from the right third. Let $g_i = S_i$ - Int D_i. Let g_o be the limiting disk $\{(x,y,z) \mid |x| \leq 1/2, |y| \leq 1/2,\ z = 0\}$. Let the

nondegenerate elements of our decomposition be $H = \{g_0, g_1, g_2, \ldots\}$. Now, in E^3, there is a copy, C, of the set $\{\sin 1/x \mid 0 < x \leq 1\} \times [0,1]$ such that $C \cap H^* = \emptyset$ and the limiting set of C is in g_0. Since C is homeomorphic to a disk minus a boundary point and P is a homeomorphism on C, $P(C) \cup P(g_0)$ is a disk in E^3/G. For our counterexample, we use $A = P(C)$ and $B = P(g_0)$. Then $A \cup B$ does not P-lift, though each of A and B does.

Example (2). Suppose that $(\text{Bd } A) \cap (\text{Bd } B)$ contains a nondegenerate element, but we violate no other hypothesis in Theorems 2.1 and 2.2. Then the following counterexample exists.

Figure 2a shows a 2-complex C in E^3. The only nondegenerate element in C is a segment ab. The image P(C) is a disk in the decomposition space. The heavy lines in the figure project onto the boudary of the disk D. Figure 2b shows the choice we make for A' and B'. Then $A = P(A')$ and $B = P(B')$ is our counterexample.

It is interesting to observe some other properties of this example. The sets A and B are disks. Their intersection is an arc in the boundary of each. The set $P^{-1}(A \cap B)$ is T-shaped and contains the P-lift of $A \cap B$ as a proper subset.

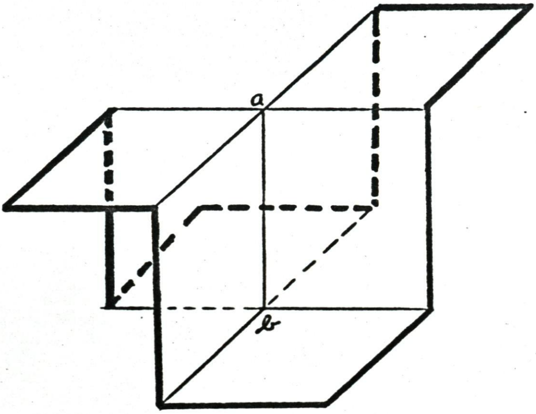


Figure 2a

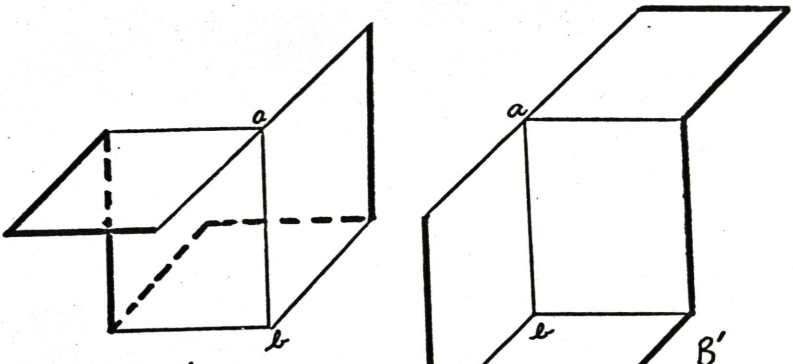


Figure 2b

<u>Example (3)</u>. We get the following counterexample if from Theorems 2.1 and 2.2 we delete that $A \cup B$ is a disk, A is a manifold, and $P(H)$ is 0-dimensional, or that there is a particular A' such that $P|A'$ is monotone. In this example there is a related counterexample for Lemma 2.1.

In E^3 let the nondegenerate elements be a set of semicircles in the $y = 0$ plane: (See Figure 3.) $\{S(r) \mid 0 < r \leq 1\}$, where each set $S(r)$ is the semicircle $\{(x,z) \mid z \geq 0,\ x^2 + z^2 = r^2\}$. We now choose A' to be the union of two arcs:

$$A' = \{(x,y,z) \mid |x| \leq 2,\ y = 0,\ z = 0\} \cup$$
$$\{(x,y,z) \mid x = 0,\ y = 0,\ -1 \leq z \leq 0\}$$

Let B' be the arc $\{(x,y,z) \mid 0 \leq x \leq 1,\ z = -x,\ y = 0\}$. Notice that since A is T-shaped and the nondegenerate elements project onto its interior, no subset of A' can be a P-lift--a subset could not both be homeomorphic to the T and project onto A.

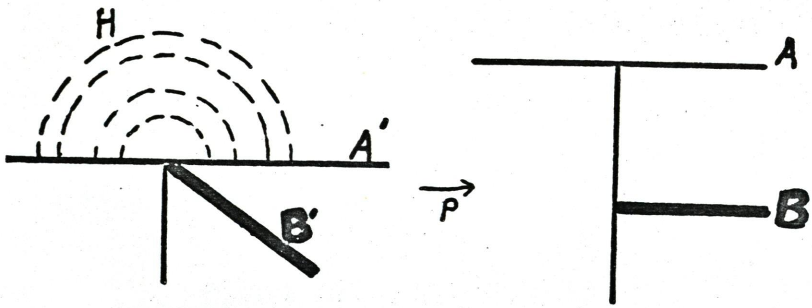


Figure 3

Example 4. If we delete the manifold requirement on A, we get this counterexample of Theorem 2.2. For Theorem 2.1, this example also deletes the conditions concerning separation of E^2 by j(g) and $H* \cap \text{Bd } A' = \emptyset$. Both conclusions of Lemma 2.3 are false when "manifold" is deleted. The only deleted hypothesis for Lemma 2.2 is the dendron condition and the conclusion is then false here.

(See Figure 4) Let S be a bounded sequence of disjoint disks in E^3/G. Let D be a disk such that $S \subset \text{Int } D$. Let $A = \text{Cl}(D - S)$ and B be a disk (with no holes punched out) such that $S \subset B \subset D \neq B$. Now $A - B \neq \emptyset$. Our decomposition has exactly one nondegenerate element, namely, g_o. The point $P(g_o)$ is in $A - B$, and g_o is the boundary of a disk in S. We now have that $A' \cup B'$ not homeomorphic to $A \cup B$, because $A \cup B$ is a disk, and $A' \cup B'$ has one hole punched out. Both A' and B' are unique P-lifts, so $A \cup B$ does not P-lift.

Examples concerned with the method of proof.

Consider the following two statements.

I. A and B each P-lift. Then $A \cup B$ P-lifts.

II. A' and B' are given P-lifts of A and B. Then $A' \cup B''$ is a P-lift of $A \cup B$.

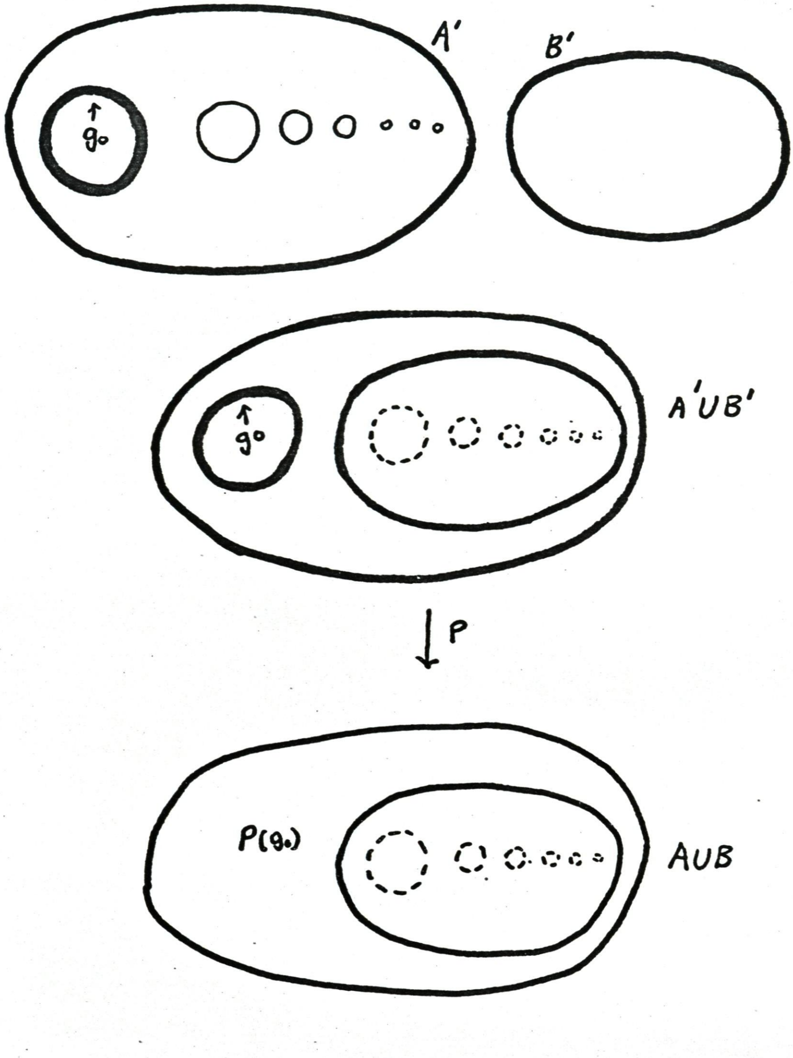


Figure 4

The proof of Theorem 2.1 consists of showing that with additional hypotheses, statement II is true. Theorem 2.2 is an example of statement I. Obviously, (not II) does not imply (not I). We have seen examples where I is not true. Our next examples are cases where II is not true, but I is true. These show for the method of proof the necessity of certain hypotheses.

Example 5. First, we have a simple example illustrating that (not II) does not imply (not I). The upper space is a disk D. The only nondegenerate element is a segment s in its interior. Let $A = P(D - s)$, $B = P(s)$, and B' be any point in s. Let A' be $D - s$. Now $A' \cup B'$ is not a P-lift, although the set $A \cup B$ does have all of D as a P-lift.

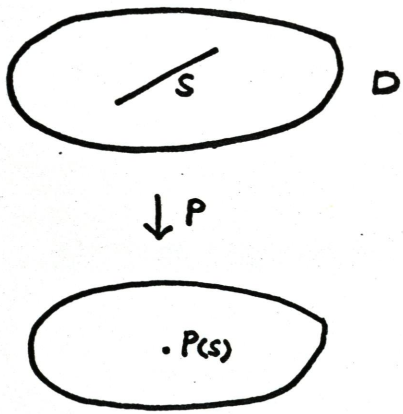


Figure 5

Example 6. If in the proof of Theorem 2.1 we do not use an A' satisfying $P|A'$ is monotone, then we get the following counterexample to $A' \cup B'$ is a P-lift of $A \cup B$. Notice that in this example, the choice of the right half of the given disk A' is itself a P-lift whose projection does have the monotone property. Hence, $A \cup B$ does P-lift to this subset of $A' \cup B'$.

(See Figure 6). In E^3 let,

$A' = \{(x, y, z) \mid |x| \leq 1, |y| \leq 1, z = 0\}$ and

$B' = \{(x, y, z) \mid x = 0, |y| \leq 1, -1 \leq z \leq 0\}$.

Let the set of nondegenerate elements H be the set of semicircles $\{ S(y, r) \mid |y| \leq 1, 0 \leq r \leq 1\}$, where

$S(y, r) = \{(x, y, z) \mid z \geq 0, x^2 + z^2 = r^2\}$.

Then the nondegenerate elements of the decomposition of A' are the pairs of points

$\{((-x, y, 0), (x, y, 0)) \mid 0 \leq x \leq 1, |y| \leq 1\}$.

Now A and B are each a disk in E^3/G and $A \cap B$ is an arc in the boundary of each. Obviously, $A' \cup B'$ is not a P-lift of $A \cup B$, since it is not homeomorphic to $A \cup B$.

Example 7. If from the hypothesis of Lemma 2.1 we delete the requirement that A be a compact manifold, we get the following counterexample to the conclusion.

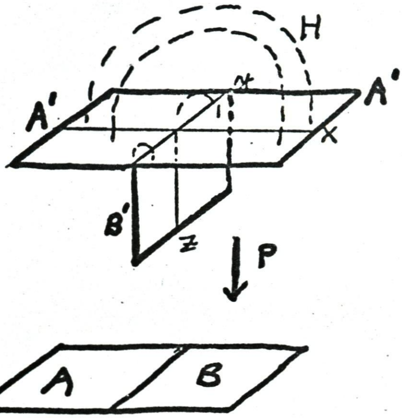


Figure 6

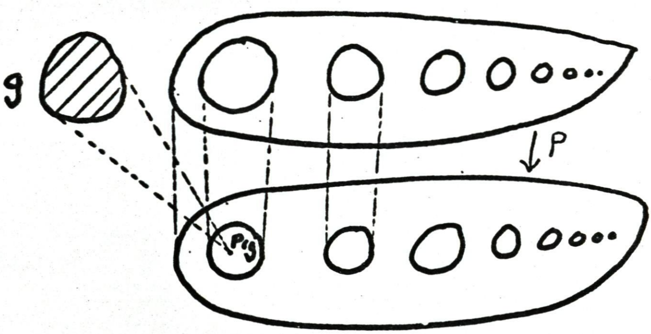


Figure 7

This example can be used to show that not every $A' \cup B'$ is a P-lift of $A \cup B$.

A is an infinite bounded sequence of disjoint disks. Let D be a disk such that $A \subset \text{Int } D$. Then let $B = \text{Cl } (D - A)$. In the upper space there is only one nondegenerate element--a disk, whose image is in the interior of one of the disks in A. This set A is the counterexample of Lemma 2.1. Note that if we choose the particular P-lift of A' that includes the nondegenerate element, then $A' \cup B'$ is not a P-lift of $A \cup B$. (See Figure 7.)

Example 8. In the proof of Theorem 2.1, we showed that a P-lift diagram for A has a P-lift diagram restricted to the subset $A \cap B$. In general, if $X \subset Y$ and Y P-lifts, it is not necessarily true that X P-lifts. Consider the following example. Let a segment s be the only nondegenerate element. Let s be in the interior of a disk D. Let $t \subset D$ be a sin $1/x$ curve with s its limiting set. Then $P(s \cup t)$ is an arc in $P(D)$. Though $P(D)$ does P-lift to D, the subset $P(s \cup t)$ does not P-lift. For this example, there does not exist a P-lift diagram restricted to the subset $P(s \cup t)$. (See Figure 8)

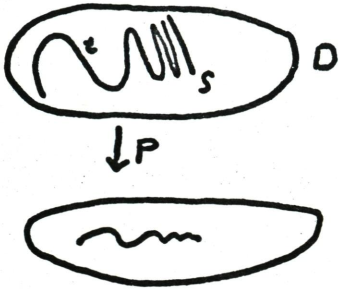


Figure 8

Remarks.

Observe that, in Example 2, $(A \cap B)' \neq A' \cap B'$. This leads us to ask:

Question. Does substitution of $(A \cap B)' = A' \cap B'$ for $A \cap B \cap P(H) = \emptyset$ in Theorem 2.1 result in a valid theorem?

As one considers possible useful notions, he is likely to consider concepts of minimal or of unique P-lifts. Examples 6 and 7 are ones in which minimal P-lift would be useful. One can define A' to be a minimal P-lift of A if, for any $x \in A'$, $A' - x$ is not a P-lift. In (7), the minimal P-lifts are unique. Monotoneity is the needed property in (6) and (7). In the following example, we have an infinite number of different minimal P-lifts, but only one is monotone. Hence, we see that using an arbitrary minimal P-lift A' will not guarantee that $P|A'$ is monotone.

Example 9. Let the nondegenerate elements in E^3 be those of example (6). Here, for the set A we use two arcs whose union is a letter T, and such that there exists the P-lift shown in the following figure.

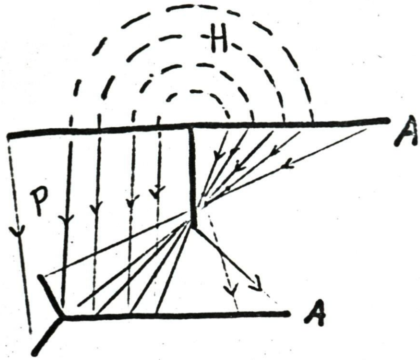


Figure 9A

Note that this P-lift is minimal, but not monotone. Following are two other minimal P-lifts of A. The one shown at the right is the minimal P-lift that is monotone.

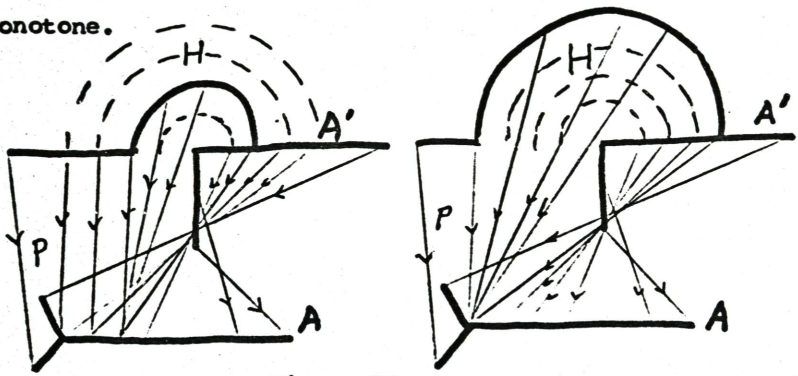


Figure 9B

The concept of unique P-lift does not guarantee

that the sum of two such P-lifts will be the P-lift of the union of the images. We see this from a slight change in Example (3). Now let the nondegenerate elements be the appropriate pairs of points instead of the semicircles. Here, unique P-lift does not give us monotone.

One also might hope that the existence of unique P-lifts for each of A and B would guarantee some niceness on their common boundary. This is not true either, as we see in the following.

<u>Example 10</u>. A' is a disk and B' is an arc, disjoint from A'. The only nondegenerate element is an arc, which intersects the interior of each of A' and B' in a segment of each. A ∪ B does not P-lift.

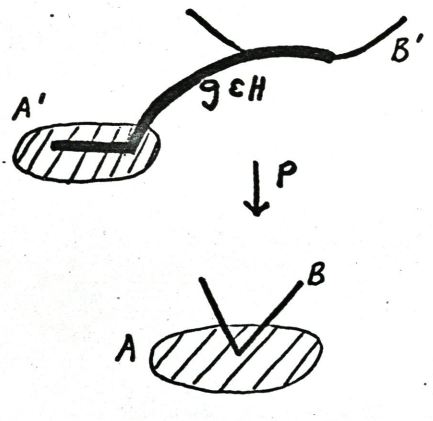


Figure 10

<u>Example 11</u>. In all the theorems and lemmas either $P|A'$ and $P|B'$ are assumed to be monotone or proved to be. Hence, the following example is interesting. In it, the existence of the P-lift depends on the fact that the decomposition A' is <u>not</u> monotone. In this example, we consider a set A and its P-lift A'.

The set A' is contained in E^2. Each nondegenerate element is a circle plus a point disjoint from the circle. There is a bounded, countable set of such elements, as shown. The P-lift A' is the union of these nondegenerate elements and an arc, as shown. Notice that the existence of the P-lift for A depends on the inclusion of the subarc α in the figure. If $P(\alpha)$ is not included in A, we observe that $A - P(\alpha) \not\approx A \approx A' \approx P^{-1}[A - P(\alpha)]$, where $\approx$ denotes homeomorphic. Hence, $A - P(\alpha)$ does not P-lift.

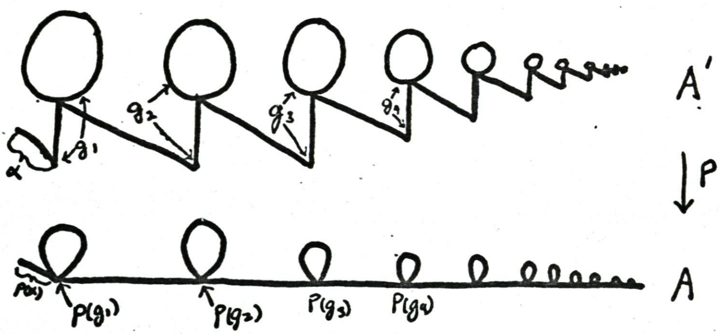


Figure 11

CHAPTER III

DECOMPOSITIONS IN WHICH THERE EXISTS A P-LIFTABLE DISK HOMEOMORPHICALLY CLOSE TO ANY GIVEN DISK

Theorem 3.1. Suppose that E^3/G is homeomorphic to E^3, H is countable, D is a disk in E^3/G, and C is a compact manifold in D. Furthermore, suppose that

(i) $(\text{Bd } C \cup \text{Bd } D) \cap P(H) = \emptyset$,

(ii) D - C has a finite number of components, and

(iii) C P-lifts.

Then, given $\epsilon > 0$, there is a disk D_ϵ^1 such that

(1) D_ϵ^1 is ϵ-homeomorphic to D,

(2) The ϵ-homeomorphism is the identity on C, and

(3) D_ϵ^1 P-lifts.

Proof. In E^3/G we use Bing's Approximation Theorem [7] to approximate D. For each $x \in D$, let f(x) for this theorem be min $\{\epsilon/2, d(x,C)\}$. We get a disk D_a that is $\epsilon/2$ homeomorphic to D and is locally polyhedral on $D_a - C$. We will now show that this implies that there is a locally finite triangulation (possibly infinite) for $D_a - C$. (Hudson [18] proves this, but we include a proof here because we wish to use properties developed for the particular triangulation in this proof.)

Let $\alpha > 0$. Let $A^1 = \text{Cl}(D_a - N_\alpha(C))$. (See Figure 12.) Since $A^1 \subset D_a - C$, for every $x \in A^1$ there exists

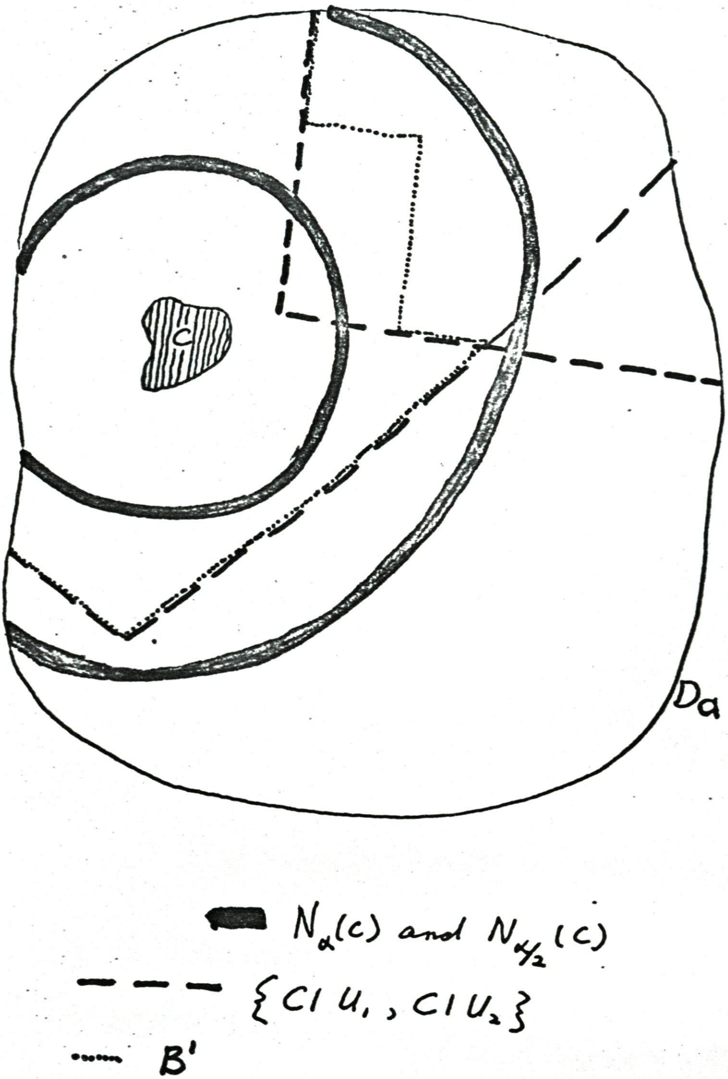


Figure 12

an open set U_x containing x such that Cl U_x is a polyhedral. Since A^1 is compact, the covering $\{U_x \mid x \in A^1\}$ of it has a finite subcovering $\{U_1, U_2, \ldots, U_{n_1}\}$. Let T_i be the triangulation of Cl U_i. The triangulations $T_1, T_2, \ldots, T_{n_1}$ induce a common triangulation T^1 of $\{\text{Cl } U_1, \text{Cl } U_2, \ldots, \text{Cl } U_{n_1}\}$. Hence, there is a complex B^1 such that

(a) $|B^1| \supset A^1$,

(b) $|B^1| \cap N_{\alpha/2}(C) = \emptyset$. We may have to subdivide simplices of T^1 to be able to satisfy this. Next, let $A^2 = \text{Cl}[D_a - N_{\alpha/2}(C)]$. Note that $A^2 \supset |B^1|$. We can find a complex B^2 such that

(a) $|B^2| \supset A^2$,

(b) $|B^2| \cap N_{\alpha/4}(C) = \emptyset$, and

(c) in the complex B^2 the triangulation of $|B^1|$ is a subdivision of B^1. Using Int $|B^1|$ as one set in the covering of A^2 will insure this condition. Let $A^i = \text{Cl } (D_a - N_{\alpha/2^{i-1}}(C))$, $i > 3$. Suppose that B^{i-1} has been found. We obtain a complex B^i such that

(a) $|B^i| \supset A^i$,

(b) $|B^i| \cap N_{\alpha/2^i} = \emptyset$,

(c) in the complex B^i the triangulation of $|B^{i-1}|$ is a subdivision of B^{i-1}, and

(d) in the complex B^i the triangulation of $|B^{i-2}|$ is the same as the triangulation of $|B^{i-2}|$ in the

complex B^{i-1}. We obtain this condition by subdividing only simplices intersecting $A^{i-1} - A^{i-2}$.

(e) No simplex in $|B^i|-|B^{i-1}|$ has diameter greater than $1/2i$. This is not necessary for the present argument; it is for later use.

Let T^i be the triangulation of B^i. Let $T = \{\sigma \mid$ there exists an i such that $\sigma \in T^i$ and $|\sigma| \subset |B^{i-1}|\}$. Observe that, given $p \in D_a - C$, there exists an integer $i > 1$ such that $\alpha/2i-1 < d(p,C)$ and hence, $p \in |B^{i-1}|$. This implies that for every $p \in D_a - C$ there exists a simplex $\sigma \in T$ such that $p \in |\sigma|$. After the ith stage the simplex $\sigma \in B^i$ such that $p \in |\sigma|$ is not subdivided, and after the i + 1 stage, no simplex λ such that $|\sigma| \cap |\lambda| \neq 0$ is subdivided. Therefore, the intersection properties (i.e., if, for any $\sigma_1 \in B^{i+1}$ $|\sigma| \cap |\sigma_1| \neq 0$, then $\sigma \cap \sigma_1$ is a common face) for $\sigma \in B^{i+1}$ are also the intersection properties for $\sigma \in T$. This implies that T is a locally finite triangulation of $D_a - C$.

Let $v_1, v_2, \ldots, v_j, \ldots$ be the vertices of the triangulation T on $D_a - C$. By moving these vertices slightly, we are going to obtain a new complex close to $D_a - C$ and <u>missing P(H)</u>.

First, we adjust v_1. Near v_1 we will choose a new vertex and call it v_1'. This v_1' will be chosen

so that:

(1) $d(v_1, v_1') < \epsilon/2$.

(2) For any $p \in$ segment $[v_1, v_1']$, the cone of p over $\mathrm{Lk}\ v_1$ is a disk that is bounded by $\mathrm{Lk}\ v_1$. Call this disk E_p. (If $v_1 = p$, then $E_p = \mathrm{St}\ v_1$.)

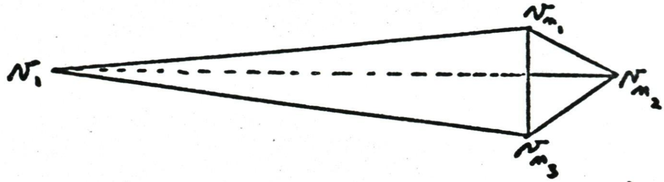
(3) $(D_a - \mathrm{St}\ v_1) \cap E_p = \emptyset$.

(4) $E_{v_1'} \cap P(H) = \emptyset$.

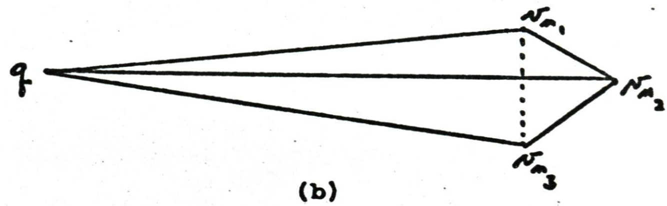
We will find a segment $[v_1, r]$ such that if p is any point in this segment, then p satisfies (1), (2), and (3). Then we will choose $v_1' \in [v_1, r]$ so that it also satisfies (4).

We easily satisfy (1) by requiring that $r \in N_{\epsilon/2}(v_1)$.

Concerning (2), consider Figures 13 and 14. $v_{n_1}, v_{n_2}, \ldots v_{n_k}$ are the vertices in $\mathrm{Lk}\ v_1$. In Figure 13 (a) suppose that v_{n_1}, v_{n_2}, and v_{n_3} lie in the plane of the paper, and v_1 is slightly behind the paper. Figure 13 (b) shows the cone over the same v_{n_1}, v_{n_2}, and v_{n_3} from a point q above the paper. If we were to choose v_1' to be q, (2) would not be satisfied. In figure 14, we assume all vertices lie in a plane. If we were to choose v_1' to be q in this case, (2) would not be satisfied. We postpone stating the condition on r that will assure us that v_1' satisfies (2).



(a)



(b)

Figure 13

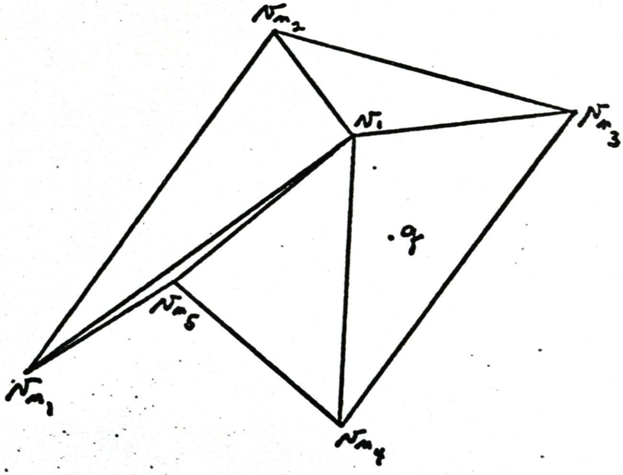


Figure 14

Now consider Figure 15.

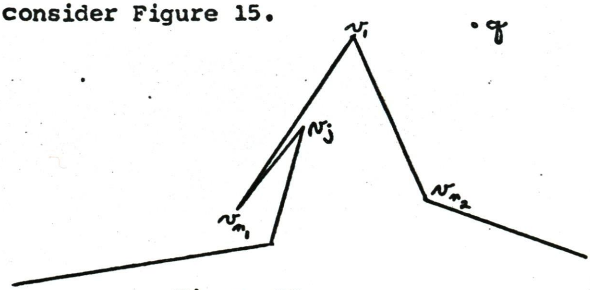


Figure 15

Not all vertices of St v_1 are shown. v_j is not a vertex of St v_1, but it is a vertex of St St v_1. If v_1' were chosen to be q, then condition (3) would be violated.

We now state the condition on r that will assure that v_1' satisfies (2) and does not violate (3) in the above manner. Let V be the set of vertices of St St v_1. V is a finite set, because T is a triangulation. There is a finite set P of planes $\{P_1, P_2, \ldots, P_n\}$ such that each contains 3 vertices of V, and there is a finite set L of lines $\{L_1, L_2, \ldots, L_m\}$ such that each contains 2 vertices of V. Let d =

$$\min \{ \min_{\substack{P_i \in P \\ v_1 \notin P_i}} d(v_1, P_i), \min_{\substack{L_i \in L \\ v_1 \notin L_i}} d(v_1, L_i) \}.$$ Choosing

$r \in N_d(v_1)$ and in general position with respect to the vertices in V assures that (2) will be satisfied and that $(\text{St St } v_1 - \text{St } v_1) \cap E_p = \emptyset$. The last condition is part of (3). For (3) we must also avoid intersections

of E_p with points outside St St v_1. This we do more easily: Let $\tilde{d} = d(\text{St } v_1, D_a - \text{St St } v_1)$. We now require that $r \in N_{\tilde{d}}(v_1)$.

We now know that if we choose v_1^* to be a point in $[v_1, r]$, it will satisfy (1), (2), and (3). Since r is in general position with respect to the points in the set V, for any $P_1, P_2 \in [v_1, r]$, it must be true that $\langle v_{n_i}, v_{n_j}, p_1\rangle \cap \langle v_{n_i}, v_{n_j}, p_2\rangle = \langle v_{n_i}, v_{n_j}\rangle$ for any simplex $\langle v_{n_i}, v_{n_j}\rangle$ in Lk v_1. (See Figure 16) Only countably many of the disjoint sets $S_p = |\langle v_{n_i}, v_{n_j}, p\rangle| - |\langle v_{n_i}, v_{n_j}\rangle|$ can contain points of the countable set P(H). There are uncountably many points p in $[v_1, r]$. Therefore, there exist uncountably many points $p_i \in [v_1, r]$ such that $S_{p_i} \cap P(H) = \emptyset$. Since there are only a finite number of vertices in Lk v_1, there exists $v_1^* \in [v_1, r]$ such that (4) is satisfied.

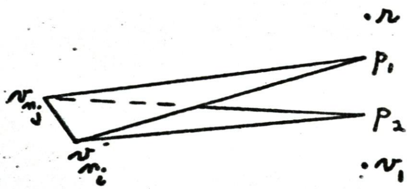


Figure 16

In the complex $D_a - C$, we now substitute $E_{v_1^*}$ for St v_1. Since we have condition (3) satisfied and the two disks $E_{v_1^*}$ and St v_1 are $\epsilon/2$-homeomorphic under a

homeomorphism that can be taken to be the identity on their common boundary, D_a is $\epsilon/2$-homeomorphic to the new set $D_a - \mathrm{St}\, v_1 + E_{v_1'}$. This set $D_a - \mathrm{St}\, v_1 + E_{v_1'}$ is a disk; call it D_{a1}. $D_{a1} \supset C$ because $C \cap \mathrm{St}\, v_1 = \emptyset$. $D_{a1} - C$ is a complex with vertices $v_1', v_2, \ldots, v_j, \ldots$. Let us now adjust v_2 to v_2'. For (1) now use $d(v_2, v_2') < \epsilon/4$. We get a disk D_{a2} homeomorphic to D_{a1}. (We postpone discussion of the closeness of this homeomorphism.) Now, we repeat the argument starting with the statement that $D_{a2} - C$ is a complex with vertices $v_1', v_2', v_3, \ldots, v_j, \ldots$ . ... The inductive step is: Suppose that $D_{a(j-1)} - C$ is a complex with vertices $v_1', v_2', \ldots, v_{j-1}', v_j, \ldots$ Adjust v_j to v_j'. For (1) use $d(v_j, v_j') < \epsilon/2^{j+1}$. ... We can now define $D_\epsilon^1 = C \cup \lim_{j\to\infty} (D_{aj} - C)$. We will show that:

(i) $D_\epsilon^1 - C$ is polyhedral.

(ii) $D_\epsilon^1 - C$ misses $P(H)$.

(iii) D_ϵ^1 is $\epsilon/2$-homeomorphic to D_a by a homeomorphism which is the identity on C.

To know that $D_\epsilon^1 - C$ is polyhedral, we must know that there is a set of simplices such that every point of $D_\epsilon^1 - C$ lies in the underlying space of some simplex and that the simplices have the intersection properties of a triangulation. Our proof here is analogous to the earlier proof that T is a triangulation. In that

proof we noted that, given a simplex, there is a stage after which this simplex is not changed. Here, there is also a stage after which a simplex is not changed again.

Suppose that $\langle v_i', v_j', v_k' \rangle$ is a simplex in $D_\epsilon^1 - C$. Suppose that $k > j > i$. Then we know that v_i' was chosen so that it is not a point of $P(H)$, that v_j' was chosen so that the segment $[v_j', v_i']$ misses $P(H)$, and that v_k' was chosen so that $|\langle v_i', v_j', v_k' \rangle| - [v_j', v_i']$ misses $P(H)$. Therefore, $D_\epsilon^1 - C$ misses $P(H)$.

The argument above showing that D_a is $e/2$-homeomorphic to D_{a1} can now be used to show that D_{aj} is $\epsilon/2^{j+1}$-homeomorphic to $D_{a(j-1)}$. Let us call this homeomorphism $f_j : D_{a(j-1)} \to D_{aj}$. Let $h_j = f_j \circ f_{j-1} \circ \cdots \circ f_2 \circ f_1$. Then $h_j : D_a \to D_{aj}$ and h_j is a homeomorphism. Let $h = \lim_{j \to \infty} h_j$. We claim that h is an $\epsilon/2$-homeomorphism of D_a onto D_ϵ^1.

That h is 1-1 is immediate. The definitions of h and D_a imply that h is onto D_ϵ^1.

Next, we check the continuity of h. Suppose $x \in C$. Since this implies that $f_j(x) = x$ for all j, we have that $h(x) = x$. Hence, h is the identity on C and h is continuous on $\mathrm{Int}_D\, C$. Consider next

$x \in D_a - C$. This implies that x has a closed neighborhood N missing C. In the notation of the beginning of the proof, there is an i_o such that $x \in N \subset B^{i_o-2}$. Since B^{i_o} has only a finite number of vertices, there is an integer m such that $h_m(x) = h_{m+j}(x)$ for all $j > 0$, $x \in N$. Hence, $h(x) = h_m(x)$ for all $x \in N$. Since h_m is a homeomorphism on D_a, h is continuous in Int N. This gives continuity of h on $D_a - C$. We now check continuity of h on $Bd_D\, C$. Given $x \in Bd_D\, C$ and $\gamma > 0$, we will show that there exists $\delta > 0$ such that for any y satisfying $d(x, y) < \delta$, it is true that $d(h(x), h(y)) < \gamma$. We note that condition (e) stated before the definition of the triangulation T implies that if x is in $Bd_D C$, then x is a limit point of the set of vertices of T. Since, by definition f_j does not move any point more than $\epsilon/2^{j+1}$ we can choose j such that $\epsilon/2^{j+1} < \gamma/3$. Observe that, if $x \in v_i, v_j, v_k$ then $h(x) \in v_i', v_j', v_k'$, and this implies, since each homeomorphism is piecewise linear, that

$d(x, h(x)) \leq \min \{d(v_i, v_i'), d(v_j, v_j'), d(v_k, v_k')\}$.

Choose a neighborhood $\hat{N}$ of $x \in Bd\ C$ such that $Cl\ \hat{N} \cap \{v_1, v_2, \ldots, v_j\} = \emptyset$. Then f_{j-1} is the identity on $\hat{N}$ and $d(x, h(x)) < \epsilon/2^{j+1}$ for $x \in \hat{N}$. Hence, if we choose $y \in \hat{N} \cap N_{\gamma/3}(x)$, then $d(y, h(y)) < \gamma/3$. Now

$$d(h(x), h(y)) \leq d(h(x), x) + d(x, y) + d(y, h(y))$$
$$< \gamma/3 \quad + \quad \gamma/3 \quad + \quad \gamma/3 \quad = \gamma$$

Therefore, we choose $\delta > 0$ such that $N_\delta(x) \subset \mathring{N} \cap N_{\gamma/3}(x)$. This implies that h is continuous for $x \in Bd_D C$.

Finally, to know that h is a homeomorphism, we must still check that h^{-1} is continuous. The arguments are similar to the above ones. If $h^{-1}(x) \in Int_D C$, we have continuity because h^{-1} is the identity. If $h(x) \in D_1 - C$, we find a neighborhood of h(x) such that h^{-1} is the inverse of the composition of a finite number of homeomorphisms in this neighborhood. If $h(x) \in Bd_D C$ we again use an "$\epsilon-\delta$" argument. Condition (e) again makes it possible to find a neighborhood, each of whose points is moved less than 1/3 of the given "ϵ". We will not carry through the details, since they are very much like the above argument.

We now have shown that h is a homeomorphism on D and on Cl(D - C). That it is an $\epsilon/2$-homeomorphism follows from the observation that if no vertex moves more than $\epsilon/2$, then no point does. This completes the proof that D_a is $\epsilon/2$-homeomorphic to D_ϵ^1, under a homeomorphism that does not move any point of C.

We will use Theorem 2.2 to show that D_ϵ^1 P-lifts. C and Cl $(D_\epsilon^1 - C)$ are the sets A and B, respectively, of Theorem 2.2. It is given that C is a compact

manifold. Since C is compact and D - C has a finite number of components, Fr_D C has a finite number of components. Each component of Bd C is a simple closed curve. (This is shown in Lemma 2.3.) Hence, Bd (D - C) has a finite number of components, each of which is a simple closed curve. This plus the fact that Cl (D - C) is contained in a disk implies that it is itself a compact manifold. Since h is a homeomorphism of Cl (D - C), we now have that Cl $(D_\epsilon^1 - C)$ is a compact manifold. C P-lifts by hypothesis. Since $[Cl(D_\epsilon^1 - C)] \cap P(H) = \emptyset$, the set $Cl(D_\epsilon^1 - C)$ lifts. Hence, it certainly P-lifts. $Bd(D_\epsilon^1 - C) = Bd\ C \cup Bd\ D$. Therefore, using hypothesis (i), we have that

$[Bd\ C \cup Bd(Cl(D_\epsilon^1 - C))] \cap P(H) = \emptyset$.

Hypothesis (ii) implies that C, $Cl(D_\epsilon^1 - C)$ and their complements each have only a finite number of components. Since H is countable, P(H) is 0-dimensional. We have satisfied the hypotheses of Theorem 2.2, so it gives us the conclusion that D_ϵ^1 P-lifts. ▯

<u>Definition</u>. (Armentrout [2], [4]) $M_1, M_2, M_3, \cdots$ is a <u>defining sequence</u> for a decomposition G if and only if $M_1, M_2, M_3, \ldots$ is a sequence of compact 3-manifolds-with-boundary in E^3 such that:

(1) For each positive integer i, $M_{i+1} \subset \text{Int } M_i$, and

(2) g is a non-degenerate element of G if and only if g is a non-degenerate component of $\bigcap_{i=1}^{\infty} M_i$.

Suppose that $g \in H$ and U is an open neighborhood containing g. We will show that then, for some i_o and j_o, there is a manifold $B_{i_o j_o}$ that is the j_oth component of the manifold M_{i_o} of the defining sequence and satisfies $g \subset B_{i_o j_o} \subset U$. Suppose not. The facts that g is contained in $\bigcap_{i=1}^{\infty} M_i$ and g is connected imply that for every i there is a j_i such that $g \subset B_{ij_i}$. If for no i, $B_{ij_i} \subset U$, then for each i there is a point $x_i \in B_{ij_i} - U$. Since the sequence $\{x_i\}_{i=k}^{\infty}$ is contained in the compact set M_k, it must converge to some point $x \in M_k$. Since x_i is an element of the manifold B_{ij} containing g, there is an arc $\alpha_i \subset B_{ij}$ such that $x_i \cap \alpha_i \neq \emptyset$ and $\alpha_i \cap g \neq \emptyset$. The sequence $\{\alpha_i\}_{i=1}^{\infty}$ has a convergent subsequence. Call it again $\{\alpha_k\}$. Its limit α is such that $x \in \alpha$ and $\alpha \cap g \neq \emptyset$. Since g is connected, $x \in g$. Therefore, $x \in U$. Since U is open, there must be some i_1 such that $x_{i_1} \in U$. This is a contradiction of the definition of x_{i_1}. Therefore, there exist i_o and j_o such that $B_{i_o j_o} \subset U$. Notice that since $H^* \cap \mathrm{Bd}\, B_{i_o j_o} = \emptyset$ this also implies that $P(H)$ is 0-dimensional when H has a defining sequence.

Theorem 3.2. Suppose that H is definable by 3-cells, D is a disk in E^3/G, and C is a compact manifold in D such that $(\mathrm{Bd}\ C) \cap P(H) = \emptyset$ and C P-lifts. Then, given $e > 0$, there is a disk D^1_ϵ such that

(1) D^1_ϵ is ϵ-homeomorphic to D,

(2) The ϵ-homeomorphism is the identity on C, and

(3) D^1_ϵ P-lifts.

Proof. Near D - C we will find a set homeomorphic to D - C and contained in the nonsingular points. It will lift. We will use Theorem 2.2 to add this set to C to get the desired set D^1_ϵ.

We first show that the assumption that H is definable by 3-cells implies that E^3/G is homeomorphic to E^3. We will use a result of O. G. Harrold [15]: If G is a monotone decomposition of S^3 such that each point of Cl P(H) has arbitrarily small neighborhoods in S^3/G with boundaries that are 2-spheres disjoint from Cl P(H), then S^3/G is a 3-sphere. We let P(g) be a point in Cl P(H), and W be an arbitrarily small open set containing P(g). $P^{-1}(W)$ is then an open set containing g. Therefore, there is some component B_{j_g} of some defining manifold such that $g \subset B_{j_g} \subset P^{-1}(W)$. But now the fact that H is definable by 3-cells implies that Bd B_{j_g} is a 2-sphere. Since Bd B_{j_g} misses Cl H^*, P(Bd B_{j_g}) is a 2-sphere that misses Cl P(H) and satisfies

Harrold's hypothesis. Hence, his conclusion gives us that E^3/G is homeomorphic to E^3.

H_{D-C} denotes the subset of H that projects into D - C. Let $\mathcal{U}$ be an open cover of Cl $P(H_{D-C})$ such that for each $P(g) \in$ Cl $P(H_{D-C})$, there is a $U_g \in \mathcal{U}$ satisfying $U_g \subset N_{\epsilon/2}(P(g))$ and $C \cap \text{Cl } U_g = \emptyset$. Then, for each $U_g \in U$, $P^{-1}(U_g)$ is an open set and contains g. There is a component B_g of some defining manifold M_{i_g} such that $g \subset G_g \subset P^{-1}(U)$. $\{\text{Int } B_g \quad g \in \text{Cl } H_{D-C}\}$ is an open cover of Cl H_{D^*C}. There is a finite subcover because Cl H_{D^*C} is compact. (It is contained in the first defining manifold, which is bounded.) Suppose that $\{\text{Int } B_{g_1}, \text{Int } B_{g_2}, \ldots, \text{Int } B_{g_k}\}$ is a minimal finite subcover of Cl H_{D^*C}. Then $B = \{B_{g_1}, B_{g_2}, \ldots, B_{g_k}\}$ is a set of 3-cells. Each is a component of some defining manifold. Hence, if $i \neq j$, then either B_{g_i} and B_{g_j} are disjoint or one contains the other. In the latter case, the subcovering was not minimal. Therefore, B is a set of disjoint 3-cells. Since Bd B_j is a 2-sphere, $P(\text{Bd } B_j)$ is also a 2-sphere. With the hypothesis that E^3/G is homeomorphic to E^3, we now have that $P(\text{Bd } B_j)$ bounds a 3-cell, namely, $P(B_j)$. Hence, $\{P(B_j) \mid j = 1, \ldots, k\}$ is a set of disjoint 3-cells that cover Cl $P(H_{D-C})$ and lie in U. It is in the union of these 3-cells and D that we will find our

new disk, D_ϵ^1.

Consider one 3-cell $A_j \in \{P(B_j) \mid j=1,\ldots,k\}$. Since $A_j \cap \mathrm{Cl}\, P(H) \subset \mathrm{Int}\, A_j$, there is a polyhedral 3-cell A_j' such that $A_j' \subset \mathrm{Int}\, A_j$ and $A_j \cap \mathrm{Cl}\, P(H) \subset A_j'$. Arbitrarily choose a point $x \in (\mathrm{Int}\, A_j') - D$. Let $d = d(x, D \cup \mathrm{Bd}\, A_j)$. The set $\mathrm{Cl}\, N_{d/2}(x)$ is a tame 3-cell inside the tame 3-cell A_j'. Hence, there is a homeomorphism h_j: $E^3/G \to E^3/G$ such that $h_j(A_j') = \mathrm{Cl}\, N_{d/2}(x)$ and, for any $y \in E^3/G - \mathrm{Int}\, A_j$, $h_j(y) = y$. Notice that for each $z \in P(H) \cap A_j$, $D \cap h_j(z) = \emptyset$. Therefore, $(h_j^{-1}(D)) \cap z = \emptyset$. Of course, h_j^{-1} is the identity on $D - A_j$. It is an ϵ-homeomorphism, since the diameter of A_j is less than ϵ.

Let $h = h_k \circ h_{k-1} \circ \cdots \circ h_1$. It is an ϵ-homeomorphism. Since $P(H_{D-C}) \subset \bigcup_j A_j$, we now have that $\mathrm{Cl}(h^{-1}(D - C) \cap P(H) = \emptyset$ and hence, that $\mathrm{Cl}\, h^{-1}(D - C)$ lifts. By our choice of the cover $\mathcal{U}$, $\mathrm{Int}\, h^{-1}(D - C)$ does not intersect C and h^{-1} is the identity on $\mathrm{Bd}\, C$. Therefore, $C \cup \mathrm{Cl}\, h^{-1}(D - C)$ is a disk. We will now use Theorem 2.2 to show that $D_\epsilon^1 = C \cup \mathrm{Cl}\, h^{-1}(D - C)$ P-lifts. We note that D being a disk and C a compact manifold imply that $\mathrm{Cl}\,(D - C)$ is also a compact manifold. Our decomposition is 0-dimensional because it is definable by manifolds. We have satisfied all hypotheses of Theorem 2.2 and, hence, it gives us the conclusion

that D^1_ϵ P-lifts. $\square$

Theorem 3.3. Suppose that H is monotone, Cl P(H) is 0-dimensional, and that Cl H* is 1-dimensional. Let $M \subset E^3/G$ be a compact 2-manifold with or without boundary and let the collection of nondegenerate elements that project into M be continuous. Suppose that, for any $\epsilon > 0$, there is a manifold M^1_ϵ that P-lifts, is ϵ-homeomorphic to M, and satisfies the containment: $M^1_\epsilon \cap P(H) \subset M \cap P(H)$. Then we conclude that, for any $\epsilon > 0$, there is another manifold M_ϵ that P-lifts, is ϵ-homeomorphic to M, and also satisfies the equality: $M_\epsilon \cap P(H) = M \cap P(H)$.

Proof. Let $\epsilon > 0$ be given. Then for $\epsilon/2$ there exists a manifold $M^1_{\epsilon/2}$ with the hypothesized properties. For simplicity of notation, let us call this manifold K_o. Since K_o is $\epsilon/2$-homeomorphic to M, let $h: M \to K_o$ be the $\epsilon/2$-homeomorphism. For each $p \in M \cap P(H)$, let $B_p = N_{\epsilon/2}(p)$. Note that $h(p) \in B_p$. In E^3 we plan to construct a new manifold M'_ϵ in $K'_o \cup P^{-1}(\bigcup_p B_p)$. From K'_o we will make fingers that reach out and pick up points that we wish to add in order to satisfy the equality in the conclusion of the theorem. Except their end points, fingers will be entirely in the degenerate points. Each finger will be made by using

a disk in the boundary of a thickened arc to replace a disk of degenerate points. When we add a finger, we will have a homeomorphism between the old and new manifolds. We will describe a construction such that we will be able to prove that the composition of an infinite sequence of the homeomorphisms is itself an $\epsilon/2$-homeomorphism. Then this $\epsilon/2$-homeomorphism composed with the $\epsilon/2$-homeomorphism h will give us our desired ϵ-homeomorphism.

First we show that arcs for the finger construction exist. At the (i + 1)th stage we will be working in an open connected neighborhood U_{i+1} that intersects both K_i' and $P^{-1}(M \cap P(H))$, where K_i' is the manifold made at the ith stage. Note that since K_i' is a compact 2-manifold and U_{i+1} is open in E^3, $K_i' \cap U_{i+1}$ is 2-dimensional. We want an arc α from a degenerate point in K_i' to a point in $P^{-1}(M \cap P(H))$. α must satisfy: Int $\alpha \subset U_{i+1} - K_i' - \text{Cl } H^*$. By hypothesis Cl H^* is 1-dimensional. U_{i+1} is an open connected subset of E^3 and therefore, no 1-dimensional set can separate it. (Hurewicz and Wallman p. 48 [19]) Hence, $U_{i+1} - \text{Cl } H^*$ is connected. Since $U_{i+1} - \text{Cl } H^*$ is again an open connected subset of E^3, if it is separated by K_i', each subset of K_i' separating $U_{i+1} - \text{Cl } H^*$ must be a 2-dimensional subset. (We have again used the statement

from Hurewicz and Wallman.) Hence, each component of $U_{i+1} - K_i' - \mathrm{Cl}\ H^*$ contains a 2-dimensional subset of K_i' in its boundary. We now cite a theorem in Hocking and Young [17] that states that in a locally connected and locally arcwise-connected space S, the set of all points on the boundary of an open set U that are accessible from U is dense in the boundary of U. E^3 is a space satisfying these hypotheses. Hence, accessible points in the boundary of $U_{i+1} - K_i' - \mathrm{Cl}\ H^*$ are dense in the boundary. Since Cl H^* is 1-dimensional, it is not dense in any 2-dimensional subset of K_i', so there must be arcwise accessible points in $K_i' - \mathrm{Cl}\ H^*$.

We said that we wanted an arc to a point in $P^{-1}(M \cap P(H)) - K_i'$. We will settle for a bit less--we will only require that the arc end at a points x close to $P^{-1}(M \cap P(H)) - K_i'$, say within $1/2^{(i+5)}$. Use $x \in [(\mathrm{Cl}\ U_i) \cap N_{1/2^{(i+5)}}(M \cap P(H))] - K_i'$, where x is a point in the arcwise connected interior of the set or x is an arcwise accessible point of $(\mathrm{Bd}\ U_i) - K_i'$. (The existence of our final construction, since it will pick up a point in each $g \subset P^{-1}(M \cap P(H))$, will show that in each g some point is actually accessible. We choose now to settle for the arc above rather than proving the accessibility of the desired points.)

We now have an arc α from x to an arcwise acces-

sible point in $K_i' - \mathrm{Cl}\, H^*$. (See Figure 17.) There is a disk D in $K_i' - \mathrm{Cl}\, H^*$ such that $\alpha \cap K_i'$ is a point, s, in Int D. (Later we shall specify how small this disk should be.) Now, by covering each point of α by an appropriately small open neighborhood, we can fatten α to obtain an open ball A satisfying:

(1) $(K_i' - D) \cap \mathrm{Cl}\, A = \emptyset$,

(2) $\mathrm{Int}\, \alpha \subset \mathrm{Int}\, A$,

(3) $\mathrm{Bd}\, A \cap H^* = x$ or $\emptyset$, and

(4) $D \subset \mathrm{Bd}\, A$.

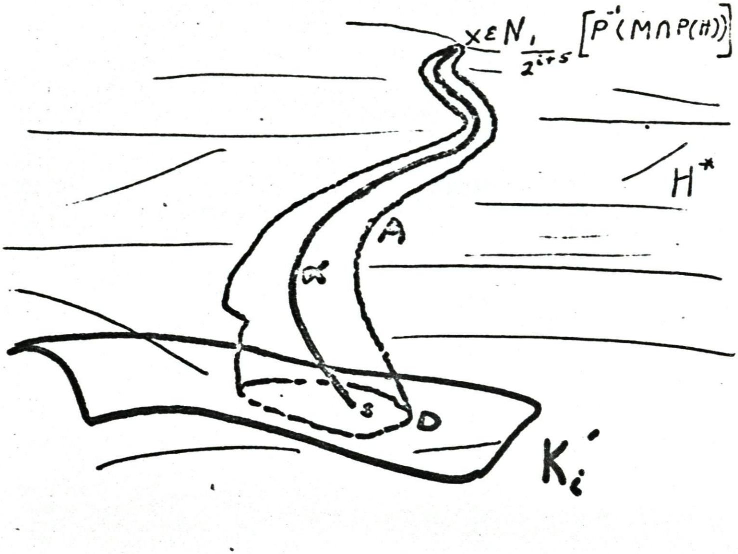


Figure 17

We use Cl ((Bd A) - D) to replace the disk D. This is the finger. We call D its base and x its tip. We

obtain a new manifold K_{i+1} that is β-homeomorphic to K_i for some β depending on the diameter of $P(U_{i+1})$. Let $h_{i+1} : K_i \rightarrow K_{i+1}$.

Our description so far has assumed that one finger is being added at the ith stage. We shall actually add a finite number of disjoint fingers at the ith stage. Each finger at the ith stage will have its base in exactly one finger of the i-1 stage. Thus, as stages progress, a finger will grow out in a tree-like fashion. Each finger will have a "territory" in which it is permitted to grow. New fingers growing out of a parent finger, which was formed at the ith stage, will have their tips $1/2^{i+2}$ dense in a particular subset S of $P^{-1}(M \cap P(H)) - K_i'$ that lies in the territory of the parent. (Later we will specify the subset S.) The base of a finger will lie in a set V such that $(h_{i-1}\, h_{i-2} \cdots h_2\, h_1)^{-1}(V)$ has diameter less than $1/2^{(i+1)}$. The set diameter of the new finger itself will be less than $1/2^{(i+1)}$.

Next, we define the territory of a new finger. Let W_i be the territory of the parent finger. Let the tips of the new fingers being formed at the (i+1)th stage be the points $x^1, x^2, \ldots, x^n$. These points are $1/2^{(i+2)}$ dense in S. From our restriction

of set diameter of a finger, the territory T^j of the j^{th} of these new fingers is contained in $U^j = W_i \cap N_{1/2^{(i+1)}}(x^j)$. If $j_1 \neq j_2$, it is quite possible for $U^{j_1} \cap U^{j_2} \neq \emptyset$, or for there to exist $g \in H$ such that $g \cap U^{j_1} \neq \emptyset$ and $g \cap U^{j_2} \neq \emptyset$. We must make the territories T^{j_1} and T^{j_2} be separated and satisfy $P(T^{j_1}) \cap P(T^{j_2}) = \emptyset$. Remember that our purpose is to be able to pick up a point in every nondegenerate element of the set S. This means that $\bigcup_{j=1}^{n} P(T^j)$ must cover $P(S)$. $P(S)$ is a 0-dimensional set and is covered by $\{P(U^j)\}$, but the later is not an open cover.

At the moment let us work only in a subset of $Cl\ H^*$ and a subset of $P(Cl\ H)$. We have hypothesized that H_M, the set of nondegenerate elements that project into M, is continuous. For each j, the lower semicontinuity of H_M implies that from the open with respect to E^3 set U^j we get the open with respect to $Cl\ H_M{}^*$ set $V^j = \{x \in g \mid g \cap U^j \neq \emptyset,\ g \in Cl\ H_M\}$. V^j is an inverse set, so $P(V^j)$ must be open with respect to $P(Cl\ H_M)$. (Notice that $P(V^j) = P(U^j) \cap P(Cl\ H_M)$.) $\{P(V^j)\}$ is an open with respect to $P(Cl\ H_M)$ cover of the desired set of points $P(S)$. Now using the 0-dimensionality of $P(S)$, we can get a refining cover $\{X^j\}$ of disjoint sets such that $X^j \subset P(Vj)$, X^j is open and closed with respect to $P(Cl\ H_M)$. (If for some j, the finger tip $x^j \not\subset Cl\ H_M^*$,

then it is possible that $X^j = \emptyset$. In this case, the corresponding finger will have no territory and never grow more. Delete it.) $P^{-1}(X^j)$ is closed with respect to Cl H_M^*. Returning now to the space E^3, we see that this implies that $P^{-1}(X^j)$ is closed with respect to E^3. The finite number of closed sets in $\{P^{-1}(X^j)\}$ can be covered by disjoint open with respect to E^3 sets W^j. Now the sets $U^j \cap W^j$ are almost the desired territory sets. They do contain all necessary points for adding, they are separated and open with respect to E^3, and each still contains its original finger tip x^j. It is possible that $U^j \cap W^j$ is not connected or does not intersect K_i'. These shortcomings can be corrected by adding tubular neighborhoods β of connected arcs in U^j - Cl H^*. These may have to lie in tunnels, γ, bored through some other $U^{j_1} \cap W^{j_1}$, but the 0-dimensionality of P(Cl H) allows this. We now have the sets $(U^j \cap W^j) - \gamma + \beta$ for appropriate sets γ and β. For each j this is the territory T^j. We have now defined the construction so that we will be able to prove that the ultimate set K_∞' is a manifold homeomorphic to K_0'.

We must also be able to show that the image $P(K_\infty')$ is a manifold and is $\epsilon/2$-homeomorphic to $P(K_0)$. The closeness is cared for by starting the construction with an i large enough that $1/2^{(i+1)}$ is less than the

Lebesgue number of some particular finite open covering of H_M* by sets of the form $P^{-1}(B_p)$, where B_p is the set defined in the first paragraph of this proof. Concerning the other condition, we would know that $P(K_\infty^!)$ is a manifold if we knew that no more than one point in any g is added. For this, we want two things. Suppose $x \in g_o$ is a finger tip. Then we need: (1) No point $y \in g_o$ is also a finger tip, and (2) The points $\{x_k\}$ from the sets $\{g_k\}$, where $\{g_k\} \to g_o$, must satisfy $\{x_k\} \to x$. Both these we will satisfy be deleting from the set $P^{-1}(M \cap P(\mathrm{Cl}\ H)) - K_i^!$ the set $\{g - [g \cap N_{2d_g}(x)] \mid g \in H \text{ and } d_g = d(g,x)\}$. (See FIgure 18.) The remaining subset is the set S that we have been using. It is $\{g \cap N_{2d_g}(x) \mid g \in H$ and and $d_g = d(g, x)\}$. Deletion is done only within the territory of finger tip x. Continuity of $\mathrm{Cl}\ H_M*$ assures us that for any sequence $\{g_k\} \to g_o$ there exists a sequence $\{y_k\} \to x$, where $y_k \in g_k$. Hence, this deletion does assure that (2) is satisfied. Notice that this

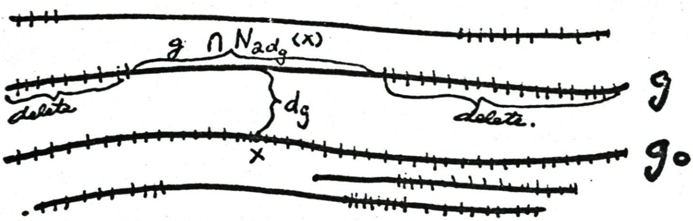


Figure 18

deletion of points from Cl H_M^* does not alter the continuity of the remaining subset.

We now prove that the composition $(\ldots h_3\, h_2\, h_1) = f : K_o' \to K_\infty'$ is a homeomorphism. Consider first the continuity of f. We can rewrite f as $f^{i+1}f_i$ where $f^{i+1} = \ldots h_{i+2}h_{i+1}$ and $f_i = h_i h_{i-1}\ldots h_1$. Let $\epsilon_1 > 0$ be given. There exists an integer k such that $1/_{2^k} < \epsilon_1/4$. Then

$\sum_{i=k}^{\infty} \frac{1}{2^{i+1}} < \frac{\epsilon_1}{4}$. Now, $1/_{2^i}$ is the maximum diameter of the territory for forming fingers at the ith stage. Since the infinite composition f^{i+1} never moves a point out of a territory it is once in, this summation shows that $d(p, f^{k+1}(p)) < \epsilon_1/4$ for any $p \in f_k(K_o')$. Rewritten, $d(f_k(x), f^{k+1}f_k(x)) < \epsilon_1/4$ for any $x \in K_o'$. On the other hand, consider f_k. It is a homeomorphism because it is the composition of a finite number of homeomorphisms. Hence, given $\epsilon_1/2$, there is a $\delta > 0$ such that if $d(x,y) < \delta$ then $d(f_k(x), f_k(y)) < \epsilon_1/2$. Putting the above statements together, we find that if $x, y \in K_o'$ such that $d(x, y) < \delta$, then $d(f(x), f(y)) = d(f^{k+1}f_k(x), f^{k+1}f_k(y)) \leq d(f^{k+1}f_k(x), f_k(x)) + d(f^{k+1}f_k(y), f_k(y)) + d(f_k(x), f_k(y)) < \epsilon_1/4 + \epsilon_1/4 + \epsilon_1/2 = \epsilon_1/2$. Hence, f is continuous.

That f is 1-1 follows from the use of territories for growth. Territories are separated sets. This ensures us that limit points of different territories are distinct.

We now show the continuity of $f^{-1}: K'_\infty \to K'_0$. Let $\epsilon_2 > 0$ be given and $x \in K'_\infty$. There are three cases:

(1) $x \notin \text{Cl } H^*$,

(2) $x \in \text{Cl } H^*$ and $x \in$ only finitely many fingers, and

(3) $x \in \text{Cl } H^*$ and $x \in$ infinitely many fingers.

Case (1): Since x is a point and Cl H* is closed, $d(x, \text{Cl } H^*) > 0$. There is an i such that $d(x, \text{Cl } H^*) > 1/2^{i-2}$. At the ith stage, each finger tip lies in $N_{1/2^{i-1}}(\text{Cl } H^*)$ and the territories of finger tips have diameter $< 1/2^i$. Therefore, x does not lie in the territory of any finger tip after stage i-1. Therefore, $h_i, h_{i+1}, \ldots$. are the identity. $h_{i-1} h_{i-2} \ldots h_2 h_1$ is a homeomorphism. We now use $\gamma = \min(d(x, \text{Cl } H^*), \epsilon_2)$ to determine δ such that if $d(x, y) < \delta$, where $x, y \in K'_1$, then $d((h_{i-1} \ldots h_1)^{-1}(x), (h_{i-1} \ldots h_1)^{-1}(y)) < \gamma$. This allows us now to state that if $x, y \in K'_\infty$ such that $d(x, y) < \delta$, then $d(f^{-1}(x), f^{-1}(y)) < \gamma \le \epsilon_2$. We have continuity for case (1).

Case (2) is very similar. If x is an element of a sequence of i fingers, we again have the identity for stages after i. In the argument for case (1) in place of d(x, Cl H*) use the distance between x and the territory of the nearest finger at the i+1 stage.

Case (3): For the given point x there is an infinite sequence F_1, F_2, F_3,... of fingers--one at each stage--containing x. Given $\epsilon > 0$, choose an integer i such that $1/2^{i-1} < \epsilon$. Let the finger F_i at the ith stage have the territory T_i. This territory is by definition an open set. Therefore, $d(x, \text{Bd } T_i) > 0$. We now use $\delta = d(x, \text{Bd } T_i)$. If $d(x, y) < \delta$ for $x, y \in K_\infty^*$, then $y \in F_i$ and therefore y came from the same finger base as x did at the i-1 stage. This base by the construction comes from a set in K_0^* of diameter $< 1/2^{i-1} = \epsilon$. Therefore $d(f^{-1}(x), f^{-1}(y)) < \epsilon$. We have continuity in case (3). f now is a homeomorphism and we have completed the proof of the theorem. $\square$

Remark 1. In Theorem 3.3 it may be necessary to hypothesize that the collection of nondegenerate elements that project into M is continuous. Without this hypothesis, I conjecture that the following is a counterexample.

We again base an example on a 2-disk whose preimage is not a manifold on one nondegenerate element. (Compare with Example (2), p. 27)

For our example we use a 2-complex C in E^3. It is shown in Figure 19. The line segment ab is a

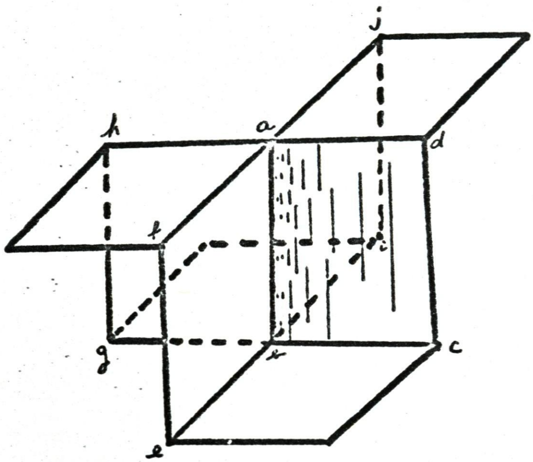


Figure 19

subset of the points in a nondegenerate element g_o. Each of the squares abcd, abef, abgh, and abij has nondegenerate vertical segments in its interior, as shown in Figure 20. There are a countable number of these segments, whose length goes to zero as the distance from ab goes to zero. Limit points of the set of segments in each square are dense in ab. Figure 21 shows the element g_o. It is a dendron which has an infinite number of branch points dense in a vertical

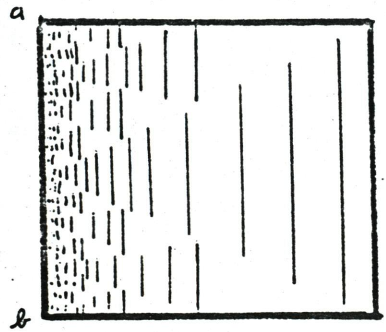


Figure 20

segment s containing a and b in its interior. The points a and b are not branch points. At each branch point there are an infinite number of segments intersecting s and lying in a horizontal plane.

The image P(C) = M is a disk because the decomposition takes segment ab to a point and each square to a triangle. In the image space Cl P(H) is 0-dimensional because H* = Cl H* and H is countable. In the

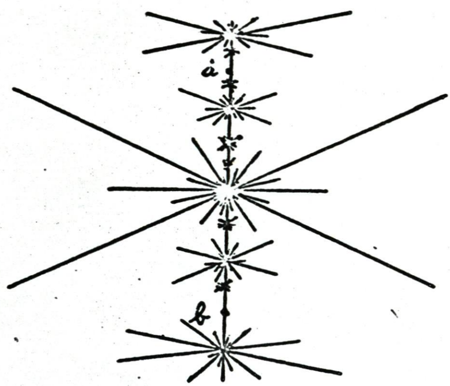


Figure 21

domain Cl H^* is 1-dimensional because it is the countable sum of closed 1-dimensional sets. The set of nondegenerate elements that project into M is the whole set H and it is not continuous on g_o. By Theorem 3, there is a P-lift M_ϵ^{1}' satisfying the hypotheses of Theorem 2.3.

Now suppose that there exists M_ϵ satisfying the equality in the conclusion of Theorem 2.3. Certainly $ab \subset M_\epsilon'$, because a dense set of points of ab are in M_ϵ' and it is closed. Let p be any branch point in the interior of ab. I conjecture that there do not exist a neighborhood N of p and a disk D containing p in its interior such that for $g \in H$, $g \cap N \neq \emptyset$ implies that $g \cap D \neq \emptyset$ and such that P(D) is a disk. The following makes this conjecture seem plausible. Any such disk D must contain a point from each $g \cap N$ that lies in the intersection of a square with N. Since D is a disk, it certainly can't contain a neighborhood of p in each of the four squares. It could pick up points of a closed 0-dimensional subset of H^* by spiraling in as illustrated in horizontal cross section in Figure 22. The sticklers on g_o make such a spiraling be not locally connected if the spiral misses g_o on each revolution. On the other hand, if a stickler were intersected then, since it lies in

q_0, P(D) would not be a disk.

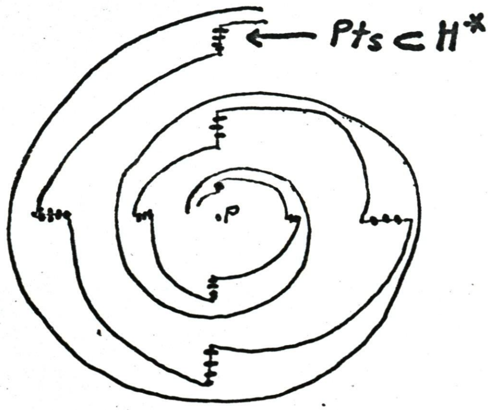


Figure 22

Remark 2. In Theorem 3.3 we hypothesized that Cl H* is 1-dimensional. This, of course, implies that no element is 2-dimensional. If 2-dimensional elements were allowed, we would have a counterexample. (See Example (1), p. 25.) Now let A ∪ B be a given manifold M. By Theorem 3.1 there is a manifold M_ϵ^1 satisfying the hypotheses of Theorem 3.3. (A manifold satisfying the requirements for M_ϵ^1 is A - $N_{\epsilon/2}$ (B).) Fairly obviously, any continuum connecting M_ϵ^1, and a point in the limiting set, but missing all other nondegenerate elements, will not be locally connected. Hence, in this example, the equality in the conclusion of Theorem 3.3 cannot be satisfied for a disk.

Corollary 1. Suppose that E^3/G is homeomorphic to E^3; H is countable; $H = \mathrm{Cl}\ H$; and that for each $g \in H$, $\dim g = 1$. Let D be a disk contained in E^3/G and let the collection of nondegenerate elements that project into D be continuous. Then, for any $\epsilon > 0$, there is another disk D_ϵ that P-lifts, is ϵ-homeomorphic to D, and satisfies the equality $D_\epsilon \cap P(H) = D \cap P(H)$.

Proof. H is countable and $H = \mathrm{Cl}\ H$ imply that $\mathrm{Cl}\ P(H)$ is 0-dimensional. Since the union of a countable number of closed n-dimensional sets is n-dimensional, $\mathrm{Cl}\ H^*$ is 1-dimensional. Now Theorem 1 gives us, for any $\epsilon > 0$, a disk D^1_ϵ that P-lifts, and is ϵ-homeomorphic to D. The method of proof of Theorem 1 is to find a new disk that misses $P(H)$. Hence, this disk D^1_ϵ certainly satisfies the containment $D^1_\epsilon \cap P(H) \subset D \cap P(H)$. We have the hypotheses of Theorem 3.3 satisfied, so it now gives us our desired conclusion. $\square$

Remark. D^1_ϵ and D_ϵ are not necessarily tame. The finger construction can result in a Fox-Artin [14] type wild sphere. I conjecture that with controlling directions of arcs and fingers, it is possible to

construct a tame D_ϵ.

<u>Corollary 2</u>. Suppose that H is definable by 3-cells and Cl H* is 1-dimensional. Let D be a disk in E^3/G and let the collection of nondegenerate elements that project into D be continuous. Then, given $\epsilon > 0$, there is a disk D_ϵ such that:

(1) D_ϵ is ϵ-homeomorphic to D,

(2) D_ϵ P-lifts, and,

(3) $D \cap P(H) = D_\epsilon \cap P(H)$.

Proof. The decomposition satisfies Theorem 3.2 with $C = \emptyset$. By Theorem 3.2 there exists a disk D_ϵ^1 that satisfies (1) and (2). Because H is definable by 3-cells, H is monotone and Cl P(H) is 0-dimensional. Hence, Theorem 3.3 is satisfied. From it, we get our conclusion.

<u>An example of a disk D that P-lifts and such that, for some $\epsilon > 0$, there is no disk D_ϵ that is ϵ-homeomorphic to D and lifts</u>. In Theorems 3.1 and 3.2, if we use $C = \emptyset$, then the D_ϵ^1 that we find in our method of proof actually lifts, rather than only P-lifting. In Theorem 3.3, we modified the P-lift of the disk to pick up exactly one point in each nondegenerate element whose image intersected the original D. Here again

we have the rather special situation in which we state that there is a P-lift because we have shown that a true lift exists. We are led to wonder whether, when there is a disk D that P-lifts, there is always, for any $\epsilon > 0$, another disk D_ϵ, which is ϵ-homeomorphic to D and actually lifts. The following example shows this not to be true.

Of course, our example must not satisfy all the hypotheses of Theorem 3.1 or 3.2. In this example E^3/G will not be homeomorphic to E^3 and H will not be definable by 3-cells. The only nondegenerate element of G will be a θ-curve. The P-lift D' will be chosen to contain only the middle arc of the θ. Then there will exist an $\epsilon > 0$ such that there can be no D_ϵ that is ϵ-homeomorphic to D and has a true lift.

Let the notation abc denote an arc with endpoints a and c and an interior point b. If abc and adc are arcs such that abc $\cap$ adc $= \{a, c\}$, then let abcda denote the simple closed curve that is the union of the arcs.

In E^2 let abc, adc, and aec be three arcs whose union is a θ-curve such that d lies in the bounded component of E^2 - abcea. (See Figure 23.) We can choose these three arcs so that abcea is a circle and adc is a line segment. Let bfe be a polygonal arc

in the unbounded component of E^2 - abcea such that a lies in the bounded component of E^2 - efbce.

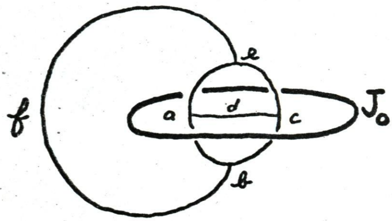


Figure 23

Let J_1 denote the simple closed curve baefb, and J_2 denote the simple closed curve bcefb. Now, in E^3, let J_o be a polygonal simple closed curve that links by Bing's linking definition in [7] J_1 and J_2 and such that J_o bounds a disk D' satisfying $D' \cap (J_1 \cup J_2 \cup adc) =$ adc. We can choose J_o such that D' is in a plane. Now any disk with J_o as its boundary must intersect both J_1 and J_2.

Let the θ-curve (abc $\cup$ adc $\cup$ aec) be the only nondegenerate element in our decomposition. Call it θ. In E^3, let U_θ be an open tubular neighborhood of θ such that U_θ is the interior of a solid double torus that misses J_o. Let J_3 and J_4 be simple closed polyhedral curves such that J_3 links aecda and aecba (See Figure 24 .) and J_4 links abcda and aecba. These can be chosen so that they miss efb, U_θ, and the disk D'. Let U_{J_o} be an open tubular neighborhood of J_o such

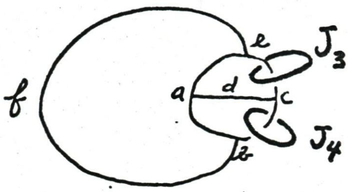


Figure 24

that U_{J_o} misses $(U_\theta \cup efb \cup J_3 \cup J_4)$.

We will require that e be chosen sufficiently small that:

(1) $N_\epsilon(P(\theta)) \subset P(U_\theta)$

(2) $N_e(D)$ misses $P(J_3 \cup J_4)$

(3) $N_\epsilon(P(J_o)) \subset P(U_{J_o})$

Suppose there does exist a disk D_ϵ that lifts and such that $h: D \to D_\epsilon$ is an ϵ-homeomorphism in E^3/G. Then in E^3 there is a homotopy taking Bd D' onto Bd D'_ϵ. Call the homotopy λ. For any $x \in$ Bd D the point $h(x)$ is in $N_\epsilon(x)$. This implies that the homotopy λ can be taken to be in $P^{-1}(N_\epsilon(\text{Bd } D))$. Hence, by (3) and the definition of U_{J_o}, the image in E^3/G of λ misses $P(U_\theta \cup efb \cup J_3 \cup J_4)$.

By our choice of e, Bd D'_ϵ links J_1 and J_2. The condition that D'_ϵ is a lift implies that D'_ϵ contains at most one point of θ. Therefore, D'_ϵ intersects arc efb and does not contain both points e and b. If it contains one, suppose it is e. Since D_ϵ is ϵ-homeomorphic to D, the disk D'_ϵ is contained in $P^{-1}(N_\epsilon(D))$.

Let us apply Bing's Approximation Theorem [7] to the disk D'_ϵ. Use for his function $f(x)$ the minimum of $d(x, \theta)$ and $d(x, \text{Bd } P^{-1}N_\epsilon[h^{-1}P(x)])$. The latter distance is positive, because the ϵ-neighborhood is open and contains x. Use of this distance assures us that the image of the new disk will still be ϵ-homeomorphic to D. To assure that we add no new intersections of the disk and θ, the distance $d(x, \theta)$ is used. Now the theorem gives us a disk $\tilde{D}'_\epsilon$ that is locally polyhedral everywhere except possibly on the point e and such that $\tilde{D}'_\epsilon$ satisfies all conditions we have stated for D'_ϵ. For simplicity of notation, let us assume that the original D'_ϵ was already locally polyhedral except possibly on the point e.

Consider a 2-sphere S^2 that is the boundary of a cylinder with bottom B, top C, and sides the annulus A, i.e., $S^2 = A \cup B \cup C$. There is a map $\phi: S^2 \to P^{-1}(N_\epsilon(D))$ such that $\phi(C) = D'_\epsilon$, $\phi(B) = D'$, $\phi(A)$ is the homotopy λ between Bd D' and Bd D'_ϵ, $\phi \mid C \cup B$ is a homeomorphism, and $\phi \mid B$ is piecewise linear. Since $\phi(B \cup A)$ misses point e, we can choose a polyhedral simple closed curve J in D'_ϵ such that e is contained in the interior of a disk $E \subset D'_\epsilon$, Bd $E = J$, and E misses $\phi(B \cup A)$.

We now wish to change $\phi(S^2)$ to a 2-sphere that is

near $\phi(S^2)$, agrees with it in neighborhoods of e and adc, and is locally polyhedral except possibly at e. For this we will use an extension of Dehn's Lemma. Burgess and Cannon [11] quote and prove the following, which they call Bing's extension of Dehn's Lemma.
If D is a polyhedral disk, f is a map of D into a triangulated 3-manifold M^3, U is an open set in E^3 containing f(Int D), and f is nonsingular in some neighborhood of Bd D, then there is a homeomorphism f' of D into f(Bd D) $\cup$ U such that f' is locally piecewise linear except on Bd D.
In their proof they use Bing's Approximation Theorem for Surfaces [7], which is quoted on p. 6 of this thesis.
In the same paper, Bing adds: If we had supposed that M is locally polyhedral at each point of a closed point we could have chosen M^* and h so that h is the identity on N. Incorporating this stronger statement by Bing into Burgess and Cannon's proof cited above, it can be shown that $[\phi(S^2)] - E$ can be approximated by a disk F such that:
(1) F agrees with $[\phi(S^2) - E]$ near arc adc and Bd E,
(2) F is a polyhedral,
(3) F lies in $U_\theta \cup P^{-1}(N_\epsilon(D))$, and
(4) F misses θ - adc and Int E.
The union of this new disk F and the disk E is the 2-sphere that we wished to find near $\phi(S^2)$. The proper-

ties of the 2-sphere $E \cup F$ are that it:

(1) contains the point e and the arc adc;

(2) is locally polyhedral off the point e;

(3) lies in $U_\theta \cup P^{-1}(N_\epsilon(D))$; and

(4) is pierced by bae and bce at a and c, respectively.

By the generalized Jordan theorem, $E \cup F$ separates E^3 into two components--a bounded one C_b and an unbounded one C_u. We will show that the closure of C_b is a 3-cell. As a step toward this we use the following theorem of Kinoshita [20]: Let A be an arc in S^3 and S be a locally polyhedral 2-sphere with one singularity (with respect to being locally polyhedral) at p such that (1) $A \cap S = p$, and (2) $A - p$ is contained in a trivial (i.e., simply connected) complementary domain of S. If A is tame, then S is also.

We will apply this using our tame arc ea for his A, our $E \cup F$ for his S, and our point e for his point p. (If e is not in D_ϵ', then we already have that $E \cup F$ is polyhedral and hence, tame.) If we knew that C_b were trivial, then the theorem would now give us that $E \cup F$ is tame and therefore that $C_b \cup E \cup F$ is a 3-cell. To get that C_b is trivial, we use a theorem of Harrold and Moise [16] that states that if a 2-sphere in E^3 is locally polyhedral except at one point, then both complementary domains are simply connected. Hence, we

have the conclusion that $C_b \cup E \cup F$ is a 3-cell.

Arcs bae and bce pierce disk F at a and c, respectively, and intersect F in no other point. The line segment adc lies in F. No other points of θ can lie in F. Since the point e is the only point of θ that can lie in E, it must be that the interiors of arcs ae and ec do not lie in $E \cup F$. We wish to show that one of the simple closed curves adcea and adcba lies in $C_b \cup E \cup F$. Case I: Suppose that b lies in C_b. Then adcba lies in $C_b \cup E \cup F$. Case II: Suppose that b lies in C_u. Then ab and cb lie in C_u. Then piercing at a and c implies that ae and ce lie in C_b. Therefore, adcea lies in $C_b \cup E \cup F$. Hence, both cases imply that the 3-cell $C_b \cup E \cup F$ contains one of the simple closed curves adcba and adcea.

Now suppose that adcea lies in $C_b \cup E \cup F$. The simple closed curve adcea bounds a polyhedral 2-disk δ such that Int $\delta \subset$ Int C_b. By the definition of linking, J_3 contains a point in Int δ and, hence, in Int C_b. Since J_3 does not intersect $E \cup F$, it must be that J_3 lies entirely in one component of $E^3 - (E \cup F)$. Because the 2-sphere $E \cup F$ lies entirely in $U_\theta \cup P^{-1}(N_\epsilon(D))$ and J_3 misses $U_\theta \cup P^{-1}(N_\epsilon(D))$, there is a ray missing $U_\theta \cup P^{-1}(N_\epsilon(D))$ and connecting J_3 with infinity. Hence, J_3 lies in C_u. This contradicts

the fact that we earlier found a point of J_3 in Int C_b. Hence, if adcea lies in the closed 3-cell $C_b \cup E \cup F$, we have a contradiction. There is a similar contradiction if adcba lies in $C_b \cup E \cup F$. Hence, the claim concerning our example is proven. $\square$

CHAPTER IV

A DECOMPOSITION SPACE CONTAINING A DISK THAT IS NOT HOMEOMORPHICALLY CLOSE TO A P-LIFTABLE DISK

We have found properties of the decomposition that guarantee that in a neighborhood of any disk D in E^3/G there is a P-liftable disk D_ϵ, which is ϵ-homeomorphic to D.

It is instructive to compare the question of the existence of a disk D_ϵ with a result of Armentrout in [3]: Suppose that M is a 3-manifold with boundary and G is a cellular decomposition of M such that M/G is a 3-manifold with boundary. Suppose K is a 2-manifold with boundary in M such that K misses H_G. Then P[K] is tame in M/G if and only if K is tame in M. For our use here, we will let $M = E^3$ and K be a disk. Note Armentrout's strong hypothesis that K miss the nondegenerate elements; this will not be satisfied in our example. Since tameness for a disk is equivalent to bicollarability, this theorem implies that if the disk K is bicollarable, then P[K] is. Although such bicollared disks have uncountably many disks on each side, we can certainly not assume that such disks map onto disks under P. We are concerned in this thesis with a neighborhood property that is

quite different from tameness.

We will describe a decomposition of E^3, and then specify an $\epsilon > 0$ and a disk in the decomposition space. The decomposition will have the following properties:

(1) E^3/G is homeomorphic to E^3,

(2) Each $g \in H$ is a tame arc,

(3) H is continuous and closed,

(4) P(H) is a Cantor set,

(5) H is not countable and is not definable by 3-cells.

A decomposition is called toroidal if it has a defining sequence M_1, M_2, M_3,... such that every component of M_i is a solid torus. It is called an (m, n) toroidal decomposition if it is an iteration of the embedding of m solid tori that essentially wrap n times around. The example we will discuss is not toroidal, but it is almost a (2, 1) toroidal decomposition in the sense that there is a defining sequence in which each M_i is the union of one component that is a cube-with-several-handles and other components that are solid tori.

Before we describe our decomposition space, we will make some definitions.

Let g_{p_1} and g_{p_2} be two line segments that have their endpoints in two parallel planes, I and II, and are perpendicular to them.

<u>Definition</u>. The countably infinite set of arcs $\{h_i \mid -\infty < i < \infty\} \cup \{g_{p_1}\} \cup \{g_{p_2}\}$ between planes I and II shown in Figure 25 is said to be <u>knit from the point p_1 to the point p_2</u>.

Although it is not necessary for this definition that p_1 and p_2 be at different heights, they are so shown in anticipation of a later step in the construction. Figure 26 illustrates a generalization of the knitting construction.

<u>Definition</u>. A Cantor set $\hat{H}$ of arcs is said to be <u>knit from a point p_1 to a point p_2</u> if $\hat{H}$ can be realized by the following modification of a countably infinite set of arcs knit from p_1 to p_2. For each h_i, let $N(h_i)$ be a tubular neighborhood of h_i, and let these be such that the members of the set $\{g_{p_1}\} \cup \{g_{p_2}\} \cup \{N(h_i) \mid -\infty < i < \infty\}$ are pairwise disjoint. In each tubular neighborhood replace the arc h_i by a Cantor set H_i of arcs, each one of which intersects I and II in the same manner as h_i. We also say that a Cantor set $\tilde{H}$ of arcs is knit from a point p_1 to p_2 if there exists a Cantor set $\hat{H}$ satisfying the above definition and if, for each $\tilde{g} \in \tilde{H}$ there is a $\hat{g} \in \hat{H}$ that is contained in $\tilde{g}$, and for each $\hat{g} \in \hat{H}$ there is a $\tilde{g} \in \tilde{H}$ containing $\hat{g}$.

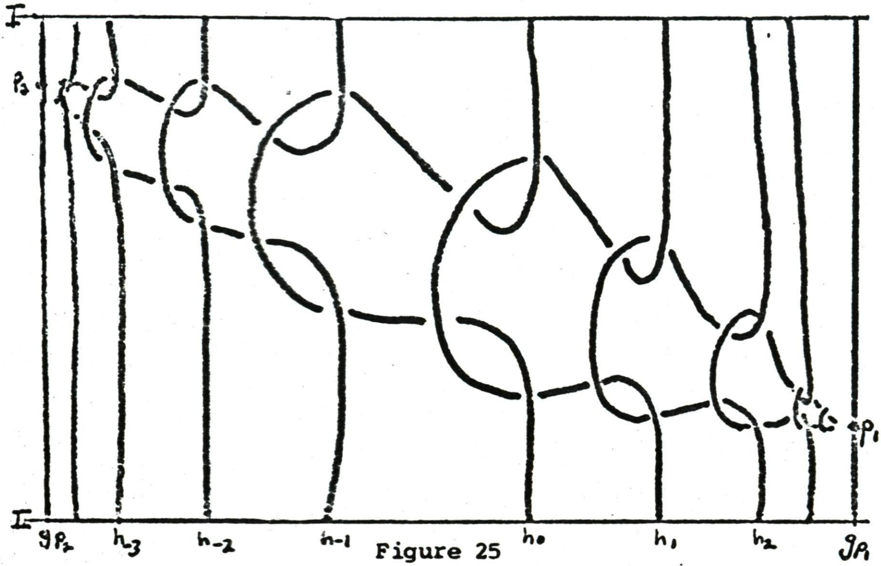


Figure 25

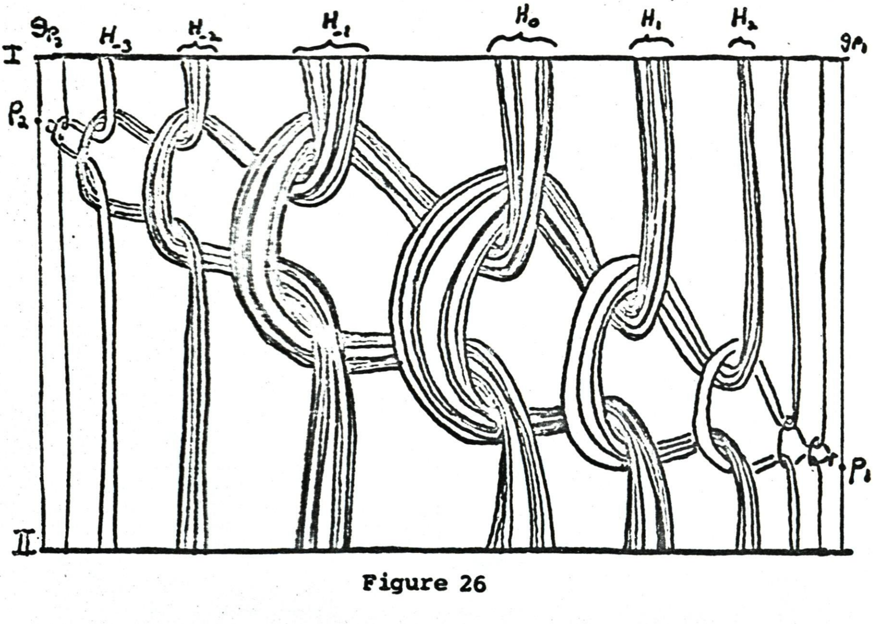


Figure 26

Next we describe a Cantor set of arcs which we will modify into a knit set. The first stage is the two disjoint tori pictured in Figure 27. We decompose each by a (2, 1) toroidal decomposition. Figure 28 indicates the first embedding of two linked tori that essentially wrap once around in each. Let P be a plane that cuts each solid torus as indicated in the figure. We can assume that the nondegenerate elements of these two toroidal decompositions intersect P in a standard Cantor set $\mathcal{C}$ of points in an interval line segment. Figure 29 shows this plane and some of the points with their usual numerical representation in base 3.

In Figure 30 we have separated the tori on P. The two copies of P are now labelled planes I and II. Between I and II we indicate the Cantor set of straight line segments connecting copies of $\mathcal{C}$ in I and II. For this Cantor set of segments we substitute the knit Cantor set of Figure 26 in the manner indicated in the following table, in which [a,b] indicates the segments containing points in $\mathcal{C}$ in this interval and H_i is a Cantor set of arcs in Figure 26.

...	H_{-3}	H_{-2}	H_{-1}	H_0	H_1
...	[.0002,.001]	[.002,.01]	[.02,.1]	[.2,.21]	[.22,.221]

H_2	...
[.222,.2221]	...

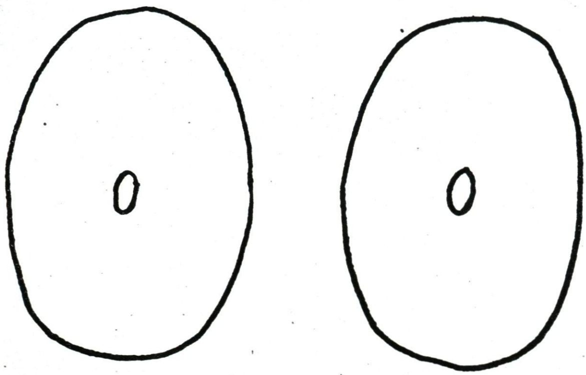


Figure 27

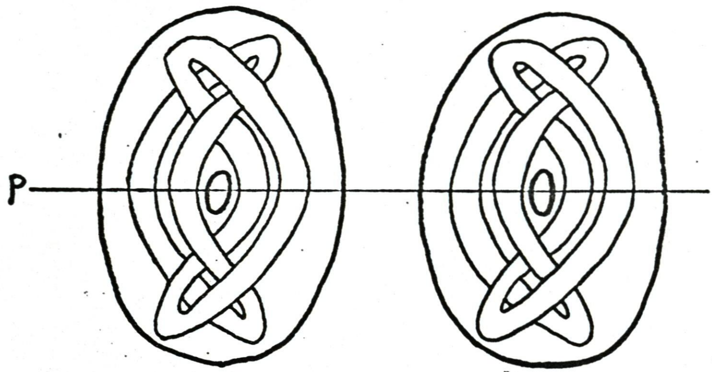


Figure 28

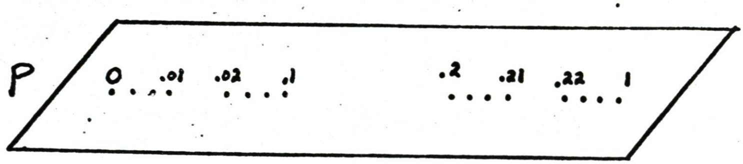


Figure 29

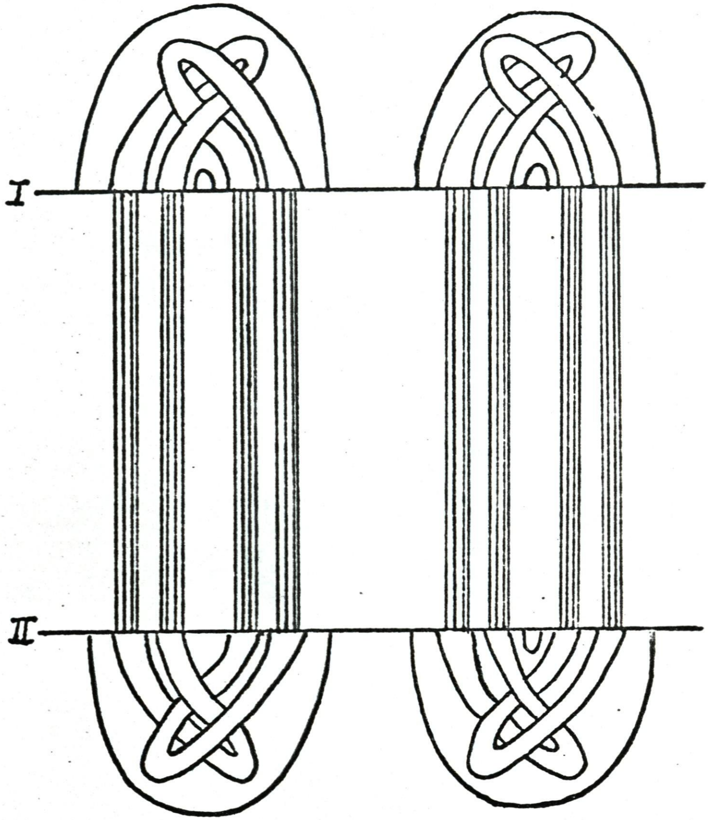


Figure 30

We have changed the collection of nondegenerate elements described by the toroidal decomposition in such a manner that they are knit between p_1 and p_2. With the knitting inserted, the collection is no longer toroidal. Figure 31 shows the second stage of a defining sequence for this knit Cantor set, $\tilde{H}$.

Map $\tilde{H}$ into E^3 in such a way that the limiting elements, which contain p_1 and p_2, are identified and the map is an embedding for all other elements. In Figure 32, parts of some elements are shown. Notice that p_1 and p_2 are mapped to different points in the identification of the two nondegenerate elements. Also in Figure 32 is the 2-complex C, which can be specified by:

$\{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 1, z = 0\}$,

$\{(x, y, z) \mid -1 \leq x \leq 0, -1 \leq y \leq 0, z = 0\}$,

$\{(x, y, z) \mid 0 \leq x \leq 1, -1 \leq y \leq 0, z = 1\}$,

$\{(x, y, z) \mid -1 \leq x \leq 0, 0 \leq y \leq 1, z = 1\}$,

$\{(x, y, z) \mid x = 0, -1 \leq y \leq 1, 0 \leq z \leq 1\}$, and

$\{(x, y, z) \mid -1 \leq x \leq 1, y = 0, 0 \leq z \leq 1\}$.

The identified elements are the element g_o, which is the segment of the z-axis: $-1 \leq z \leq 2$. The points p_1 and p_2 are $z = 1/3$ and $z = 2/3$. The set $\tilde{H}^* \cap C$ is contained in the line segments $\{g_o\}$,

$\{(x, y, z) \mid x = y, 0 \leq x \leq 1, z = 0\}$, and

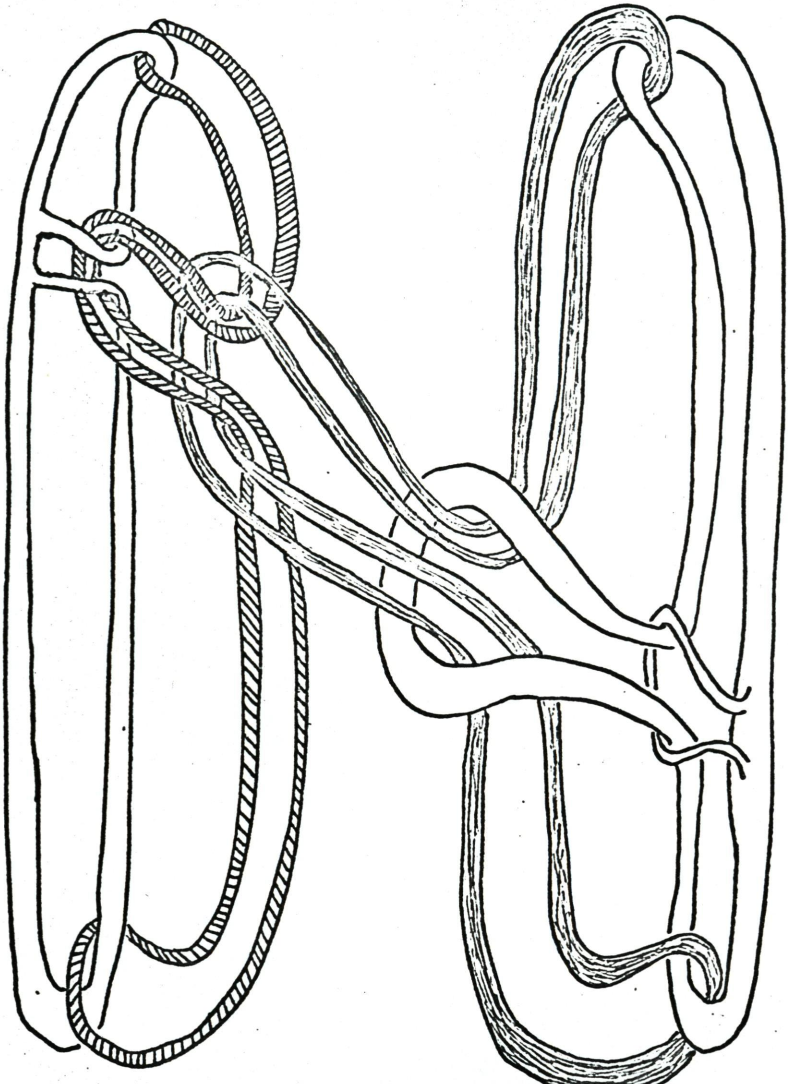


Figure 31

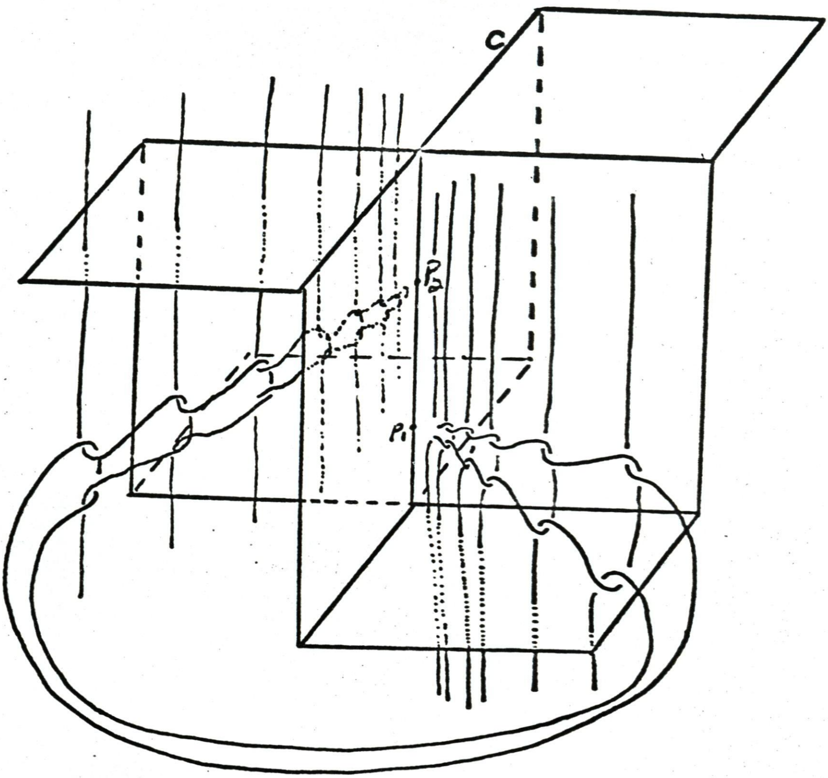


Figure 32

$\{(x, y, z) \mid x = -y, -1 \leq x \leq 0, z = 1\}$.

Finally, we add nondegenerate elements in $y < 0$ in such a way that we have symmetry with respect to the z-axis. All the nondegenerate elements are indicated in Figure 33. They are the set H of nondegenerate elements of the decomposition G for which we make the claim in this paper. In E^3/G, $P(C)$ is a disk. It is the disk D for the claim.

We must also specify the ϵ for the claim. There is a defining sequence of manifolds for this decomposition. The first stage M_1 is shown in Figure 34. Figure 35 shows the intersection of C with this manifold. The set $C \cap \text{Bd } M_1$ has five components, each of which is a simple closed curve. Call them c_j with $j = 0, 1, 2, 3$, and 4. There is an infinite triangulation of $E^3 - H^*$ such that $C - H^*$ and Bd M_1 are complexes in this triangulation. Note that H^* is closed and misses Bd M_1. Let r_j denote the regular neighborhood in this triangulation of c_j. For each j, the set r_j is a solid torus that misses H^*. Notice that in each handle of M_1 there is a simple closed curve that misses these tori. Choose the value for ϵ to be a positive distance such that for each component of c_j, the set $P^{-1}N_\epsilon(P(c_j))$ lies in r_j, and $P^{-1}N_\epsilon(P(c_j))$ misses $(\text{Bd } M_1) - \bigcup_{j=0}^{4} r_j$. These conditions for ϵ imply that for any disk D_ϵ' that projects onto a disk

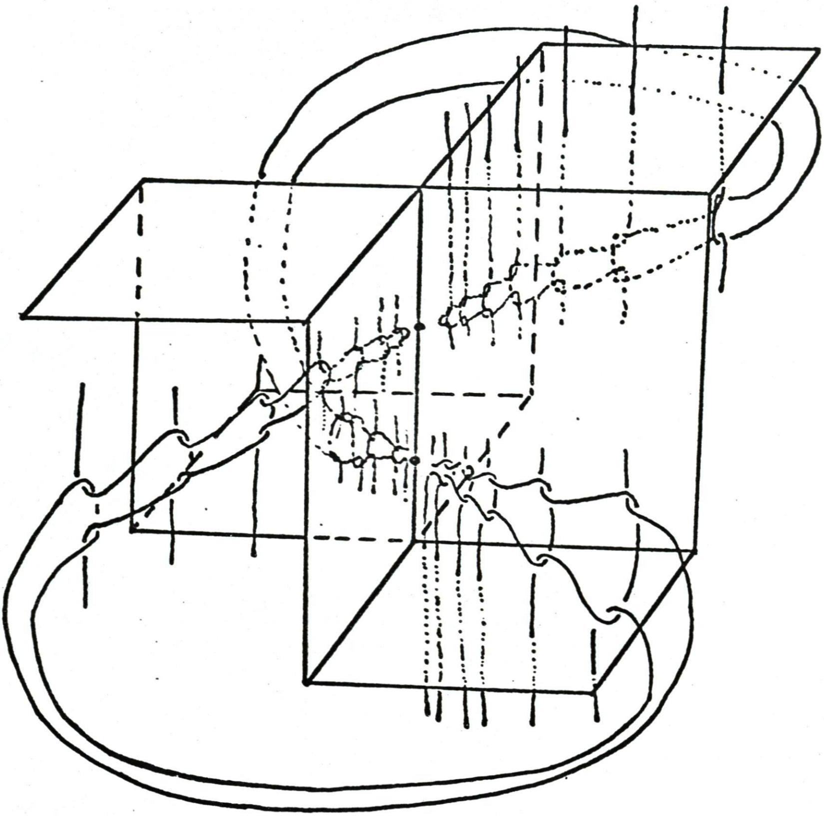


Figure 33

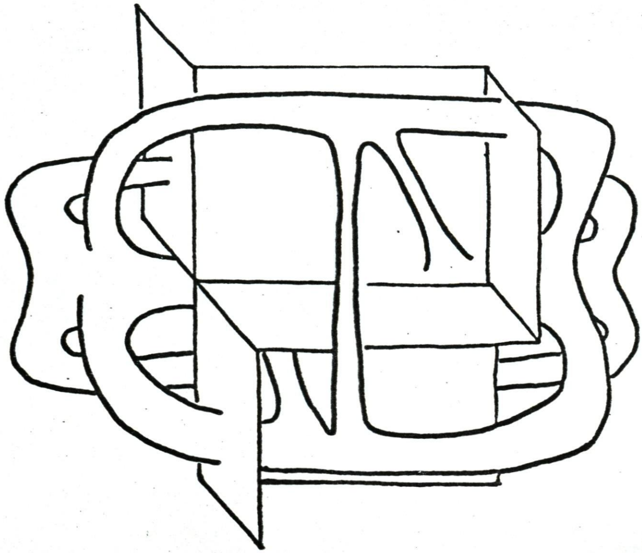


Figure 35

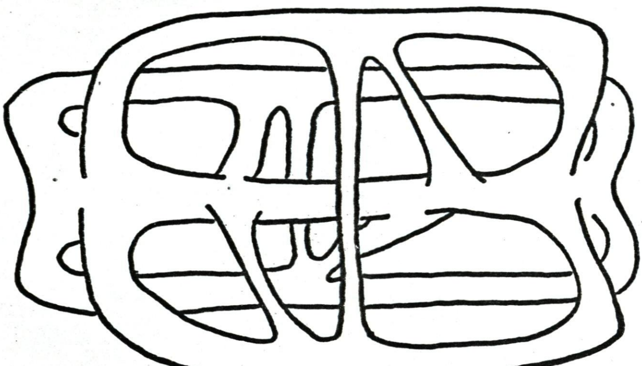


Figure 34

D_ϵ that is ϵ-homeomorphic to D, it is true that $D'_\epsilon \cap \mathrm{Bd}\, M_1$ is contained in the regular neighborhood of $C \cap \mathrm{Bd}\, M_1$. This completes the description of the example.

Theorem 4.1. There is no P-liftable disk D_ϵ that is ϵ-homeomorphic to D.

Proof. Fort [13] altered the defining manifolds for Bing's dogbone space [18] to guarantee nice properies for each nondegenerate element. One of these properties is that each nondegenerate element lies between a pair of parallel planes and has an endpoint in each plane. The change is a homeomorphism of each defining manifold and is such that the resulting decomposition is homeomorphic to Bing's original decomposition. It is possible to construct a (2,1) toroidal decomposition so that it has this property. Let us assume, also, that for our knit example that we have constructed its defining sequence $M_1, M_2, M_3, \cdots$ in such a way that each $g \in H$ lies between a pair of parallel planes R and S and has an endpoint in each plane.

We first will prove a lemma for a toroidal decomposition and then generalize it to our almost toroidal example.

Definition. A disk D is said to be meridional in a solid torus T if Bd D is a nontrivial simple closed curve in Bd T and D is trivial in T. We will

also say that a subdisk $D \subset D_\epsilon'$ is a meridional subdisk of D_ϵ' in a toroidal component T of an element M_i of the defining sequence if:

(a) Bd D is a simple closed curve J in Bd T,

(b) J bounds a meridional (as defined immediately above) disk d in T, where d is not necessarily in D_ϵ', and,

(c) The disk in D_ϵ' bounded by J contains no subdisk satisfying (a) and (b).

This latter definition allows us to call a disk meridional even if in its interior it contains a subdisk that protrudes outside T. Condition (c) is included so that a set of nested subdisks each with boundary in Bd T do not count as more than one.

There is an analogous definition of a trivial subdisk of D_ϵ' in a toroidal component T.

Lemma 4.1. Let $N_1, N_2, N_3, \ldots$ be the defihing sequence for a (2, 1) toroidal decomposition G_T with nondegenerate elements H_T. Let E' be a disk in E^3 that projects onto a disk in E^3/G_T. Assume that E' is locally polyhedral off H_T^*. Suppose that D_a and D_b are disjoint subdisks in E' and that they are meridional disks in the solid torus N_1. Then, for any $g \in H_T$, $g \cap E' \neq \emptyset$.

Notation. For any positive integer n, let T with

a subscript of n digits, each of which is a one or two, be a toroidal component of N_n. As is conventional, the kth digit of the subscript indicates the component of N_k in which the given torus lies. Thus, $T_{1221} \subset T_{122}$. When we are concerned with the tori imbedded in a given torus at the next stage or the next few stages, the subscripts agree except in a small number of final digits. A notation introduced by Casler [12] is then convenient. We let <u>$n\alpha$</u> denote a subscript of n digits, and append digits to $n\alpha$. Thus, $T_{n\alpha 12} \subset T_{n\alpha 1} \subset T_{n\alpha}$. For our knit example, we add to this notation a superscript $j = 1, 2, 3$, or 4 that denotes which quadrant of the 2-complex C is intersected by the torus $T^j_{n\alpha}$. To denote the nontoroidal component, i.e., the component containing the central element g_o, we use the notation $Q_{n\alpha}$, where $n\alpha$ consists entirely of ones.

<u>Proof of Lemma 4.1</u>. The defining sequence for the given decomposition G_T is not unique. Let us choose for it a (2,1) toroidal defining sequence such that each element of the sequence is polyhedral and in general position with respect to $E' - (\text{Bd } D_a \cup \text{Bd } D_b)$. Without loss of generality of the lemma, the defining sequence can be so specified that there are two parallel planes R and S such that each plane contains

one end point of each nondegenerate element. We will denote the chosen defining sequence by $N_1, N_2, N_3, \cdots$. The proof of Lemma 4.1 depends on the following lemma.

Lemma 4.2. Suppose that a disk D is locally polyhedral off H_T^*; in general position with respect to Bd N_n for every positive integer n; and that for some torus $T_{k\alpha}$, D is a meridional disk in $T_{k\alpha}$. Then

(1) For i equals either 1 or 2, D contains two disjoint subdisks d and e that are each meridional in $T_{k\alpha_i}$.

(2) If, for $j \neq i$, $D \cap T_{k\alpha_j} = \emptyset$, then it is possible to choose d and e to satisfy the following: Let A be a component of $T_{k\alpha_i} - (d \cup e)$. Then $A \cup D$ links $T_{k\alpha_j}$.

Proof of Lemma 4.2. The proof of (1) is a modification of Bing's Theorems 1-4 in [9], in which he is using a (2, 2) toroidal decomposition. The modifications of his theorems are here labeled Lemmas A - D, respectively.

Lemma A. If D is a disk that is locally polyhedral off H_T and in general position with respect to Bd $T_{k\alpha}$ and J is a component of $D \cap$ Bd $T_{k\alpha}$, then J either bounds a disk on Bd $T_{k\alpha}$ or J circles Bd $T_{k\alpha}$ once longitudinally and no times meridionally, or J circles Bd $T_{k\alpha}$ once meridionally and no times longitudinally.

Proof. Bing's Theorem 1 is an analogous statement for a polyhedral disk. For our locally polyhedral disk there is an approximation by a polyhedral disk that agrees with the given disk D in the neighborhood of Bd T. Since this theorem concerns a single torus, Bing's proof of it applies to our decomposition with its different embedding. $\square$

<u>Lemma B</u>. The fundamental group of $E^3 - (T_{k\alpha i} \cup T_{k\alpha j})$ is a free group on two generators. A simple closed curve that circles Bd $T_{k\alpha}$ meridionally cannot be shrunk to a point in $E^3 - (T_{k\alpha i} \cup T_{k\alpha j})$.

Proof. We must modify Bing's proof of his analogous theorem to fit our different embedding of $T_{k\alpha i}$ and $T_{k\alpha j}$. Following his method, we draw Figure 36. Each arrow represents a loop that goes from the eye, then along the arrow behind the torus, and back to the eye. Certainly the fundamental group can be generated by the four loops a, b, x, and y. At each of the points where $T_{k\alpha i}$ and $T_{k\alpha j}$ intersect in the perspective drawing, we have a relationship of the generators. Figure 37 indicates the manner in which we get one of these. The two arrows represent homotopic curves in $E^3 - (T_{k\alpha i} \cup T_{k\alpha j})$. One arrow is equivalent to the loop one gets by going from the eye, around x, to the

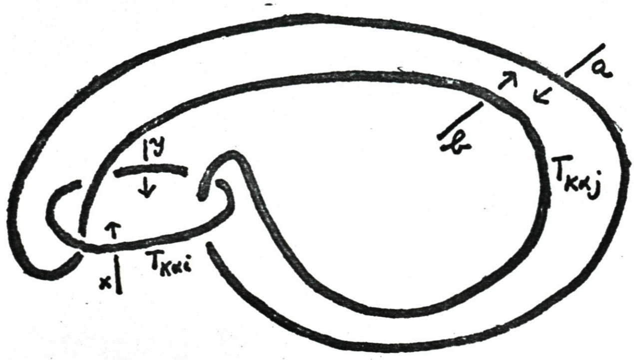


Figure 36

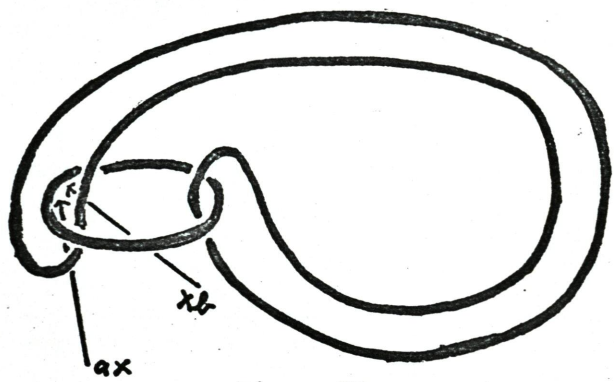


Figure 37

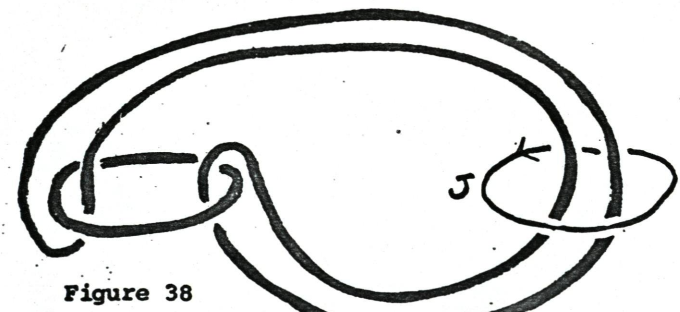


Figure 38

eye, around b, and back to the eye. It is xb. The other is the sum of loops a and x. Hence, we have $xb = ax$. Similarly, we get this relationship again at the other "front" perspective intersection and on each of the "back" perspective intersection the relationship $bx = yb$ results. Solving these, we find that $b = x^{-1}ax$ and $y = x^{-1}axa^{-1}x$. Hence, our fundamental group is a free group generated by a and x.

In Figure 38 we show a simple closed curve J that circles Bd $T_{k\alpha}$ meridionally. It corresponds to $ab^{-1} = ax^{-1}a^{-1}x$. Since this is not trivial in the free group, J can not be shrunk to a point in $E^3 - (T_{k\alpha i} \cup T_{k\alpha j})$. This completes the proof of Lemma B. $\square$

<u>Lemma C</u>. If D is a disk that is locally polyhedral off H_T and in general position with respect to the defining sequence, and D is meridional in $T_{k\alpha}$, then either $D \cap \text{Bd } T_{k\alpha i}$ contains a simple closed curve that circles Bd $T_{k\alpha i}$ meridionally or $D \cap \text{Bd } T_{k\alpha j}$ contains a simple closed curve that circles Bd $T_{k\alpha j}$ meridionally.

Proof. Bing's proof of the analogous theorem depends on his previous two theorems. Since Lemmas A and B are true for our embedding, this theorem is true for our embedding. $\square$

<u>Lemma D</u>. If D is a disk that is locally polyhedral off H_T and in general position with respect to

Bd $T_{k\alpha}$, then D contains two mutually exclusive subdisks such that each of the subdisks is meridional in $T_{k\alpha i}$ or each of the subdisks is meridional in $T_{k\alpha j}$.

<u>Proof</u>. Bing's proof of his analogous statement depends on his previous three theorems. Since we have the above three lemmas, this lemma is also true for our embedding.

<u>Continuation of proof of Lemma 4.2</u>. Lemma D is statement (1) of Lemma 4.2

To prove statement (2) we use the covering space of $T_{k\alpha}$. Figure 39 shows the covering space $\tilde{T}_{k\alpha}$ and the linked copies of $T_{k\alpha i}$ and $T_{k\alpha j}$ embedded in it.

Since Bd D does not link $T_{k\alpha j}$, there is a polyhedral disk F in $T_{k\alpha} - T_{k\alpha j}$ such that $F \cap T_{k\alpha j}$ is a longitudinal simple closed curve in Bd $T_{k\alpha j}$, the disk F does not intersect the disk D, and F is in general position with respect to $T_{k\alpha}$, $T_{k\alpha i}$, and $T_{k\alpha j}$. One of the copies of F in $\tilde{T}_{k\alpha}$ is shown in Figure 39. In view of the linking of each copy of $T_{k\alpha j}$ with each of its two adjacent copies of $T_{k\alpha i}$, F intersect each of these adjacent copies of $T_{k\alpha i}$ in a set containing a meridional disk of $T_{k\alpha i}$. Denote these meridional disks by f_1 in the left copy of $T_{k\alpha i}$ and f_2 in the right copy, as shown in Figure 39. The disk f_2 in the covering space $\tilde{T}_{k\alpha}$ is copied in the torus $T_{k\alpha}$. Hence, in every copy of

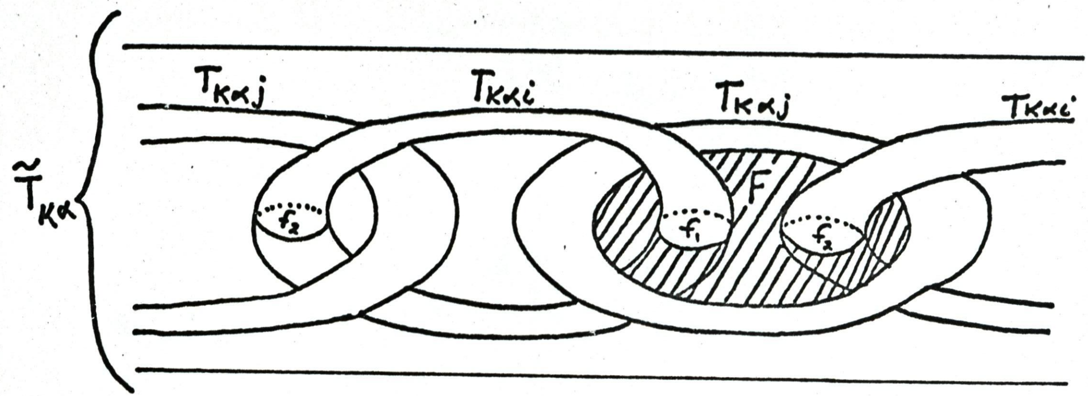

$\tilde{T}_{K\alpha}$
$T_{K\alpha j}$
$T_{K\alpha i}$
$T_{K\alpha j}$
$T_{K\alpha i}$
F
f_2
f_1
f_2

Figure 39

$T_{k\alpha i}$ in $\tilde{T}_{k\alpha}$ there is a copy of f_2. In the left copy of $T_{k\alpha i}$ a copy of f_2 is shown in Figure 39. Let β be an arc in F connecting a point $r_1 \in f_1$ with a point $r_2 \in f_2$. In $\tilde{T}_{k\alpha}$ let γ be an arc in $T_{k\alpha i}$ such that the end points of γ are r_1 and r_2 and Int γ misses f_1 and f_2. Let δ be a similar arc in the other component of $T_{k\alpha i} - (f_1 \cup f_2)$. Let J_a and J_b be simple closed curves that are the unions of the images of β and γ and of β and δ, respectively, in the torus $T_{k\alpha}$. Each of J_a and J_b is a longitudinal simple closed curve in $T_{k\alpha}$. Hence, each must intersect D. At least one meridional disk of $T_{k\alpha i}$ must contain a point of $D \cap J_a$, and at least one meridional of $T_{k\alpha i}$ must contain a point of $D \cap J_b$. These are the meridional disks that we choose for d and e in the statement (2) of Lemma 2. Now, by our requirement that d and e are in different components of $T_{k\alpha i} - (f_1 \cup f_2)$, we find that $D \cup A$ (where A is defined in the lemma) intersects F in the one meridional disk f_1, but $D \cup A$ misses f_2 or vice versa. Hence, there is a simple closed curve in $D \cup A$ that links a longitudinal simple closed curve in $T_{k\alpha j}$. This implies that $D \cup A$ is linked with $T_{k\alpha j}$. $\square$

<u>Proof of Lemma 4.1 continued</u>. Assume that there is some $g_1 \in H_T$ that misses E'. Then, for some subscript

$n\alpha i$, there are tori $T_{n\alpha}$ and $T_{n\alpha i}$ containing g_1 and such that E' contains a meridional disk in $T_{n\alpha}$, but does not contain a meridional disk in $T_{n\alpha i}$. Assume that $n\alpha i$ is the subscript of the first toroidal component containing g_1 for which this statement is true. Recall that the torus T_1 contains the two given subdisks D_a and D_b; if $n > 1$, then $T_{n\alpha}$ contains at least two disjoint meridional subdisks of $D_a \cup D_b$ because $T_{n\alpha}$ links neither Bd D_a nor Bd D_b. We use Lemma 4.2 to show that there are at least four disjoint meridional subdisks of E' in $T_{n\alpha j}$, where $j \neq i$. From these subdisks we choose a set of exactly four that satisfy Lemma 4.2. Similarly, at every subsequent stage we choose a minimal set of subdisks satisfying Lemma 4.2 and contained in those chosen at the previous stage.

We will show that, when four or more subdisks are meridional in a toroidal component, each subdisk folds in some manner so as to intersect some tori at later stages many times and others not at all. (In a related manner, Bing [6] folds nondegenerate elements.)

For any integer $h > n$ and any toroidal component of $N_h \cap T_{n\alpha j}$ that has at least four meridional subdisks of E', the following analysis applies. In a toroidal component let D^1 and D^2 denote two of the meridional subdisks that each have two subdisks in the same

component at the next stage. We will show that one of D^1 and D^2 intersects plane R and the other intersects plane S. If we suppose that for either R or S, this intersection does not occur, then the subdisks can be ordered using the following method. Suppose neither D^1 nor D^2 intersects R. There is an oriented simple closed curve J that intersects R in 2 points and pierces each meridional disk once. Let α be the component of $J - R$ lying below R. The closure of α is an oriented arc that intersects each meridional disk once. We will say that the disk intersected first by α lies closest to R.

Now, there is no $g \in H_T$ that intersects both D^1 and D^2. Let $T_{k\alpha 12}$ denote the first toroidal component that has no meridional disk in D^1. We can assume without loss of generality that $T_{k\alpha 12}$ misses D^1 and that $T_{k\alpha 12} \cap D^2$ contains meridional disks. (Figure 40) We now use the second statement in Lemma 4.2. Let D of that lemma be a meridional subdisk $D^1_{k\alpha} \subset D^1 \cap T_{k\alpha 1}$. Then d and e of the lemma are subdisks of $D^1_{k\alpha}$. By the lemma, the upper (as pictured in Figure 40) component A of $(T_{k\alpha 1} - (d \cup e))$ must be such that $A \cup D^1_{k\alpha}$ links $T_{k\alpha 2}$. The construction of the example now requires that $D^1_{k\alpha}$ intersect R or S in order to miss $T_{k\alpha 12}$. Let $D^2_{k\alpha}$ denote the meridional subdisk in $D^2 \cap T_{k\alpha 1}$ that has a

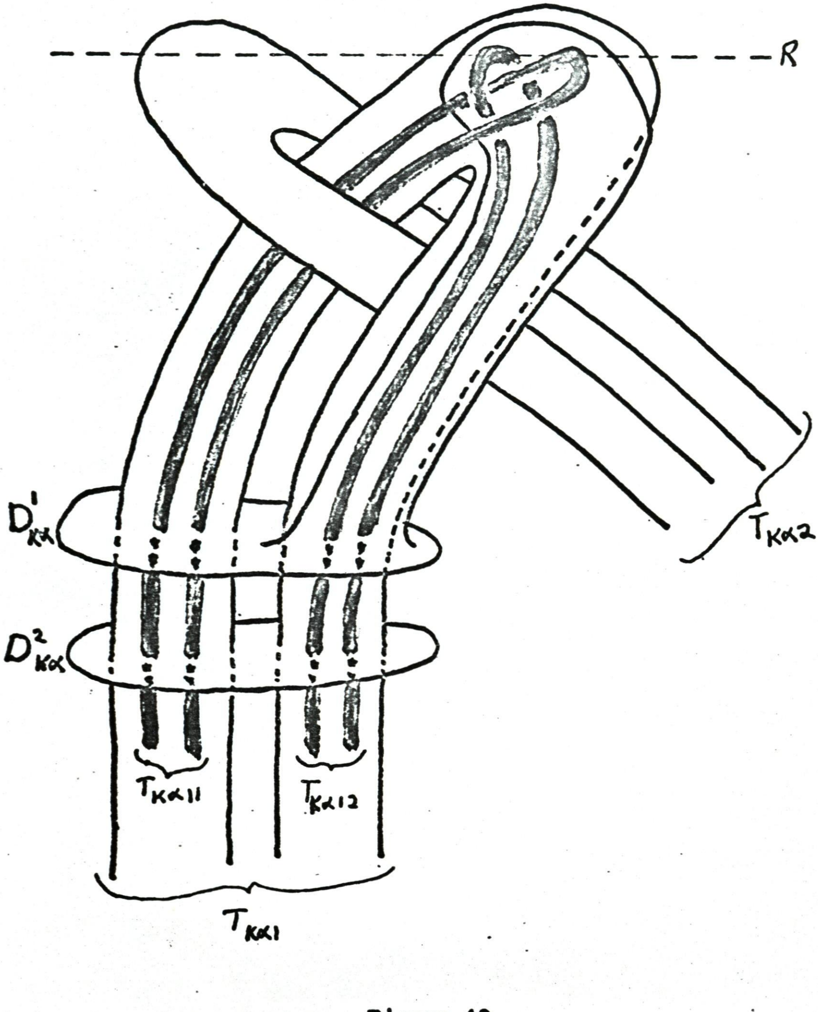

R
$D^1_{K\alpha}$
$D^2_{K\alpha}$
$T_{K\alpha 2}$
$T_{K\alpha 11}$
$T_{K\alpha 12}$
$T_{K\alpha 1}$

Figure 40

meridional subdisk in $T_{k\alpha 12}$. Assume that $D^1_{k\alpha}$ is closer to R than $D^2_{k\alpha}$ is to R. Then, since $D^1_{k\alpha}$ and $D^2_{k\alpha}$ are disjoint, it must be that $D^1_{k\alpha}$ intersects R rather than S.

Now suppose that there is a $g_2 \in H_T$ such that for every T containing g_2 the meridional disk closer (or closest) to R intersects g_2. Then, the other meridional disks at any stage must ultimately not intersect toroidal components containing g_2. Let us now assume that such an element g_2 lies in $T_{k\alpha 11}$. There must be a first stage, m ($m > k + 1$), such that $D^2_{k\alpha} \cap N_m$ has a meridional disk in the toroidal component containing g_2, but $D^2_{k\alpha} \cap N_{m+1}$ does not. Repeating the above argument, a subdisk of $D^2_{k\alpha}$ must intersect R or S. Now, if more than minimal sets of subdisks of $D^1_{k\alpha}$ and $D^2_{k\alpha}$ exist between stages k and m, we can assume that the choice we make of subdisks to use in these stages is such that those of $D^2_{k\alpha}$ are farther from R than those of $D^1_{k\alpha}$ are from R. Hence, a subdisk of $D^2_{k\alpha}$ must intersect S rather than R. We have now shown that if $D^1_{k\alpha}$ intersects R, then at some stage $D^2_{k\alpha}$ must intersect S. At the stage immediately after this intersection with S, there must be at least two subdisks of $D^2_{k\alpha}$ in the toroidal component not containing g_2. We can now repeat our whole argument with this pair replacing the original pair in $T_{n\alpha j}$. This sort of analysis leads to the conclusion that there is

a nested sequence E', D_1, D_2, D_3,... of disks such that each lies in the interior or the previous, and ($\text{Int } D_i$) - D_{i+1} contains a point below plane S if i is odd and above plane R if i is even. It is not possible for there to be such a sequence in a disk. This contradiction implies that Lemma 4.1 is true. $\square$

We assume that our example is not correct; that is, assume that in E^3/G there does exist disk D_ϵ that is ϵ-homeomorphic to the disk D and is the image of a disk $D_\epsilon' \subset E^3$. We can assume that Bing's approximation Theorem [7] has been used to make D_ϵ' be locally polyhedral off H^*. It can be made to be in general position with respect to Bd M_i for each i.

<u>Lemma 4.3</u>. For each $g \in H$, g intersects D_ϵ'.

Proof. The proof is a generalization of that of Lemma 4.1. We need to generalize the concept of a disk being meridional in a component of an element of the defining sequence. Based on Figure 31, the central component of M_i for $i > 1$ can be taken to be a cube-with-8-handles. For simplicity of a 2-dimensional drawing, we will use a cube-with-12-handles for the central component in each M_i for $i > 1$. In Figure 41, the central components are shown for M_i and M_{i+1}. Only two of the four tori linking the cube-with-12-handles of M_{i+1} are shown. Instead of the method of

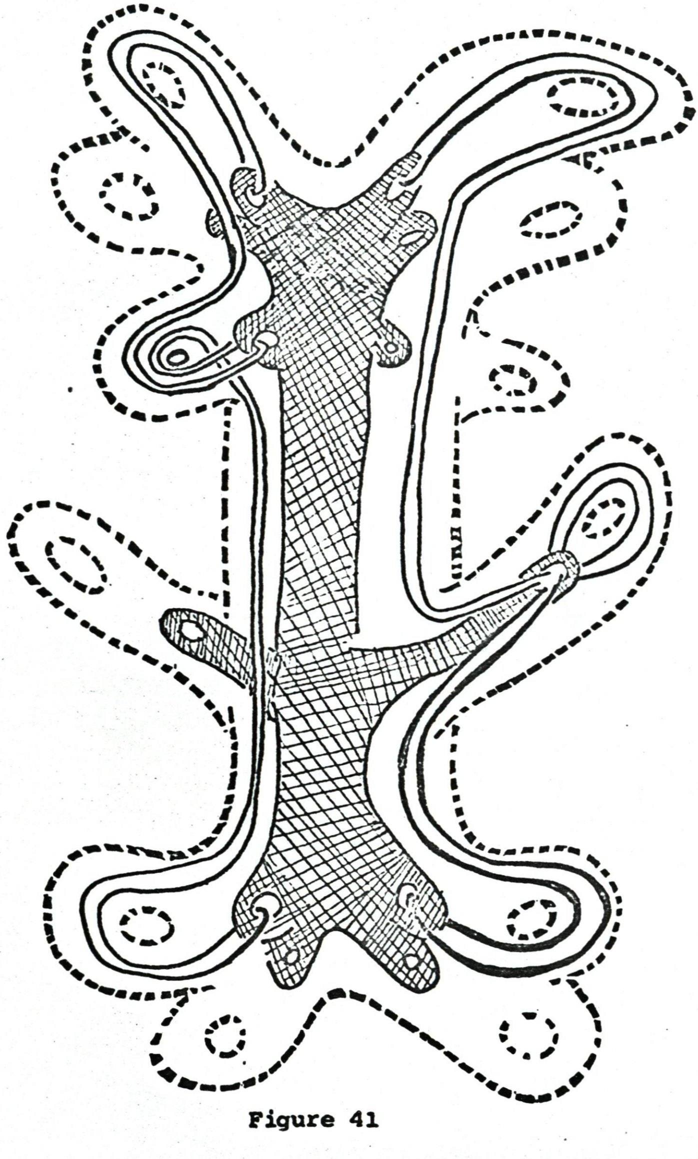


Figure 41

proof used below, it would be possible to work with iterations of the left-hand and right-hand manifolds shown in Figure 31. The problems encountered in such an argument would be very analogous to those dealt with below.

<u>Definitions</u>. A disk D is said to be <u>meridional</u> in M_1 if and only if Bd D is homotopic in Bd M_1 to the boundary of $C \cap M_1$. A disk D is said to be <u>meridional</u> in a cube-with-12-handles $Q_{n\alpha 1} \neq M_1$ if

(1) Bd D is a simple closed curve in Bd $Q_{n\alpha 1}$, and

(2) For quadrant j = 1, 2, 3, or 4, there is a meridional (as defined above for a torus, allowing it to contain subdisks which are trivial in Bd M_i) subdisk in any torus $S^j_{n\alpha 1}$ that satisfies the following:

 (a) $S^j_{n\alpha 1}$ is isotopic in $Q_{n\alpha 1}$ to the torus that is shown in Figure 42 for j = 1 and in Figure 43 for j = 4. For j = 2 and 3, tori in analogous positions are used. (This condition implies that $S^j_{n\alpha 1}$ links $T^j_{n\alpha 2}$.)

 (b) $S^j_{n\alpha 1}$ contains $Q_{n\alpha 111}$, $T^j_{n\alpha 12}$, and misses $T^k_{n\alpha 12}$ for $k \neq j$. (This condition implies that part, but not all of $Q_{n\alpha 11}$ lies in $S^j_{n\alpha 1}$.)

 (c) $S^j_{n\alpha 1}$ is in general position with respect to D'_ϵ.

The component $Q_{n\alpha 1} \neq M_1$ is said to contain a <u>full set</u>

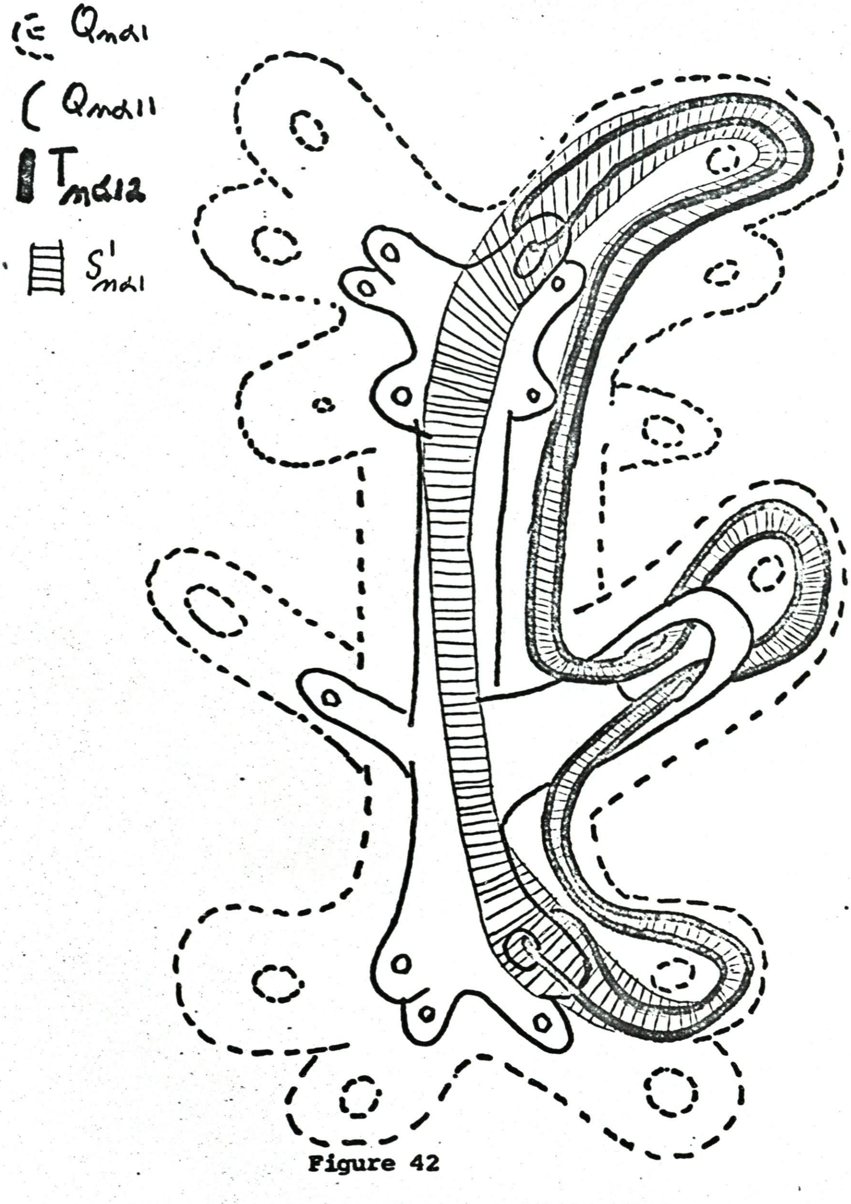


Figure 42

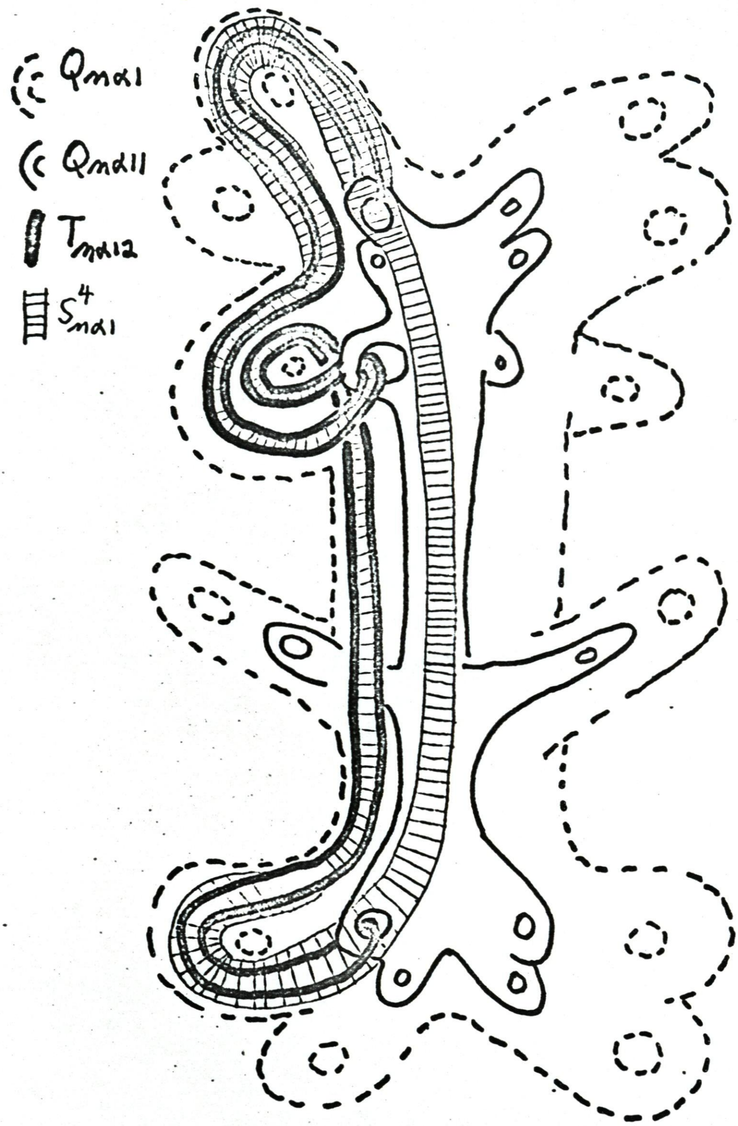


Figure 43

<u>of meridional disks</u> of D'_ϵ if $D'_\epsilon \cap Q_{n\alpha 1}$ contains subdisks satisfying the above definition for each $j = 1, 2, 3,$ and 4. For M_1 the five mutually exclusive subdisks of $D'_\epsilon \cap M_1$ are said to form two <u>full sets of meridional disks</u>. One set consists of the single "central" disk and the other set is the other four disks.

We begin the proof of Lemma 4.3 by assuming that there exists $g_1 \in H$ that does not intersect D'_ϵ. Let $g_1 = \bigcap_{i=1}^{\infty} R_i$, where each R_i is a component of the element M_i of the defining sequence. There must exist some first integer i such that for $j = i$ the following are true, but for $j = i + 1$, they are not true.

(1) If R_j is a torus, it contains a meridional subdisk of D'_ϵ.

(2) If R_j is a cube-with-12-handles it contains a full set of meridional subdisks of D'_ϵ.

There are the following possibilities:

Case 1: R_{i+1} is a torus that links a torus.

Case 2: R_{i+1} is a cube-with-12-handles.

Case 3: R_{i+1} is a torus that links a cube-with-12-handles.

We dispense with case 1 easily--this reduces to Lemma 4.1.

In cases 2 and 3, R_i is a cube-with-12-handles. Since, for $i > 1$, no simple closed curve in R_i links

Bd D_ϵ', for each meridional subdisk in R_i there must be another with its boundary isotopic in Bd R_i to the boundary of the given meridional subdisk. Thus, meridional subdisks occur in pairs.

In case 2, let R_{i+1} be denoted by $Q_{n\alpha 1}$. By assumption, this does not have a full set of meridional subdisks. Therefore, there must be an integer a = 1, 2, 3, or 4 such that for j = a, $Q_{n\alpha 1}$ does not have a meridional subdisk. But since $Q_{n\alpha}$ has a full set of meridional subdisks, it has a meridional subdisk for j = a. Hence, any torus $S^a_{n\alpha}$ contains a pair of meridional subdisks of D_ϵ', but no torus $S^a_{n\alpha 1}$ contains a meridional subdisk of D_ϵ'. In a torus $S^a_{n\alpha}$ it would be useful if $T^a_{n\alpha 2}$ were linked to a torus $S^a_{n\alpha 1}$. Then, by Lemma 4.2, $T^a_{n\alpha 2}$ would have to contain four meridional disks of D_ϵ'. Since the decomposition inside $T^a_{n\alpha 2}$ is (2, 1) toroidal, Lemma 4.1 would lead to a contradiction. This method does not work immediately because there is no $S^a_{n\alpha 1}$ contained in any $S^a_{n\alpha}$. Consider the $S^a_{n\alpha 1}$ and $S^a_{n\alpha}$ shown in Figure 44. We will add to $S^a_{n\alpha 1} \cap S^a_{n\alpha}$ a cylinder L so that we get a torus $\bar{S}^a_{n\alpha 1}$ that is linked with $T^a_{n\alpha 2}$ in the interior of a slightly enlarged $S^a_{n\alpha}$. We will show that the fact that $S^a_{n\alpha 1}$ does not contain a meridional subdisk implies that we can arrange sets so that the new torus $\bar{S}^a_{n\alpha 1}$ does not contain a meridional

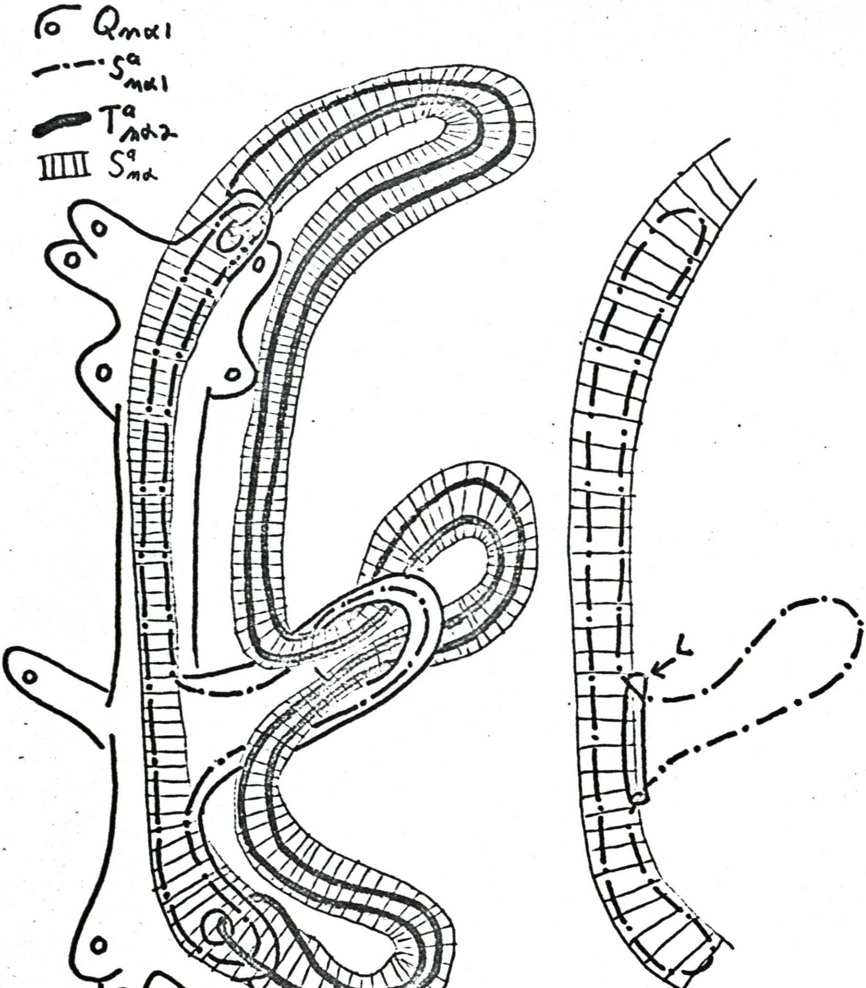


Figure 44

subdisk contained in one of the meridional disks of $S^a_{n\alpha}$. First, we note that because there is no meridional subdisk of D'_ϵ in $S^a_{n\alpha 1}$, we can move D'_e so that it still satisfies all requirements on D'_ϵ and also totally misses $S^a_{n\alpha 1}$. Then there is a disk W (Figure 44) such that

(1) $W \cap S^a_{n\alpha 1} = \text{Bd } W$,

(2) Bd W is a longitudinal simple closed curve in $S^a_{n\alpha 1}$, and

(3) $W \cap D'_\epsilon = \emptyset$.

Now we find the desired cylinder L near $W \cap \text{Bd } S^a_{n\alpha}$. This completes Case 2.

In Case 3, let $R_{i+1} = T^b_{n\alpha 2}$ and let $S^b_{n\alpha}$ and $S^b_{n\alpha 1}$ be any tori satisfying the definitions. The torus $S^b_{n\alpha}$ contains two meridional disks of D'_ϵ and $T^b_{n\alpha 2}$ contains none. We will show that this implies that there are four meridional subdisks in $S^b_{n\alpha 1}$. This is not immediate, because $S^b_{n\alpha 1}$ is not contained in $S^b_{n\alpha}$. In order to get such a condition, we will alter $S^b_{n\alpha}$ and $T^b_{n\alpha 2}$. Denote the altered tori by $\hat{S}^b_{n\alpha}$ and $\hat{T}^b_{n\alpha 2}$, respectively. (See Figure 45.) (We will also push subsets of $D'_\epsilon - H^*$ in such a way that the set still satisfies the definition of D'_ϵ.) The new tori will have the following properties:

(1) $\hat{S}^b_{n\alpha} \supset S^b_{n\alpha 1} \cup \hat{T}^b_{n\alpha 2}$,

(2) $S^b_{n\alpha 1}$ and $\hat{T}^b_{n\alpha 2}$ are linked in $\hat{S}^b_{n\alpha}$,

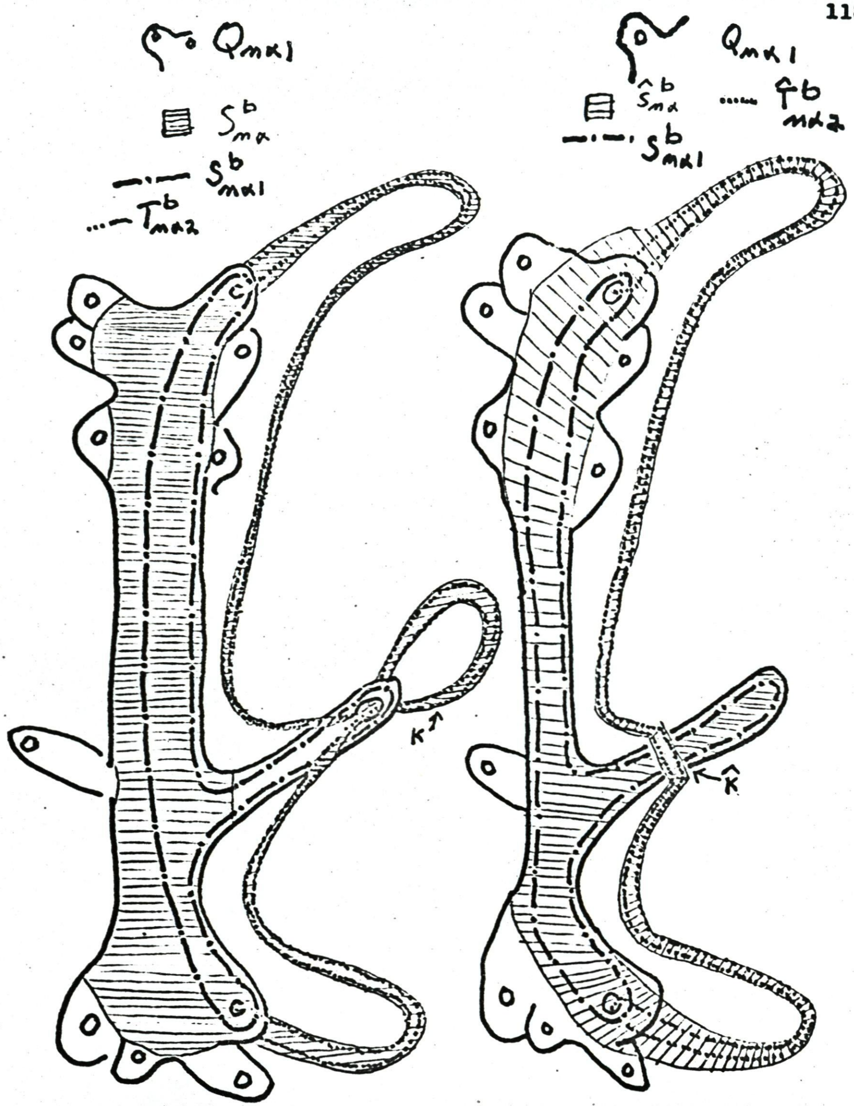

$Q_{m\alpha 1}$
$S^b_{m\alpha}$
$S^b_{m\alpha 1}$
$T^b_{m\alpha 2}$
C
G
K
$Q_{m\alpha 1}$
$\hat{S}^b_{m\alpha}$
$\hat{T}^b_{m\alpha 2}$
$S^b_{m\alpha 1}$
$\hat{K}$

Figure 45

(3) $\hat{S}^b_{n\alpha}$ and $\hat{T}^b_{n\alpha}$ are in general position with respect to D'_ϵ, and,

(4) If δ is a meridional subdisk of $S^b_{n\alpha} \cap D'_\epsilon$, then there is a meridional subdisk $\hat{\delta}$ of $\hat{S}^b_{n\alpha}$ such that $\hat{\delta}$ has no meridional subdisks in $\hat{T}^b_{n\alpha 2}$ and $\delta \cap H^* = \hat{\delta} \cap H^*$.

The steps in constructing these altered tori are:

(a) Near the knitting, a cylinder $\hat{K} \subset E^3 - D'_\epsilon$ is chosen that will be used to replace a cylinder K of the torus $S^b_{n\alpha}$, and thereby remove a loop of knitting. The cylinders K and $\hat{K}$ have the same ends, and Bd $\hat{K}$ is in general position with respect to D'_ϵ.

(b) If there is any meridional disk of D'_ϵ in $S^b_{n\alpha}$ that has its boundary in $K \cap$ Bd $S^b_{n\alpha}$, then push $D'_\epsilon - H^*$ in the neighborhood of the boundary in such a way that

(i) the intersections of D'_ϵ with $T^b_{n\alpha 2}$ and $S^b_{n\alpha 1}$ are unaltered,

(ii) no meridional disk has its boundary in $K \cap$ Bd $S^b_{n\alpha}$,

(iii) D'_ϵ is not pushed into $\hat{K}$, and

(iv) D'_ϵ is in general position with respect to all manifolds in the construction.

(c) Make the substitution of $\hat{K}$ for K. Substitute for $K \cap D'_\epsilon$ its homeomorphic copy in $\hat{K}$, thereby forming an altered disk D'_ϵ. Similarly, substitute a copy of $K \cap T^b_{n\alpha 2}$ in $\hat{K}$, thereby forming the new torus $\hat{T}^b_{n\alpha 2}$.

Alter slightly for general position, if necessary. This construction implies that $\hat{T}^{b}_{n\alpha 2}$ does not contain a meridional subdisk of D'_{ϵ}.

Now note the knitting protrusion (Figure 45) on $Q_{n\alpha}$ in the bth quadrant and not contained in $S^{b}_{n\alpha}$.

(d) If there is a meridional subdisk δ of $S^{b}_{n\alpha} \cap D'_{\epsilon}$ such that Bd δ intersect the protrusion, then in Bd $S^{b}_{n\alpha}$ push Bd δ off the protrusion. Do this in such a way that D'_{ϵ} remains a disk with desired general position properties and the intersections with $S^{b}_{n\alpha 1}$ and $\hat{T}^{b}_{n\alpha 2}$ are unaltered.

(e) Add the protrusion. As illustrated, fill in the hole through which the knitting loop K extended. The resulting torus is $\hat{S}^{b}_{n\alpha}$. It has two meridional disks that do not have meridional subdisks in $\hat{T}^{b}_{n\alpha 2}$, so there must be four meridional subdisks of D'_{ϵ} in $S^{b}_{n\alpha 1}$.

We can repeat this argument in successive stages. By arguments similar to those in the proof of Lemma 4.1, only one of the four meridional disks in $S^{b}_{n\alpha 1}$ can intersect g_{o}. This implies that there must at some stage be a torus with four meridional subdisks. Use of Lemma 4.1 then leads to a contradiction. Hence, Case 3 cannot occur, and Lemma 4.3 is proven. $\square$

Notation. Let $\hat{C}$ be the component of $C \cap M_1$ that

intersects g_o.

Recall that ϵ was chosen so that $P^{-1}N_\epsilon(P(c_j))$ lies in r_j and $P^{-1}N_\epsilon(P(c_j))$ misses $\text{Bd } M_1 - \bigcup_{i=0}^{4} r_j$. (p. 92) The simple closed curve denoted by c_o is now Bd $\hat{C}$. Its regular neighborhood in a particular triangulation of $E^3 - H^*$ is r_o. Hence, r_o is a solid torus that misses H^*, and contains c_o. (See Figure 46.)

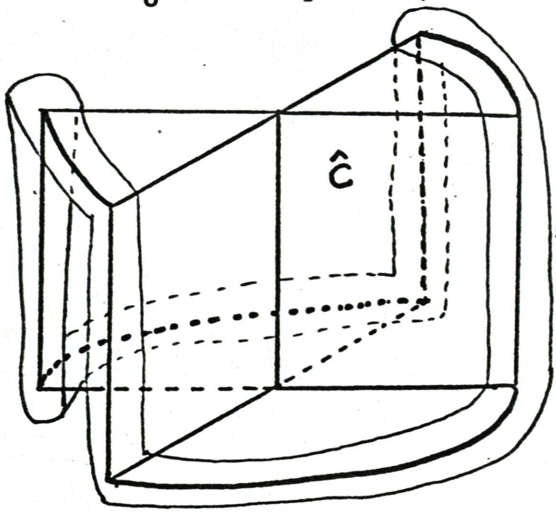


Figure 46

<u>Notation</u>. Let $\hat{D}_\epsilon'$ denote the meridional subdisk of D_ϵ' in M_1 such that $\text{Bd } \hat{D}_\epsilon' \subset r_o$. For later use, note that $\hat{C} \cup \hat{D}_\epsilon' \cup r_o$ separates E^3.

<u>Lemma 4.4</u>. For any $g \in H$, g intersects $\hat{D}_\epsilon'$ if and only if it intersects $\hat{C}$.

Proof. Note that g intersects $\hat{C}$ is equivalent to $g \in H \cap Q_{11}$. Suppose that there is an element g_1 of

$H \cap Q_{11}$ that does not intersect $\hat{D}'_{\epsilon}$. Lemma 4.3 then implies that g_1 intersects D'_{ϵ}. Now suppose that g_1 intersects the component of $D'_{\epsilon} \cap M_1$ that is a meridional disk lying in the kth quadrant. Call this component D^k. By arguments similar to those of Lemma 4.3, we find that for every $g \in H \cap Q_{11}$, it is true that g intersects D^k. For each quadrant $m = 1, 2, 3$, or 4 consider a simple closed curve that contains g_o and passes through the handle of M_1 that is vertical in Figure 34 and lies in the mth quadrant. Notice that Bd $\hat{D}'_{\epsilon}$ links each of these. This implies that $\hat{D}'_{\epsilon}$ must intersect some element of g lying in $Q_{11} \cup T^m_{12}$. We have just shown that all elements of $Q_{11} \cap H$ miss $\hat{D}'_{\epsilon}$. Therefore, $\hat{D}'_{\epsilon}$ must intersect an element in $H \cap T^m_{12}$. Repeating the argument of Lemma 4.1, we find that $\hat{D}'_{\epsilon}$ must intersect all elements in $H \cap T^m_{12}$. Hence, the component D^m of $D'_{\epsilon} \cap M_1$ in the mth quadrant intersects no elements in $H \cap T^m_{12}$. Therefore, D^m must intersect those elements in $H \cap Q_{11}$. Again using the method of Lemma 4.3, we find that D^m intersects every $g \in Q_{11} \cap H$. Since this is true for $m = 1, 2, 3$, and 4, we have four intersections with each $g \in H \cap Q_{11}$. This contradicts the assumption that $P(D'_{\epsilon})$ is a disk. Therefore, g intersects $\hat{C}$ implies that g intersects $\hat{D}'_{\epsilon}$.

Now suppose that there is an element g_2 that inter-

sects $\hat{D}'_\epsilon$ and misses $\hat{C}$. Then $g_2 \not\in H \cap Q_{11}$. Let T^k_{12} be the component of M_2 containing g_2. By the argument of Lemma 4.1 it can be shown that $\hat{D}'_\epsilon$ intersects every $g \in H \cap T^k_{12}$. Then the component D^k of $D'_\epsilon \cap M_1$ in the kth quadrant must intersect all elements of $H \cap Q_{11}$. This in turn implies that $\hat{D}'_\epsilon$ must intersect all elements of $H \cap T^m_{12}$ for each $m = 1, 2, 3$, and 4, and leads to the same contradictions as above. Therefore, g intersects $\hat{D}'_\epsilon$ implies that g intersects $\hat{C}$. $\square$

<u>Notation</u>. Recall that H consists of a countable collection of Cantor sets of arcs knit in a specific manner described above. Let $\mathcal{A}$ consist of g_o plus exactly one element from each of these Cantor sets of arcs. Notice that $\mathcal{A}$ is the union of two countable sets, each of which is knit from $p_1 \in g_o$ to $p_2 \in g_o$. Associated with the set $\mathcal{A}$ there is a decomposition of E^3 with $\mathcal{A}$ as the set of nondegenerate elements. Let us call this decomposition $G_{\mathcal{A}}$. Let $P_{\mathcal{A}}: E^3 \to E^3/G_{\mathcal{A}}$.

<u>Lemma 4.5</u>. If there exists a disk D'_ϵ, then $P_{\mathcal{A}}(D'_\epsilon)$ is also a disk.

Proof. For each $g \in H$, $g \cap D'_e$ is a connected set that does not separate D'_ϵ and $\mathrm{Cl}\, H^* \subset \mathrm{Int}\, D'_\epsilon$. Hence, the same is true for each $g \in \mathcal{A}$, and $\mathrm{Cl}\, \mathcal{A}^* \subset \mathrm{Int}\, D'_e$.

$\mathcal{G}$ is an upper semicontinuous decomposition. A theorem of Laurence C. Siebenmann [28] gives us that the decomposition $\{g \cap D'_\epsilon \mid g \in \mathcal{G}\}$ is shrinkable using McAuley's definition [22], and one of McAuley's theorems [22] can be used to show that D'_ϵ mod this decomposition of D'_ϵ is a disk. But this is $P_{\mathcal{G}}(D'_\epsilon)$, so we have shown that it is a disk. $\square$

It is instructive at this point to see why this <u>countable</u> knit set $\mathcal{G}$ does not suffice for the example. The requirement that two points be ϵ-close in the image space is equivalent to the requirement that they lie in the same component of an element of the defining sequence at an appropriate stage. There are disks that agree with the 2-complex C except inside a central component of the defining sequence. Let the disk Z shown in Figure 47 be such a disk. Let it be chosen so that there is one $g_1 \in \mathcal{G}$ such that g_1 intersects Z in 3 points as shown in Figure 48. We can now remove one of these intersections by making the bubble on Z shown in Figure 49. It can be very close to g_1. The second bubble shown in Figure 50 removes a second intersection. With these changes in Z_1 we now have a disk whose image is a disk ϵ-homeomorphic to D. Of course, in the knit Cantor set construction that we chose to make, such bubbles would intersect nondegenerate

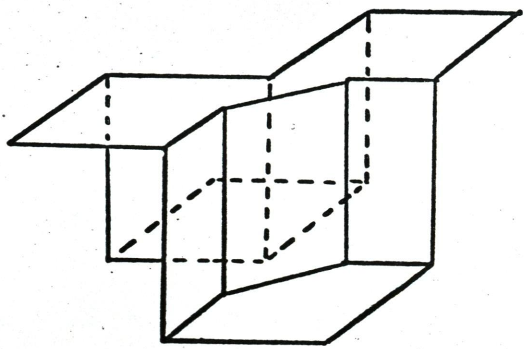


Figure 47

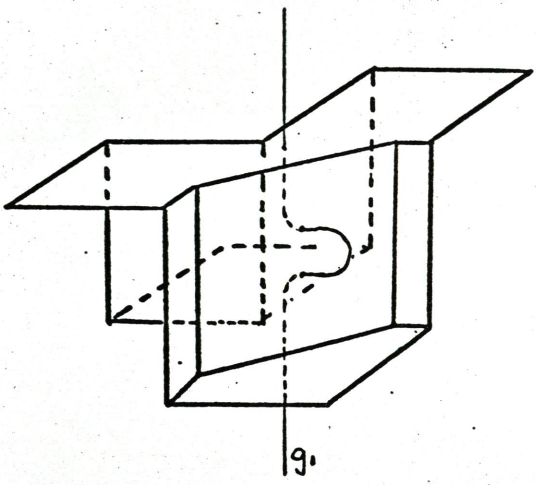


Figure 48

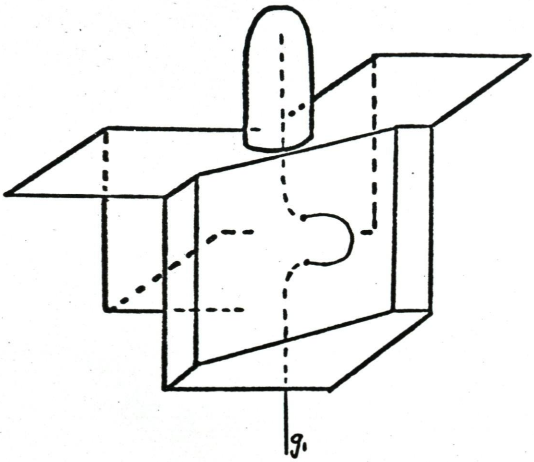


Figure 49

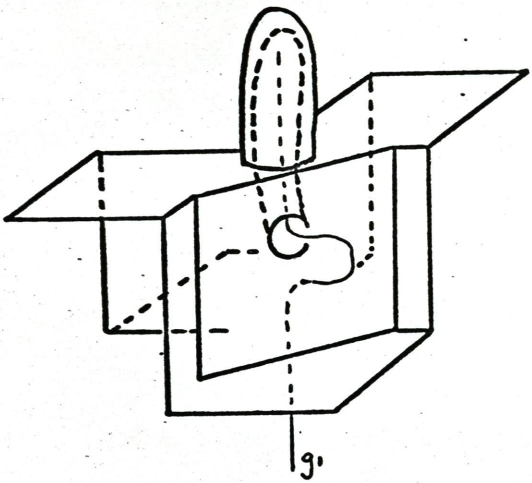


Figure 50

elements.

<u>Definition</u>. We will say that there are four <u>quadrants</u> of E^3, denoted for $i = 1, 2, 3$, and 4 by Q^i, where Q^i is the set resulting from crossing the z-axis with the ith quadrant of the x-y plane.

Let p_R and p_S be the endpoints of g_o lying in planes R and S, respectively. Let q_1 and q_2 be points in g_o between p_1 and p_2 and such that the order of these points in g_o is $p_R, p_2, q_2, q_1, p_1, p_S$. Let W_2 be the open half space of E^3 above the z-coordinate of q_2 and W_1 be the open half space of E^3 below the z-coordinate of q_1.

<u>Lemma 4.6</u>. The existence of the disks D'_ϵ and $\hat{D}'_\epsilon$ implies the existence of disks E and $\hat{E}$ with the following properties:

(1) $\hat{E} \subset \text{Int } E$.

(2) $\text{Bd } \hat{D}'_\epsilon = \text{Bd } \hat{E}$.

(3) The $P_{\mathcal{Q}}$-images of E and $\hat{E}$ are disks.

(4) E is locally polyhedral off g_o.

(5) For any $A \in \mathcal{Q}$, $A \cap D'_\epsilon \neq \emptyset$ implies that $A \cap E$ is a single point contained in $A \cap D'_\epsilon$.

Proof. Construction of the disks E and $\hat{E}$ depends on the use of the following set of toroidal components For each $A^i_j \in \mathcal{Q} - \{g_o\}$ lying in Q^i there is the torus T^i_j containing A^i_j used in the definition of A^i_j. (We are

now using T^i_j as a simpler notation for $T^i_{n\alpha 2}$ when $n\alpha$ consists of j digits, each of which is a one.) These tori are mutually exclusive. The disk D'_ϵ intersects each T^i_j in two disjoint meridional disks δ_a and δ_b of T^i_j. One of these meridional disks, δ_a, contains a subset of A^i_j and the other one, δ_b, misses $\mathcal{a}^*$ completely.

The disk δ_b can be approximated by a polyhedral disk that is within $1/2^j$ of δ_b, agrees with δ_b on Bd T^i_j, and misses $\mathcal{a}^*$. This approximation is done using Bing's statement quoted on p. 78 of this thesis. The disk δ_a can also be approximated within $1/2^j$ of δ_a by a disk that agrees with δ_a on Bd T^i_j. Furthermore, the approximation can be chosen so that it intersects $\mathcal{a}^*$ in exactly one point p and p lies in $A^i_j \cap D'_\epsilon$. We find this approximation in the following manner. We can assume that the above form of Bing's Approximation Theorem has been used to make δ_a be locally polyhedral off A^i_j. To the element A^i_j add a line segment at each end in such a manner that

(i) these segments, a_1 and a_2, lie in Int T^i_j,

(ii) a_1 and a_2 miss the approximation of δ_b,

(iii) neither a_1 nor a_2 intersects D'_ϵ except possibly in a point of $A^i_j \cap D'_\epsilon$, and

(iv) in the latter case in (iii), choose a_1 and a_2 so that they lie on opposite sides of δ_a.

Now let B be a polyhedral 3-cell such that

(i) $[T^i_j] \cap [N_{1/2^{j+1}}(A^i_j \cap \delta_a)] \supset B \supset \text{Int } B \supset A^i_j \cap D'_\epsilon$,

(ii) B misses the approximated disk δ_b,

(iii) B is in general position with respect to δ_a, and

(iv) Bd B is pierced by $a_1 \cup A^i_j \cup a_2$ in exactly two points, x_1 and x_2, and these points lie in different components of $T^i_j - (\delta_a \cup \delta_b)$. (It can be shown by an argument similar to the proof of Lemma 4.3 that if there are two components of $A^i_j - D'_\epsilon$, then they lie in different components of $T^i_j - (\delta_a \cup \delta_b)$.)

With these conditions, the set $\delta_a \cap$ Bd B consists of a finite number of simple closed curves. Let J_o be the one of these that bounds a disk r in δ_a containing no other one of this set of simple closed curves and containing the set A^i_j. (See Figure 51.) Now in B there is a polyhedral disk s such that

(i) $s \supset A^i_j \cap D'_\epsilon$,

(ii) Bd s = Bd r , and

(iii) s misses $\delta_a - r$, $A^i_j - D'_\epsilon$, and Int $(a_1 \cup a_2)$.

Replacing r by s, we get a polyhedral approximation that is the desired closeness and agrees with δ_a on Bd T^i_j. We wish now to change the intersection from the entire set $A^i_j \cap D'_\epsilon$ to a single point in this set. We do this with a piecewise linear homeomorphism that moves only points in a small neighborhood N of $A^i_j \cap D'_\epsilon$. This

homeomorphism is the one obtained by first taking $N \cap s$ to the flat disk containing the line segment shown in Figure 52, performing the rotating homeomorphism indicated by the dotted lines, and then mapping back into T^i_j. We now have the desired approximation for δ_a. It is pierced by A^i_j in a point of $A^i_j \cap D'_\epsilon$.

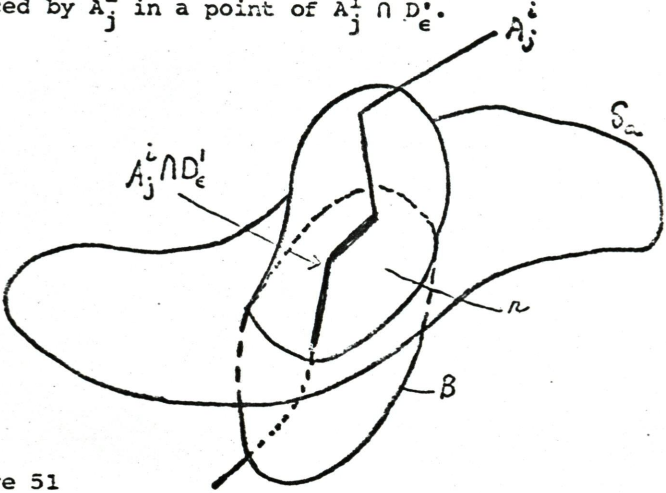


Figure 51

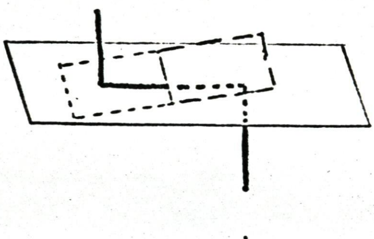


Figure 52

By making the above approximations in each T^i_j, we get E and $\hat{E}$. $\square$

<u>Lemma 4.7</u>. For each quadrant Q^i, either

(a) There exists an arc α^i in $\hat{E}$ satisfying:

(i) One endpoint is in $(\mathrm{Bd}\ \hat{E}) \cap \mathrm{Int}\ Q^i$.

(ii) The other endpoint is $p_1 \in g_o$ if $i = 1$ or 3 and $p_2 \in g_o$ if $i = 2$ or 4.

(iii) Near g_o the arc α^i is contained in W_1 if $i = 1$ or 3 and W_2 if $i = 2$ or 4.

(iv) $\alpha^i \subset (\mathrm{Int}\ Q^i) \cup W_1$ or $(\mathrm{Int}\ Q^i) \cup W_2$.

or,

(b) In Q^i the disk E can be altered so that it satisfies

(1) - (5) above and also

(6) For any $A \in \mathcal{A}$, the endpoints of A lie in the closure of the unbounded component of $E^3 - (\hat{C} \cup \hat{E} \cup r_o)$.

Proof. By defining a positive orientation of the simple closed curve $\mathrm{Bd}\ \hat{C}$, it is possible to define the sides of C and E as upper and lower sides. If $C \cap A_j^i \neq E \cap A_j^i$, then either the component of A_j^i between C and E (a) pierces one of C and E from the upper side and the other from the lower side, or (b) pierces both from their lower or both from their upper sides. Now consider the construction shown in Figure 53. The disks D_a and D_b are the meridional subdisks that we have just constructed from meridional subdisks of D'_ϵ. The nondegenerate element A_j^i intersects D_a. The dotted lines indicate a construction that interchanges subdisks in the interiors of D_a and D_b in such a way that the new meridional disk bounded by $\mathrm{Bd}\ D_b$

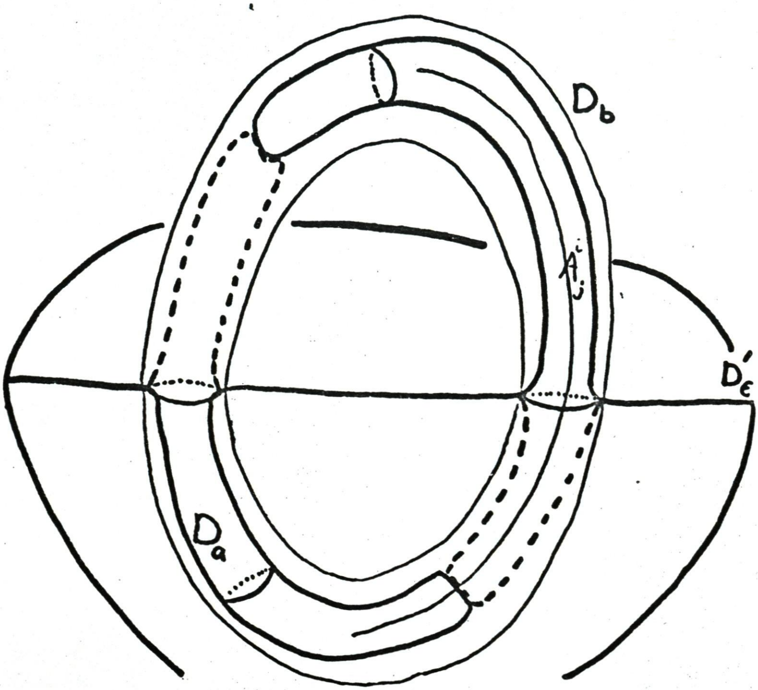


Figure 53

now intersects A_j^i in one point and the new meridional subdisk bounded by Bd D_a now misses α^*. Observe that if property (6) in the conclusion of this lemma were not true for A_j^i, then performing this construction would make property (6) be true for that A_j^i.

If this construction were made in an infinite number of tori, the resultant set might not be a disk. We will now show that if it were necessary to perform the construction an infinite number of times, then the original set E contains the arc α^i and thus satisfies

condition (a).

Let $Q_{n\alpha}^{\bullet}$ be the nontoroidal component that contains T_j^i two stages later. Let $D_1 \subset D_\epsilon'$ be the central meridional disk of $Q_{n\alpha}$. It intersects T_j^i. Let D_{11} be the central meridional disk of $Q_{n\alpha 1}$ and D_{12} be the meridional subdisk of $Q_{n\alpha 1}$ in quadrant Q^i. It is D_{12} that intersects T_j^i. In T_j^i let $D_{12k} \subset D_{12} \subset D_1$ be the meridional subdisks that contain A_j^i. (The boundaries of these subdisks of D_ϵ' were not changed in constructing E.) With respect to g_o extended to $\pm\infty$, orientations of Bd D_1 and Bd D_{11} are the same as the orientation of Bd $\hat{C}$. This can be shown in a manner similar to Lemma 4.4. Similarly, it can be shown that with respect to the extended A_j^i the orientations of Bd D_1 and Bd D_{12} must also be the same as that of Bd $\hat{C}$. It is possible that the orientation of Bd D_{12k} with respect to the extended A_j^i is not the same as the orientation of Bd $\hat{C}$. Now there is a homeomorphism that takes D_{12} to a flat disk but doesn't move the extended A_j^i near $\pm\infty$. It gives us Figure 54. This shows that D_{12} must contain points above plane R and below plane S.

Now, let us suppose that we are working in quadrant Q^1. If we need the infinite sequence of constructions to reverse orientations, then there is an infinite sequence of points $x_1, x_2, x_3, \ldots$ such that each x_n lies

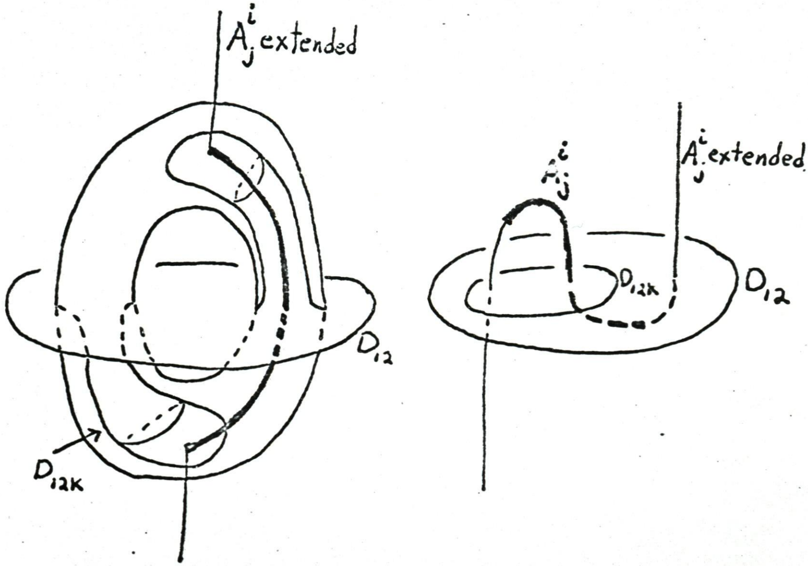


Figure 54

in Int Q^1 and below plane S. The limiting point for this sequence is the point p_S, which is in W_1. If we needed the infinite sequence of constructions, then p_S also lies in E. Each of the points x_n in the above sequence lies in $W_1 \cap E$. Since E is a disk, it is locally connected. Therefore, there is an arc B_a from p_S to one of the points x_n and such that B_a lies in $W_1 \cap E$. From x_n to a point in (Bd $\hat{E}$) $\cap$ (Int Q^1) there is another arc B_b in Int Q'. The union of B_a and B_b is the desired arc α^1. In the other quadrants, similar arcs can be found. For later use, if it happens that

the arc α^3 intersects α^1, then the intersection is in $W_1 \cap E$. Replace the subarc of α^3 between this point and p_S by the corresponding subarc of α^1. It is not possible for α^1 or α^3 to intersect α^2 or α^4. This completes Lemma 4.7. $\square$

Let us assume that in each quadrant Q^i conclusion (a) holds or that E has been altered to satisfy conclusion (b).

Fox and Artin [10], in defining their wild arcs, described arcs in a cylinder and then copied this cylinder into each of an infinite number of sections of an ellipsoid of revolution. We will provide a similar description of the knit portion of $\mathcal{a}$. In Figure 55 a cube is shown containing arcs. In Figure 56 a pyramid is divided into an infinite number of frusta and the point p_2. Each frustum is a homeomorphic copy of the cube in Figure 55. They are joined in such a way that we have the sequence of arcs knit to p_2 shown in Figure 56. Similarly, a pyramid can be used to form a set of arcs knit from p_1. Such a pair of pyramids with their bases identified define knitting from a point p_1 to a point p_2. Let π_1 and π_2 be piecewise linear copies in E^3 of this defining pair of pyramids. We can orient π_1 and π_2

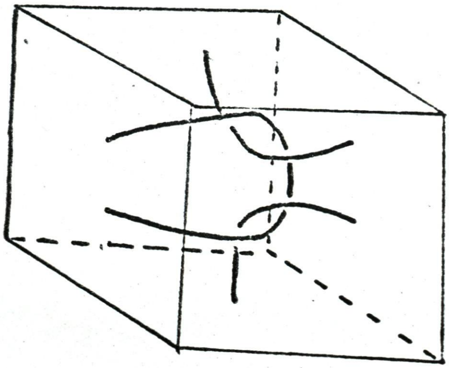


Figure 55

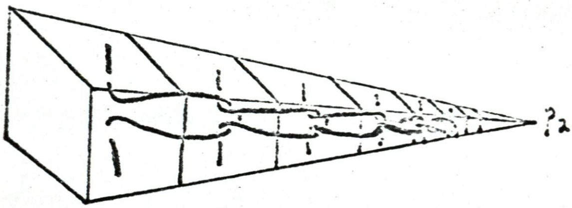


Figure 56

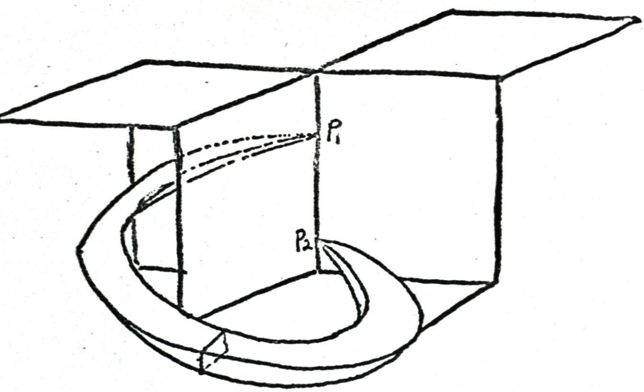


Figure 57

in E^3 in such a way that they contain our knitting. In Figure 57 we show π_1 and C. We can assume that π_1 and M_1 are such that $\pi_1 \subset M_1 \subset E^3$ and that $\pi_1 \cup g_o$ contains a simple closed curve that passes through the "front" handle of M_1. (Figure 35) Similarly, π_2 lies in the "back" handle.

We will be concerned with the location of frustum faces that have their interiors in Int π_1 and Int π_2.

Lemma 4.8. If in a quadrant Q^i conclusion (b) of Lemma 4.7 holds, then no frustum face in Int π_1 or Int π_2 in Q^i lies in a bounded component of $E^3 - (\hat{C} \cup \hat{E} \cup r_o)$.

Proof. We first show that there is such a face of π_1 in the unbounded component of $E^3 - (\hat{C} \cup \hat{E})$ in each of quadrants Q^1 and Q^4, or π_2 for Q^2 and Q^3. In each quadrant there is the $A_1^i \in \mathcal{A}$ that does not intersect $\hat{C}$. There is a face f_1^i containing points of A_1^i. Lemma 4.4 implies that A_1^i does not intersect $\hat{E}$. The endpoints of A_1^i lie in the closure of the unbounded component of $E^3 - (\hat{C} \cup \hat{E})$. Hence, A_1^i does also. Now, with $f_1^i \cap A_1^i$ in the unbounded component, we need only make replacements of subsets of $\hat{E}$ that may intersect f_1^i in order to have the whole face f^i in the unbounded component.

Now suppose that one of two faces f_m^i and f_n^i lies in the unbounded component and the other lies in a bounded

component of $E^3 - (\hat{C} \cup \hat{E} \cup r_o)$. In proving this lemma we need be concerned with only those elements of $\mathcal{A}$ that intersect the faces f_m^i and f_n^i and the finite set of faces between them in π (where π denotes π_1 or π_2, appropriately). Let the elements of $\mathcal{A}$ that intersect these faces be called $A_1, A_2, \ldots, A_k$, as shown in Figure 59.

In a copy of Figure 52 we label the arcs uv, wx, and yz as shown. (Figure 58) In Int uv arbitrarily choose two subarcs. These are L_1 and L_2 in the figure.

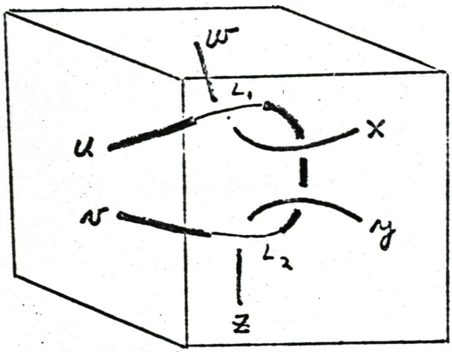


Figure 58

Copying this cube into each frustum of π between f_m^1 and f_n^1 gives us the light arcs as copies of L_1 and L_2 in Figure 62. We wish to show that in each frustum we can push the copy of L_1 so that it contains a point in the copy of wx and similarly to push the copy of L_2 so that it contains a point in the copy of yz. In these pushes we do not wish to add other intersections with

nondegenerate elements or intersections with the disks. This will make the altered $\bigcup_{i=1}^{k} A_i$ be homeomorphic to the set in Figure 63.

In Figure 58 we have added arcs in the boundary of the cube. This gives us three linked simple closed curves.

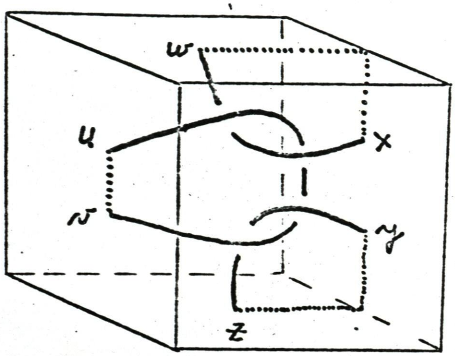


Figure 59

Let this figure be the copy of the cube Z in a particular frustum Y. We can assume that E is in general position with respect to Bd Y. Then E $\cap$ Bd Y consists of simple closed curves and E $\cap$ Y is punched disks. If in Y one of these punched disks separates uv from wx or yz, then we will need to make alterations inside Y. We will change E and $\hat{E}$ to disks that we will call E_1 and $\hat{E}_1$. We require that E_1 and $\hat{E}_1$ satisfy conditions (1) - (6) except that the intersection point with an element A $\in$ $\mathcal{A}$ is no longer necessarily in A $\cap$ $D_e^{\bullet}$. Note that it is true that $P_{\mathcal{A}}(E_1)$ and $P_{\mathcal{A}}(\hat{E}_1)$ are disks. We will find

that if a frustum face in Int π lies in a bounded component of $E^3 - (\hat{C} \cup \hat{E}_1 \cup r_o)$, then $P_\alpha(E_1)$ is not a disk. This contradiction will prove the lemma for the original $\hat{E}$.

Suppose that in $\hat{E} \cap$ Bd Y there is a simple closed curve J_1 such that J_1 bounds a disk in Bd Y containing exactly one of the points u, v, w, x, y, or z. Then for the A_i containing that point it is possible to push in the neighborhood of A_i and remove this intersection of E with Bd Y. In this manner all such intersections of E^- with faces of frusta can be removed and we then know that any remaining intersections bound disks in Bd Y containing more than one point of the set u, v, w, x, y, and z. We have removed intersections of the sort shown in Figure 59.

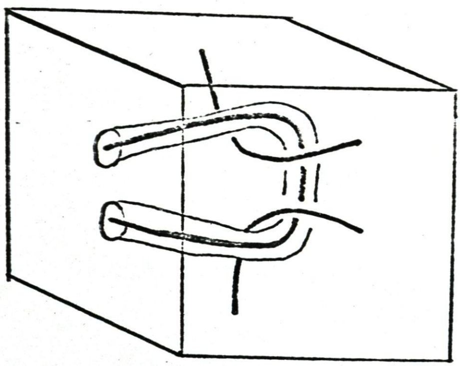


Figure 60

We now have the sets E_1 and $\hat{E}_1$. The only remaining manner in which a punched disk contained in E_1 can separate Y into two components such that one component contains uv is indicated in Figure 61.

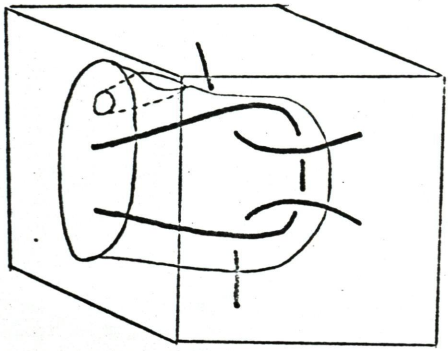


Figure 61

Now, the linking of the simple closed curves in Figure 59 requires that a point of each wx and yz lies in the component containing uv. Pushing $\hat{E}_1$ a bit inside Y now allows the desired alterations.

Figure 62 is homeomorphic to the grid in Figure 61. Notice the positions of points b_m, b_n, c_m, and c_n. The disk E_1 intersects each $A_i \in \mathcal{A}$ for $1 \leq i \leq k$ in one point, which lies in the heavy part of the arc $a_i b_i b_{i+1} c_{i+1} c_i d_i e_i$. The points a_i and e_i are the endpoints of a nondegenerate element. Recall that they lie in the unbounded component of $E^3 - (\hat{C} \cup \hat{E}_1 \cup r_o)$.

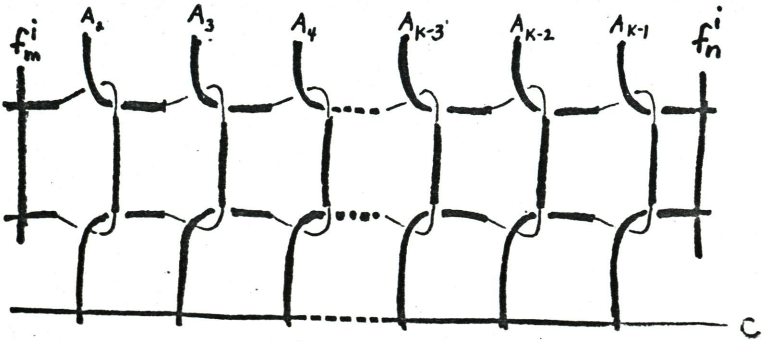


Figure 62

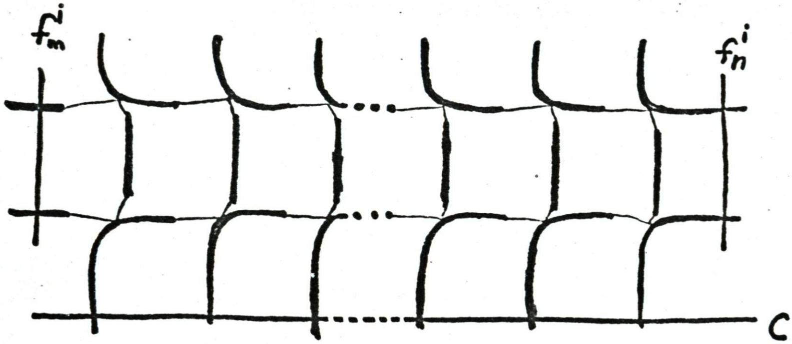


Figure 63

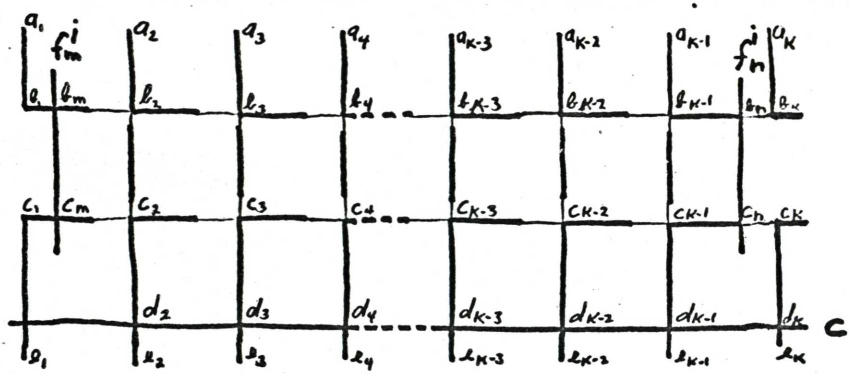
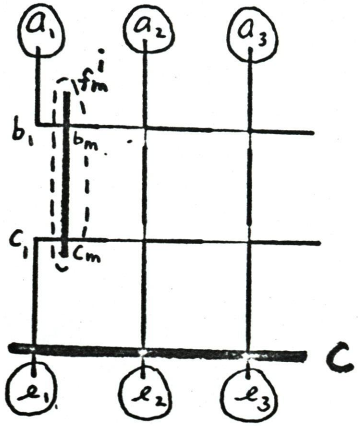


Figure 64

Suppose that one face, f_m or f_n, lies in the unbounded component and the other face lies in a bounded component.

First, suppose that f_m^i lies in a bounded component K. Then the arc $a_1b_1b_m$ intersects $\hat{E}_1$. In Figure 62 notice that points or sets that we know lie in the unbounded component are encircled in a solid line and those that we know lie in a bounded component are encirled in a dotted line. The intersection of $a_1b_1b_m$ is the only allowed intersection of $\hat{E}_1$ with A_1. Hence, as shown in Figure 63, b_2 and c_2 of A_1 lie in K. Therefore, a_2b_2 intersects $\hat{E}_1$ and b_3 and c_3 must lie in K. Continuing across, as shown in Figure 64, we find that b_n and c_n lie in a bounded component. This proves the lemma in the case that f_m^i lies in a bounded component.

Now, suppose that f_n^i lies in a bounded component L. We then have Figure 65. As shown in Figure 65, $a_{k-1}b_n$ intersects $\hat{E}_1$. If $a_{k-1}b_{k-1}$ intersects $\hat{E}_1$, then $a_{k-2}b_{k-1}$ must intersect $\hat{E}_1$. (Figure 66) If, for each i such that $2 \leq i \leq k - 1$, the segment a_ib_i intersects $\hat{E}_1$, then we have Figure 67. Notice that since $\hat{E}_1$ intersects a_2b_2, it must be that c_2 lies in L. We now have a contradiction in that it appears that $\hat{E}_1$ must intersect both b_nb_2 and c_nc_2, which lie in the same nondegenerate element. Now if, for some i, the segment $b_{i-1}b_i$



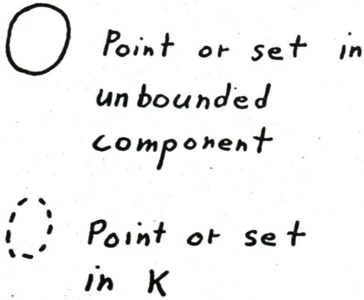


Figure 65

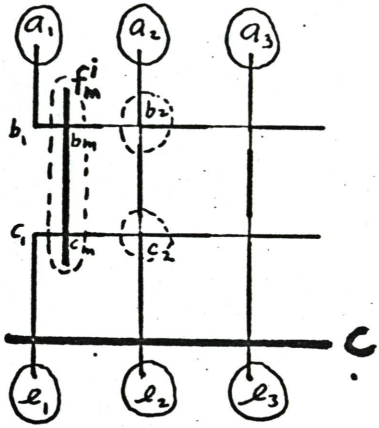


Figure 66

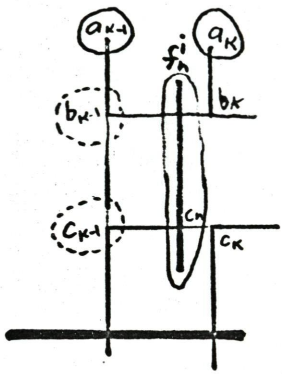


Figure 67

Point or set in unbounded component

Point or set in L ...... $\hat{E}_1$

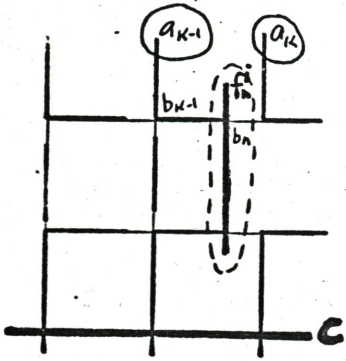


Figure 68

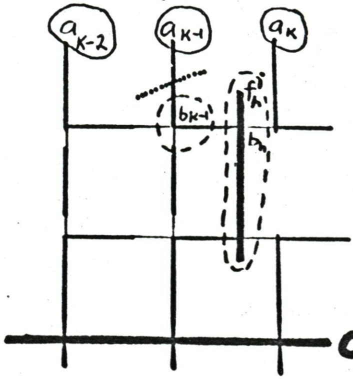


Figure 69

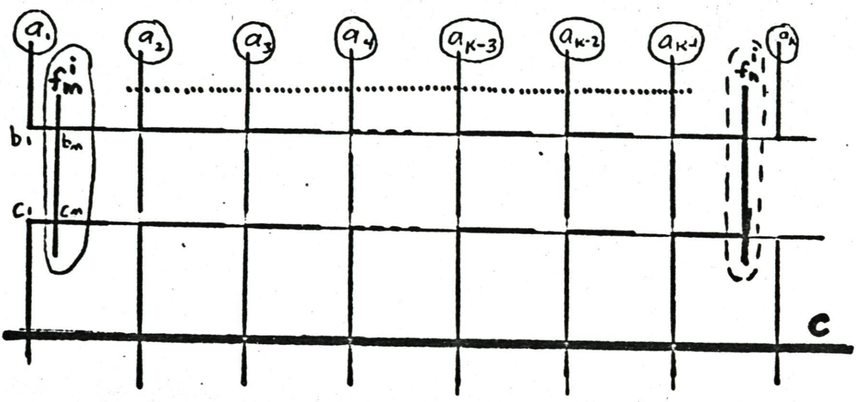


Figure 70

Point or set in unbounded component

Point or set in L

— $\hat{E}_1$

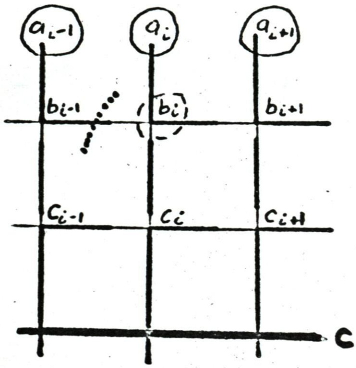


Figure 71

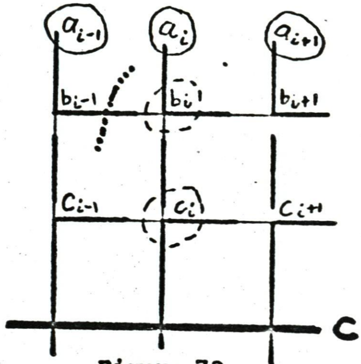


Figure 72

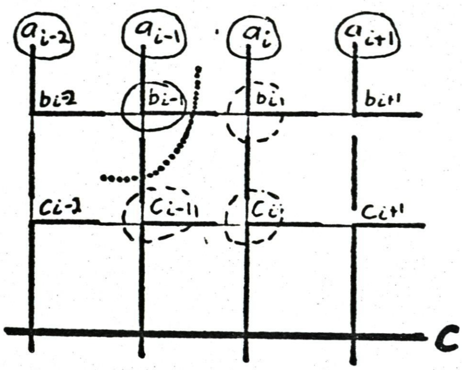


Figure 73

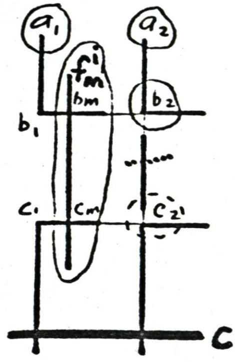


Figure 74

intersects $\hat{E}_1$, then we have Figure 68. Since $b_{i-1}b_i$ intersects $\hat{E}_1$, the segment c_ic_{i-1} cannot. (Figure 69) Therefore, $b_{i-1}c_{i-1}$ must intersect $\hat{E}_1$, as shown in Figure 70. Continuing across, for $j < i-1$, the nondegenerate element A_j intersects $\hat{E}_1$ in b_jc_j and, hence, cannot intersect $\hat{E}_1$ in any other segment. We have now Figure 71. Since b_2c_2 intersects $\hat{E}_1$, the segment c_2c_m must not intersect $\hat{E}_1$. This contradicts the assumption that f_m^i lies in L, and proves Lemma 4.8. $\square$

<u>Lemma 4.9</u>. In each quadrant Q^i there is an arc α^i, as defined in Lemma 4.7.

Proof. If there is no such arc in a quadrant Q^i, then near g_o in Q^i the disk $\hat{E}$ lies entirely above the half space W_1 in Q^1 or Q^3, or entirely below the half space W_2 in Q^2 or Q^4. But this then implies that there is a face of a frustum in a bounded component of $E^3 - (\hat{C} \cup \hat{E})$. This contradiction proves the lemma. $\square$

We can now complete the proof of our knit example. The set $g_o \cup \bigcup_{i=1}^{4} \alpha^i$ separates $\hat{E}$ into four sets whose closures are disks. Each has the segment q_1q_2 of g_o in its boundary. Four such disks can not be subdisks of a disk. This contradiction proves that there can be no disk D_ϵ in our example. $\square$

Remarks concerning properties of the decomposition.

At the beginning of this chapter we stated five properties of the decomposition. We now comment about each.

(1) E^3/G is homeomorphic to E^3.

Theorem (Bing [10]). Suppose G is a monotone upper semi-continuous decomposition of E^3, and H is the sum of the non-degenerate elements of G. Then the decomposition space of G is topologically E^3 if each component of H is an element of G, and for each open set U containing H and each $\epsilon > 0$, there is a homeomorphism h_ϵ of E^3 onto itself which is fixed on $E^3 - U$ and takes each component of H onto an ϵ set.

We will apply this to our decomposition. Let an open set U containing H* and an $\epsilon > 0$ be given. By the argument on p. 53 of this thesis, there is an element M_i of the defining sequence such that $M_i \subset U$. There is a homeomorphism h_1 that takes the nontoroidal component of M_i to diameter $< \epsilon$. This, of course, may stretch other components of M_i. Each other component is a torus and in each of these tori the decomposition is (2,1) toroidal. In [6] Bing proves that there exists a homeomorphism h_e that takes each nondegenerate element of a (2,1) toroidal decomposition to diameter less than ϵ and is fixed off the torus.

Hence, there is a homeomorphism h_2 that is fixed off the stretched toroidal components of M_i and takes each nondegenerate element in them to diameter less than ϵ. Therefore, $h_2 h_1$ is our desired homeomorphism. This proves that E^3/G is homeomorphic to E^3 in our example.

(2) Each $g \in H$ is a tame arc.

This is obviously true for g_o. Recall Fort's work [13] refered to earlier. If we assume that defining manifolds for a (2,1) toroidal decomposition are specified in a similarly nice manner, then each nondegenerate element of that decomposition is polyhedral. In the knit decomposition of our example consider each toroidal component T that is linked to a nontoroidal component. The nicely specified manifolds for the above (2,1) decomposition can be piecewise linearly mapped into T. Hence, each nondegenerate element can be polyhedral. This implies that it is a tame arc.

(3) H is continuous and closed.

This follows from the fact that each element of H spans the pair of planes R and S. Hence, H* is equal to Cl H*, which is closed. Also, the spanning implies that no nondegenerate element properly contains the limiting set of other nondegenerate elements. This

implies that H is continuous.

(4) P(H) is a Cantor set.

This follows because more than one component of the defining sequence is imbedded in each one at the previous stage, and H* is equal to Cl H*.

(5) H is not countable and is not definable by 3-cells.

This follows from Theorems 3.1, 3.2, and 4.1.

Concerning a Question asked by Armentrout. In [2] Armentrout asks (Question 8): Suppose G is a point-like decomposition of E^3. If S is a 2-sphere in E^3/G, does there exist a 2-sphere S' in E^3 such that P[S'] is a 2-sphere homeomorphically close to S? I conjecture that the answer is negative, and that the following example is a counterexample.

Analogous to the example we have defined and proven for the similar statement concerning a disk, we will define an upper semicontinuous decomposition of E^3, a 2-complex that is not a sphere in E^3 but projects onto a sphere S in the decomposition space, and an $\epsilon > 0$. We claim that there is no 2-sphere $S' \subset E^3$ such that its image P(S) is ϵ-homeomorphic to S.

Notice first that it is not possible in E^3 to take two copies of the disk counterexample, connect the

boundaries of the two copies of the 2-complex C, and have a counterexample for this sphere question. The knitting of the nondegenerate elements prevents this. The example that we are about to construct changes the knitting so that we can do something similar to adding two copies of C.

We first define a 2-complex K, whose image will be the 2-sphere. Start with the 2-complex C that we defined for the disk example. Its center-line is the segment of the z-axis from 0 to 1. Let C' be the reflection of C in the horizontal plane at $z = -2$. These 2-complexes C and C' are shown in Figure 75 The points p_3 and p_4 in C' are reflections of p_1 and p_2, respectively. In Figure 76 four vertical T-shaped surfaces are added to $C \cup C'$. The resulting 2-complex is K. Notice that K is not a manifold on the z-axis, but K - (z-axis) is homeomorphic to a 2-sphere minus two points.

We now define a decomposition of E^3 into points and four knit Cantor sets. Each of the four knit Cantor sets is a copy of the set $\tilde{H}$ that was defined in the disk example. Each of the quadrants Q^1, Q^2, Q^3, and Q^4 of E^3 (as quadrants were defined in the disk example) contains a copy of $\tilde{H}$ knit from a point in C or C' to a point in the other. Figure 74 shows an

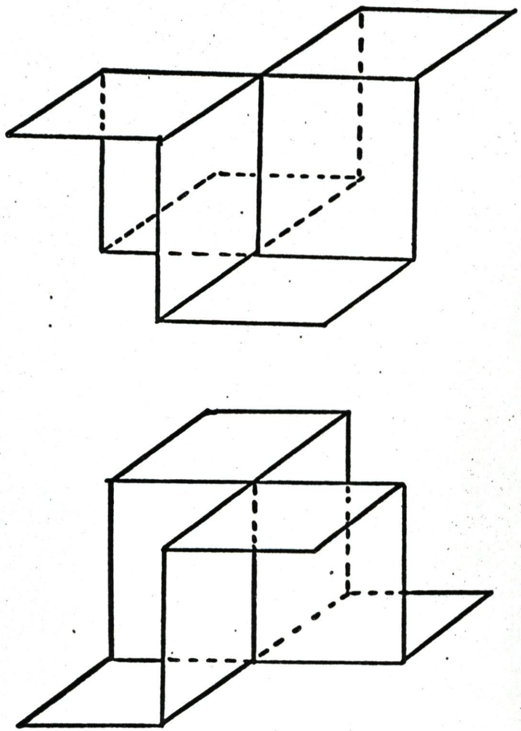


Figure 75

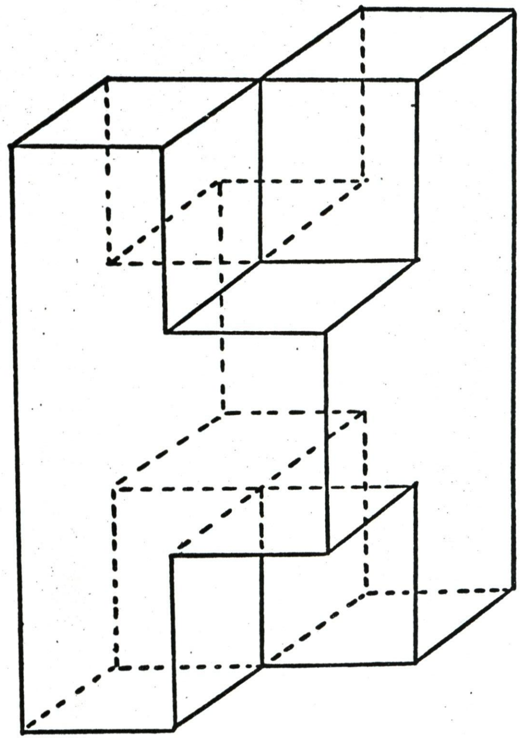


Figure 76

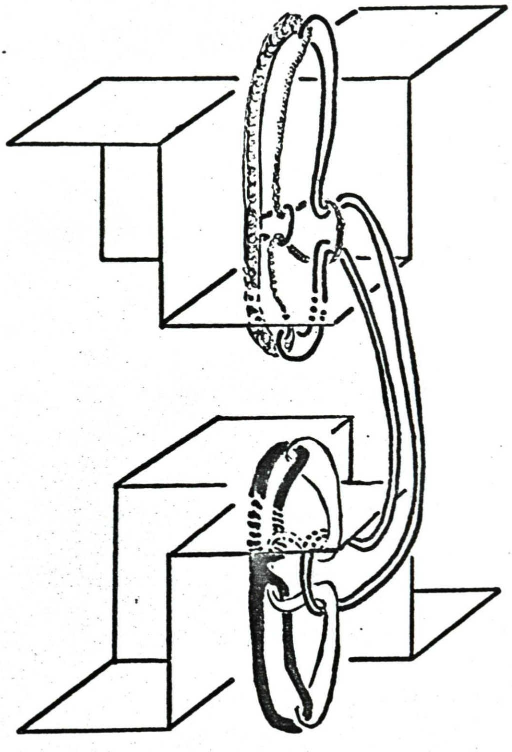


Figure 77

element of the defining segment of $\tilde{H}$ and its orientation with respect to K. One limiting element of each knit Cantor set is the segment g_o defined by $-1 \leq z \leq 2$ and the other limiting element is g_o' defined by $-6 \leq z \leq -3$. In Q^1 and Q^3 the knitting is from p_1 to p_3; and in Q^2 and Q^4 it is from p_4 to p_2. Recall the pairs of pyramids π_1 and π_2 that were used in the disk example proof (Figure 57). For this example analogous pairs of pyramids are shown in Figure 78. Figure 79 shows some of the nondegenerate elements that lie in Q^1. In Q^3 the knit Cantor set from p_1 to p_3 is the reflection through the z-axis of the construction in Q^1. In Q^2 and Q^4 we orient $\tilde{H}$ in such a way that the knitting is from $p_4 \in C'$ to $p_2 \in C$. For the latter two quadrants notice that the knitting lies in the bounded component of $E^3 - K$, whereas in Q^1 and Q^3 the knitting lies in the unbounded component of $E^3 - K$. Though the knitting lies in these components, every nondegenerate element except g_o and g_o' pierces K.

The definition of ϵ for this example is analogous to the definition of ϵ in the disk example.

I conjecture that there is a proof of this example analogous to the proof of the disk example.

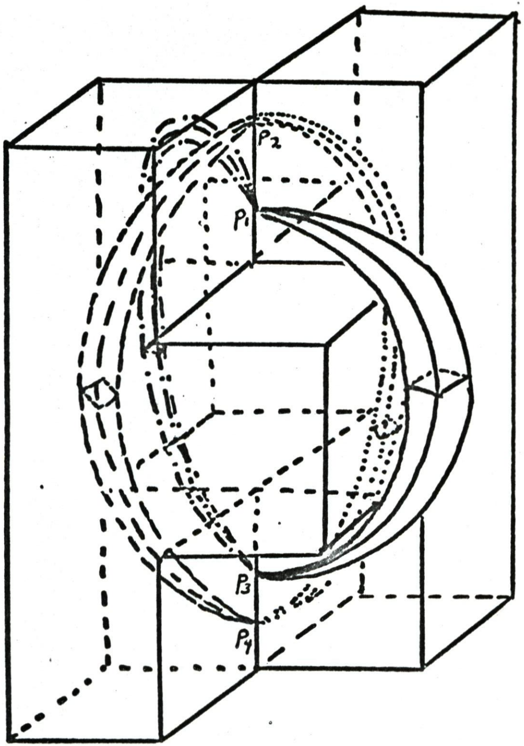


Figure 78

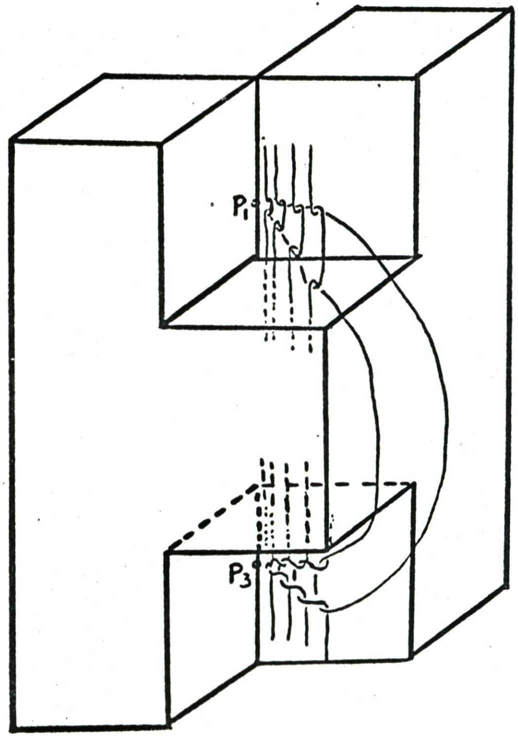


Figure 79

A related example. The following definition is made by Armentrout, Lininger, and Meyer [5]:

Definition. If G is any monotone decomposition of R^3, let H_G denote the union of the nondegenerate elements of G, and let P_G denote the projection map from R^3 onto the decomposition space R^3/G associated with G. Suppose that F and G are monotone decompositions of R^3 such that each of $\mathrm{Cl}\,(P_F[H_F])$ and $\mathrm{Cl}\,(P_G[H_G])$ is compact and 0-dimensional. Then F and G are equivalent decompositions of R^3 if and only if there is a homeomorphism h from R^3/F onto R^3/G such that $h[\mathrm{Cl}\,(P_F[H_F])] = \mathrm{Cl}\,(P_G[H_G])$.

For our knit decomposition G of Theorem 4.1, there is an equivalent decomposition F that has only a countable number of nondegenerate elements. The method of producing this countable decomposition is due to Bing [9]. Let $M_1^F, M_2^F, M_3^F, \cdots$ be the defining sequence for F. We will require that each M_i^F be homeomorphic to M_i of the decomposition G. Furthermore, the imbedding of M_i^F in M_{i-1}^F is the same as the imbedding of M_i in M_{i-1}. Recall the definition of $T^j_{n\alpha}$ on p.121. The symbol $n\alpha$ denotes a positive integer consisting of n digits, each of which is a one or a two. Let $(T^F)^j_{n\alpha}$ correspond for F to $T^j_{n\alpha}$ for G. We now require that diam $(T^F)^j_{n\alpha} < 1/2^k$, where k is the number of digits two in $n\alpha$. Suppose now

that the disks in $C \cap T^{j}_{n\alpha}$ are required to lie in a set of diameter less than $1/2^n$. It is then possible in E^3 to require that $(T^F)^{j}_{n\alpha} \cap C = T^{j}_{n\alpha} \cap C$. That F has only a countable number of nondegenerate elements follows from analogy with Bing's example [9]. Essentially, this argument is that a nondegenerate component of Cl H must lie in the intersection of a sequence of defining sequence components with at most a finite number of twos in any subscript. There are a countable number of such sequences.

Theorem (Armentrout, Lininger, and Meyer [5]). Suppose that F and G are monotone decompositions of R^3 such that Cl $P_F[H_F]$ and Cl $P_G[H_G]$ are compact 0-dimensional sets. Suppose that F has a defining sequence $M_1, M_2, M_3, \cdots$ and there exists a sequence $f_1, f_2, f_3, \cdots$ of homeomorphisms from R^3 onto R^3 such that (1) for each i, $f_{i+1} | (R^3 - \text{Int } M_i) = f_i | (R^3 - \text{Int } M_i)$, and (2) $f_1[M_1], f_2[M_2], f_3[M_3], \cdots$ is a defining sequence for G. Then F and G are equivalent.

This theorem is satisfied by the F and G that we have defined. Hence, they are equivalent, and we have the homeomorphism h of the definition.

With the condition we put on $(T^F)^{j}_{n\alpha} \cap C$, the disk $D = P_G(C) = h(P_F(C))$. Of course, C is not a disk, but $P_F(C)$ is a disk.

The decomposition F satisfies Theorem 3.2. Hence, there must be a P-liftable disk near $P_F(C)$.

We have shown by this pair of decompositions that equivalent decompositions may differ in the property of the existence of a P-liftable disks near any given disk.

Concluding remarks. The following definition is from Michael [25].

Definition. A collection G of closed point sets filling a metric space is said to be equi-LC^m provided that it is true that, if y is a point of an element g_0 of G and ϵ is a positive number, there is a positive number δ such that if g is an element of G, then any mapping of a k-sphere ($k \leq m$) onto a subset of $g \cap S(y,\delta)$ is homotopic to a constant on a subset of $g \cap S(y,\epsilon)$.

Observe that a knit Cantor set is equi-LC^0. We now make new definitions, which are not satisfied by a knit Cantor set at a point p_2 to which the set is knit.

Definition. A collection G of closed point sets in E^3 is said to be equi-locally connected provided that, if y is a point of an element g_0 of G and ϵ is a positive number, there is a topological 3-cell B contained in the ϵ-neighborhood of y such that if g is an element of G, then $g \cap \text{Int } B$ is connected.

Definition. A collection G of closed point sets in E^3 is said to be strongly equi-locally connected provided that, if y is a point of an element g_0 of G and ϵ is a positive number, there is a topological 3-cell B contained in the ϵ-neighborhood of y such that if g is an element of G, then $g \cap \text{Int } B$ and $g \cap B$ are connected.

Definition. A collection G of closed point sets in E^3 is said to be equi-locally connected and equi-semi-connected provided that, if y is a point of an element g_0 of G and ϵ is a positive number, there is a topological 3-cell B contained in the ϵ-neighborhood of y such that if g is an element of G, then $g \cap \text{Int } B$ is connected and $g \cap \text{Ext } B$ has no more than two components.

Question. Is the following true?

Let G be an upper semicontinuous decomposition of E^3 with nondegenerate elements H. Furthermore, suppose that H is a continuous, strongly equi-locally connected collection and that $P(H)$ is 0-dimensional in E^3/G. Then, given a disk $D \subset E^3/G$ and a positive number ϵ, there is a disk D_ϵ that is ϵ-homeomorphic to D and is the image of a disk under the projection mapping P.

Questions. Is the above true if we substitute one of the other three definitions of equi-local connectedness? Can the condition that $P(H)$ be 0-dimensional be

dropped from the hypothesis?

Notice that Bing's dogbove space [8] is an example that satisfies the hypotheses of the above statement.

REFERENCES

1. Alford, W. R. and Ball, B. J., Some almost polyhedral wild arcs, Duke Math. J., Vol 30 (1963), pp. 33-38.

2. Armentrout, Steve, Monotone decompositions of E^3, Annals of Math. Studies, Vol. 60 (1966), pp 1-25.

3. ______________, Concerning cellular decompositions of 3-manifolds with boundary, TAMS, Vol. 137 (1969), pp. 231-236.

4. ______________, UV decompositions, Proceedings of the Conference on Monotone Mappings and Open Mappings at SUNY/Binghamton (1970) (to appear).

5. Armentrout, Steve, Lininger, Lloyd C., and Meyer, Donald V., Equivalent decompositions of E^3, Pacific J. of Math., Vol. 24 (1968), pp. 205-227.

6. Bing, R. H., A homeomorphism between the 3-sphere and the sum of two solid horned spheres, Annals of Math. Vol. 56, (1952), pp. 354-362.

7. ______________, Approximating surfaces with polyhedral ones, Annals of Math., Vol. 65, (1957), pp. 456-483.

8. ______________, A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3, Annals of Math., Vol. 65, (1957), pp. 484-500.

9. ______________, Point-like decompositions of E^3, Fund. Math., Vol. 50 (1962), pp. 431-453.

10. ______________, Inequivalent families of periodic homeomorphisms of E^3, Ann. of Math. 80 (1964), pp. 78-93.

11. Burgess, C. E. and Cannon, J. W., Embeddings of surfaces in E^3, (to appear).

12. Casler, B. G., On the sum of two solid Alexander horned spheres, TAMS, Vol. 116 (1965), pp. 135-150.

13. Fort, M. K., Jr., A note concerning a decomposition space defined by Bing, Annals of Math., Vol. 65 (1957), pp. 501-504.

14. Fox, Ralph H. and Artin, Emil, Some wild cells and spheres in three-dimensional space, Annals of Math. Vol. 49 (1948), pp. 979-990.

15. Harrold, O. G., Jr., A sufficient condition that a monotone image of the 3-sphere be a topological 3-sphere, PAMS, Vol. 9 (1958), pp. 846-850.

16. Harrold, O. G., Jr., and Moise, E. E., Almost locally polyhedral spheres, Annals of Math., Vol 57 (1953), pp. 575-578.

17. Hocking, John G. and Young, Gail S., Topology, Addison-Wesley, Reading, Mass., 1961.

18. Hudson, J. F. P., Piecewise Linear Topology, W. A. Benjamin, Inc., New York and Amsterdam, 1969.

19. Hurewicz, Witold and Wallman, Henry, Dimension Theory, Princeton University Press, Princeton, 1948.

20. Kinoshita, S., On quasi-translations in 3-space, Fundamentals of Math., Vol. 56 (1964), pp. 69-79.

21. Kwun, Kyung Whan, Upper semicontinuous decompositions of the n-sphere, PAMS, Vol. 13 (1962), pp. 284-290.

22. McAuley, Louis F., Upper semicontinuous decompositions of E^3 into E^3 and generalizations to metric spaces, Topology of 3-Manifolds and Related Topics, Proc. Univ. of Georgia Inst., 1961, pp. 21-26.

23. McAuley, Louis F. Lifting certain open mappings, Proc. Nat. Acad. Sci., Vol. 53 (1965), pp. 255-260.

24. McAuley, Louis F. and Tulley, Patricia, Lifting certain light open mappings, Math. Annalen, Vol. 175 (1968), pp. 114-120.

25. Michael, Ernest, Continuous selections, I, II, and III, Annals of Math. Vol. 63 (1956), pp. 361-382; Vol. 64 (1956), pp. 562-580; Vol. 65 (1957), pp. 375-390.

26. Moore, R. L., Concerning upper semi-continuous collections of continua, TAMS, Vol. 27 (1925), pp. 416-428.

27. Papakyriakopoulos, C. D., On Dehn's lemma and the asphericity of knots, Annals of Math., Vol. 66 (1957), pp. 1-25.

28. Siebenmann, L. C., Approximating cellular maps by homeomorphisms (to appear).

29. Whyburn, Gordon Thomas, Analytic Topology, American Mathematical Society, Providence, Rhode Island, 1942.

30. Woodruff, Edythe P., Concerning the condition that a disk in E^3/G be the image of a disk in E^3, Proceedings of the Conference on Monotone Mappings and Open Mappings at SUNY/Binghamton (1970) (to appear).

ADDENDUM

Statement (made on p. 129). If there are two components of $A_j^i - D_\epsilon'$, then they lie in different components of $T_j^i - (\delta_a \cup \delta_b)$.

Proof. We will prove the analogous statement for any $g \in H$ contained in T_j^i: If there are two components of $g - D_\epsilon'$, then they lie in different components of $T_j^i - D_\epsilon'$. Note that $g - D_\epsilon'$ can have at most two components.

Suppose that the statement is not true; i.e., suppose that there exists a nondegenerate element $g \subset T_j^i$ such that two components of $g - D_\epsilon'$ lie in a component K_{j1} of $T_j^i - D_\epsilon'$. We first will show that for any toroidal component $T_m^i \subset T_j^i$ the fact that both components of $g - D_\epsilon'$ lie in the same component of $T_j^i - D_\epsilon'$ implies that both components of $g - D_\epsilon'$ lie in the same component of $T_m^i - D_\epsilon'$. The disk D_ϵ' can be approximated by a disk λ that agrees with D_ϵ' in T_j^i and on Bd D_ϵ', and is polyhedral off T_j^i. Note that Bd λ = Bd D_ϵ' is trivial in $E^3 - T_j^i$. The disk λ can be completed to a 2-sphere S that intersects T_j^i only in $D_\epsilon' \cap T_j^i$. Each component of $T_j^i - D_\epsilon'$ or $T_m^i - D_\epsilon'$ lies in either the interior or the exterior of S. Hence, if $g - D_\epsilon'$ lies in one component of $T_j^i - D_\epsilon'$, it lies in Int S or Ext S, and this implies that it lies in one

component of $T_m^i - D_\epsilon'$.

Let a and b denote the endpoints of g. They both lie in the component K_{j1} of $T_j^i - D_\epsilon'$. Let $\gamma_1 = \min\{d(D_\epsilon', a \cup b), d(a,b)\}$. Since H is a continuous collection, there is a distance γ_2 such that for any $g_k \in H$ such that $g_k \subset T_j^i \cap N_{\gamma_2}(g)$, the endpoints of g_k lie in $N_{\gamma_1}(a \cup b)$. In $N_{\gamma_2}(g)$ there is a toroidal component T_k^i of the defining sequence. All endpoints of nondegenerate elements in T_k^i lie in $N_{\gamma_1}(a \cup b)$. This implies that all nondegenerate elements in T_k^i lie in the closure of one component of $T_k^i - D_\epsilon'$.

Consider the two tori $T_{k\alpha 1}$ and $T_{k\alpha 2}$ imbedded in T_k^i. As the center of $T_{k\alpha 1}$ we can take a polyhedral simple closed curve J_1 that is the union of the following four arcs: a nondegenerate element $g_a \subset T_{k\alpha 11}$, a nondegenerate element $g_b \subset T_{k\alpha 12}$, and polyhedral arcs β_1 and β_2 lying in $T_{k\alpha 11} \cap N_{\gamma_1}(a)$ and $T_{k\alpha 12} \cap N_{\gamma_1}(b)$, respectively, and each joining an endpoint of g_a with an endpoint of g_b. There is an analogous simple closed curve J_2 in $T_{k\alpha 12}$. Both J_1 and J_2 lie in the closure of a component K_{k1} of $T_k^i - D_\epsilon'$. Let J_o be a simple closed curve linking T_k^i. If K_{k1} were known to be tame, then we could state that J_o is shrinkable in $E^3 - (J_1 \cup J_2)$. There is a polyhedral approximation $\tilde{D}$ of D_ϵ' that agrees with $D_\epsilon' \cap T_k^i$ and misses $J_1 \cup J_2$.

Now the closure of each component of $T_k^i - \tilde{D}$ is a tame 2-cell. Both J_1 and J_2 are contained in the closure of one component of $T_k^i - \tilde{D}$. This implies that the simple closed curve J_o linking T_k^i can be shrunk to a point in $E^3 - (J_1 \cup J_2)$. This contradicts Lemma B, p. 99, and proves our statement. ▯

Prof. Harry Berkowitz suggested that it may not be easy to prove the existence of the disk s (p. 129, line 16). He suggested that instead of the method implied on pp. 129-130 I do the following: In the boundary of a regular neighborhood of $\delta_a \cap A_j^i$, find a disk δ that intersects δ_a only in Bd δ and contains one point of $a_1 \cup A_j^i \cup a_2$. Replace the subdisk of δ_a bounded by Bd δ by the disk δ. Then, push a subdisk of δ in a tubular neighborhood of $a_1 \cup A_j^i \cup a_2$ to make the intersection with $a_1 \cup A_j^i \cup a_2$ lie in $\delta_a \cap A_j^i$. Details of a method to find the disk δ follow.

The 3-cell B is a regular neighborhood of $A_j^i \cap \delta_a$, but it may not contain the desired disk δ in its boundary. We will alter B in such a way that it will still be a polyhedral regular neighborhood of $A_j^i \cap \delta_a$ and will also contain a disk satisfying the requirements on δ.

Let α denote the arc $a_1 \cup A_j^i \cup a_2$. Denote the

finite set of simple closed curves in Bd B $\cap$ δ_a by $\mathcal{J} = \{J_1, J_2, \cdots, J_n\}$. These simple closed curves lie in the annulus $\delta_a - \alpha$ and also lie in the annulus (Bd B) $- \alpha$. Note that if $J \in \mathcal{J}$ is trivial in one of these annuli, then it is trivial in B $- \alpha$, and, hence, it is trivial in the other annulus. We will first remove these trivial simple closed curves from the intersection by altering Bd B.

Let $J_1 \in \mathcal{J}$ be trivial in the annuli and bound a disk $d_1 \subset \delta_a - \alpha$ such that d_1 contains no other element of $\mathcal{J}$. This simple closed curve J_1 also bounds a disk e_1 in (Bd B) $- \alpha$. (See Figure 80.) In Bd B replace e_1 by d_1. This gives us an altered 3-cell, which we continue to call B. In (the collar of B) $- \alpha$ and in the neighborhood of d_1, push d_1 slightly off δ_a in such a direction that the number of elements in $\mathcal{J}$ is n $-$ 1 or fewer.

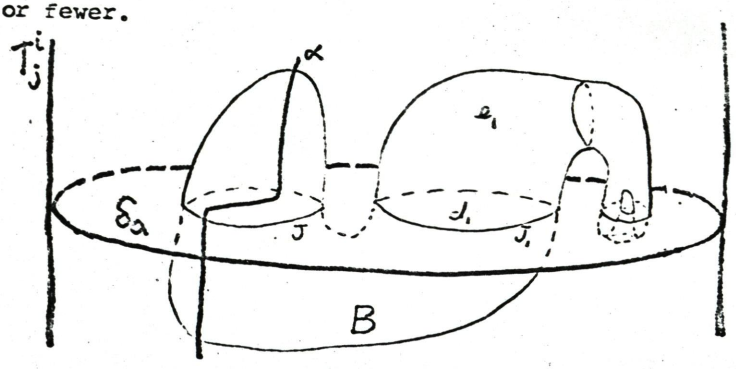


Figure 80

Repeat this construction until all simple closed curves in $\mathcal{J}$ are removed. The statement proved above (p. 163) implies that there are an odd number of elements remaining in $\mathcal{J}$. Suppose that J_2 and J_3 bound a subannulus in (Bd B) - α that contains no other element of $\mathcal{J}$. Replace this annulus by the annulus that they bound in $\delta_a - \alpha$. Push the replacement slightly off δ_a in (the collar of B) - α in such a direction as to decrease the number of elements in $\mathcal{J}$. This will remove an even number of elements from $\mathcal{J}$. Repeating this construction, we are finally left with only one element J in $\mathcal{J}$. Either disk in Bd B bounded by J now satisfies the requirements on δ.

<u>Concerning the proof of Lemma 4.7</u> (pp. 130-135). It is also possible for an endpoint of A_j^i to not lie in the closure of the unbounded component of $E^3 - (\hat{C} \cup \hat{E} \cup r_o)$ because the meridional disks D_a and D_b extend as shown in Figure 81, or farther. It is then possible to replace D_a and D_b by the meridional disks indicated in Figure 82.

If this situation occurs in a particular torus T_j^i, then $D_a \cup D_b$ has points above R and below S. Hence, as on pp. 133-134, if this occurs in an infinite number of tori in the quadrant Q^i, then α^i exists.

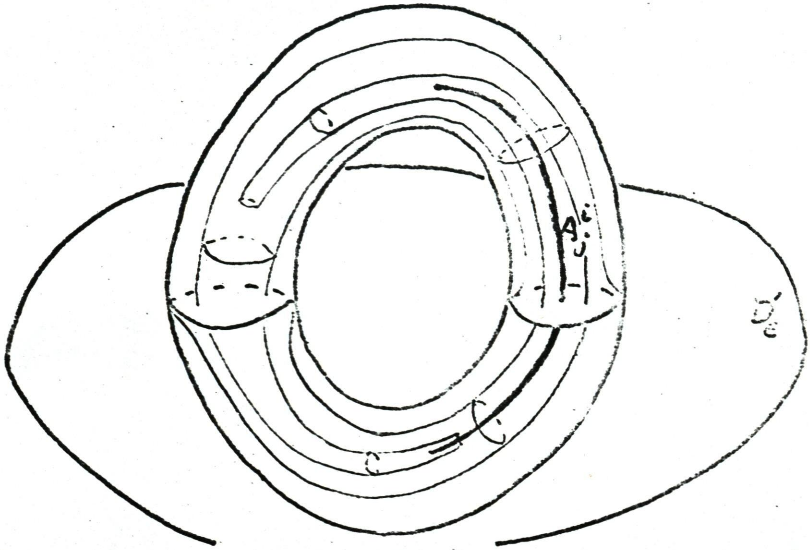


Figure 81

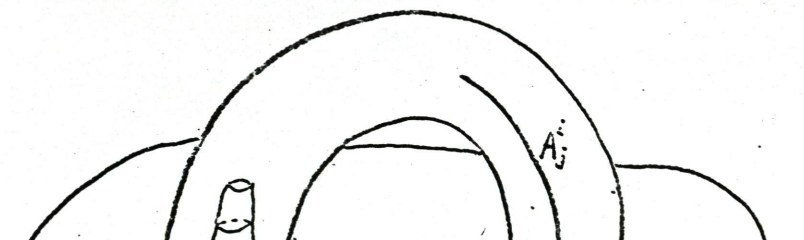


Figure 82

Page 54, line 3 should read:

... (Bd C) $\cap$ Cl P(H) = $\emptyset$ ...

Page 56, line 20-23. The statement that Cl (D - C) is also a compact manifold may not be true. It is possible that D - C contains an infinite number of sets that each intersect Bd D. Since there is a finite distance between Bd C and Cl P(H), only a finite number of these components can contain points of P(H). Let $\hat{C}$ be the union of C and those components of D - C that contain no points of P(H). Now, $\hat{C}$ and D - $\hat{C}$ are sets that satisfy the hypotheses of Theorem 2.2.