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Decomposition spaces and separation properties

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DECOMPOSITION SPACES AND SEPARATION PROPERTIES

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CHAPTER I

Introduction

1.1. Introductory Remarks.

This thesis deals primarily with decomposition spaces and the question of inheritance by a decomposition space of certain topological properties. Some new topological concepts which are introduced are of independent interest but they are explored here principally for their implications in decomposition spaces.

In Chapter II we compare McAuley's definition of an upper semicontinuous decomposition with other separation properties of the decomposition space and relations of these properties to the projection map. In contrast to Whyburn's (originally, Moore's) definition of upper semicontinuity, which is equivalent to requiring the projection map to be closed, these are purely topological properties, but some nevertheless impose conditions on quotient maps onto spaces satisfying them. Also, they are investigated in conjunction with various basis restrictions on the decomposition space (such as first countability, etc.) or conditions on the nature of the individual elements of the decomposition.

Chapter III is more narrow in scope, dealing specifically with certain shrinkability theorems of McAuley, originally asserted for decompositions which are upper semicontinuous in the sense he defined. The observation that this definition of upper semicontinuity did not yield the desired properties as supposed led to the investigations of

Chapter II. Proofs of the theorems with the amended hypotheses are supplied.

1.2. Notation and terminology

Unless the contrary is stated, the terms employed are as defined in [10]. Where concepts have had a variety of names, some effort has been made to list these and to use one of those names already appearing in the literature. Exception has been made in the form of adopting letters in place of verbal descriptions of properties for the sake of brevity.

Throughout this thesis, where X denotes a topological space, G a collection of mutually disjoint subsets covering X, the decomposition (quotient, factor) space X/G will be denoted by $\underline{I}$ and the canonical projection (quotient, factor) map by $\underline{p}:X \to I$, where $x \in p^{-1}px$. Also, we may write $x \in p(x)$, using the same name for an element of the decomposition space whether regarding it as a subset of X (an element of G) or as a point of I. The collection of non-degenerate elements of G is denoted $\underline{H_G}$, or simply $\underline{H}$.

If A is a collection of subsets of a space, $\underline{A^*}$ means the union of the members of A. Where $A \subset I$, A^* will be $p^{-1}A$, following the convention noted above. So the topology of I can be described by: A is open in I if and only if A^* is open in X. Also, A(B), where B is a set not necessarily belonging to A, is the subcollection of A consisting of those members of A which intersect B. $(A(B))^*$ is written $\underline{A^*(B)}$.

The singleton $\{x\}$ is frequently abbreviated as x, where we hope no confusion will result. For instance, $A(\{x\})$ is contracted to A(x).

$x \, \underline{lp} \, A$ means "x is a limit point of the set A" and $x \, \not{\underline{lp}} \, A$ is used for the negation.

$\Rightarrow$ is used for the logical "only if" or "implies" and $\Leftrightarrow$ or <u>iff</u> for "if and only if."

While we are concerned only with decompositions into closed subsets, where I is T_1, no such standing assumption is made, and all of the theorems are intended to stand only on the hypotheses specifically stated in them.

CHAPTER II

Upper Semicontinuity and Separation Properties

2.1. Mc, M', and M

Definition. A space X is $\underline{m_c}$ iff $x \neq y$, $x \, \ell p \, A$ and $y \, \ell p \, A \Longrightarrow$ there exists a subset B of A such that $x \, \ell p \, B$ and $y \, \not\ell p \, B$.

Definition. X is $\underline{Mc}$ iff X is T_1 and m_c, i.e., $x \neq y$, $\{x,y\} \subset \overline{A} \Longrightarrow$ there exists a subset B of A such that $x \in \overline{B}$ and $y \notin \overline{B}$.

A decomposition G of a space X is upper semicontinuous in the sense of McAuley [11] iff $\mathcal{I}$ is Mc.

Proposition 1. $T_2 \Longrightarrow Mc$

Proof. $T_2 \Longrightarrow T_1$ and if x and y are different limit points of a set A, let U and V be disjoint open sets containing x and y, respectively. Then $B = A \cap U$ gives the desired subset for Mc.

The converse of Proposition 1 is not true in general. In fact, we can state a condition which is stronger than Mc and yet fails to yield T_2 without some restriction like first countability on the space.

Definition. X is $\underline{m'}$ iff $x \, \ell p \, A \Longrightarrow$ there is a subset B of A such that $\overline{B} \sim B = \{x\}$.

<u>Definition</u>. X is <u>M'</u> iff X is T_1 and m', i.e., $x \in \overline{A} \Longrightarrow$ there is a subset B of A such that $\overline{B} = B \cup x$.

<u>Proposition 2</u>. M' $\Longrightarrow$ Mc

<u>Proof</u>. If $x \neq y$ and both are limit points of a set A, we can assume neither belongs to A by T_1. Then there is a subset B of A such that $x \, \ell p \, B$ and $B \cup x$ is closed, so $y \not\ell p \, B$.

We will show that T_2, M' and Mc are all equivalent in a first countable space; but the latter two are equivalent in the presence of a weaker base condition and in that case equivalent to another property which we call M, as it was named by McDougle [13].

<u>Definition</u>. X is <u>M</u> iff sequential limits are unique, i.e., $\{x_n\}$ a sequence, $x_n \to x$ and $x_n \to y \Longrightarrow x = y$.

<u>Definition</u>. X is <u>KC</u> iff each compact subset of X is closed.

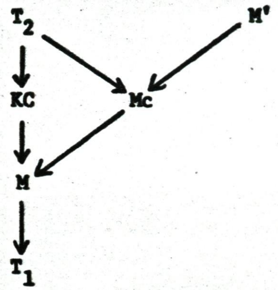
The abbreviation KC was used by Wilansky [23].

Clearly, $T_2 \Longrightarrow KC \Longrightarrow M \Longrightarrow T_1$.

<u>Proposition 3</u>. Mc $\Longrightarrow$ M

<u>Proof</u>. Suppose $x_n \to x$ and $x_n \to y$ with $x \neq y$. By T_1 we can assume the sequence x_n is not frequently constant. So $x \, \ell p \bigcup x_n$ and $y \, \ell p \bigcup x_n$. By Mc there is a set $B \subset \bigcup x_n$ for which $x \, \ell p \, B$ and $y \not\ell p \, B$. Since $x \, \ell p \, B$, we must have $B = \bigcup x_{n_i}$ for some subsequence x_{n_i}, again by T_1. And since $x_{n_i} \to y$ also, $y \, \ell p \bigcup x_{n_i}$, which is a contradiction.

We have so far:



Each of these properties is topological and each is inherited by subspaces. Eventually, we will give examples to show the converses of these implications do not hold, but it will be more illuminating if we first examine additional conditions in which they do. That the first column of implications has no converse is well-known. See, for example, [23].

M' has two very useful properties. One is that it is preserved by closed maps. Another is that even considerable weakening of M' on I guarantees that the projection map is pseudo-open.

<u>Proposition 4</u>. M' is preserved by closed maps.

<u>Proof</u>. Let X be M' and $f: X \to Y$ a closed map of X onto Y. Suppose $y \ell p A$ in Y. We can assume $y \notin A$. Since f is closed, there is a point $x \in f^{-1}y$ with $x \ell p f^{-1}A$ and since X is M', there is a subset $B \subset f^{-1}A$ satisfying $\overline{B} = B \cup x$ and $x \ell p B$. Then by continuity of f, we have $y \ell p f(B)$, while $\overline{f(B)} = f(\overline{B})$ since f is closed. But $f(\overline{B}) = f(B \cup x) = f(B) \cup y$. And since $f(B) \subset A$, this completes the proof.

<u>Corollary 4.1</u>. If X is M' and G is Whyburn-usc, then I is M'.

<u>Corollary 4.2</u>. If X is M', then any Whyburn-usc decomposition of X is McAuley-usc.

Henceforth, when we use the term upper semicontinuous (usc) without qualification, it will be understood in Whyburn's sense, i.e., p is closed. For McAuley's we use the term Mc.

2.2. <u>Pseudo-open maps</u>

<u>Definition</u>. If $f(X) = Y$ is a continuous map of X onto Y, f is <u>pseudo-open</u> (Arhangel'skii [1]), pre-closed (T'ong [17]), a P_1-mapping (McDougle [13]), iff $y \in Y$, $f^{-1}y \subset U$ open $\Longrightarrow y \in \text{int } f(U)$.

A map f of X onto Y is <u>quasi-compact</u> (quotient) iff the image of an open inverse set is open. An <u>inverse</u> set is a subset A of X such that $f^{-1}fA = A$. Since the complement of an inverse set is an inverse set, under a quasi-compact map the image of a closed inverse set is closed.

The properties of pseudo-open maps listed here seem to have been discovered independently by a number of people.

<u>Proposition 6</u>. pseudo-open $\Longrightarrow$ quasi-compact

<u>Proof</u>. Suppose $O = f^{-1}fO$ is an open inverse set. If $y \in fO$ then $f^{-1}y \subset O$ open. So $y \in \text{int } fO$. Thus fO is open.

<u>Proposition 7</u>. p is pseudo-open $\Longleftrightarrow$ ($g \in G$, $g \subset U$ open in $X \Longrightarrow$ there is a set V open in X such that $g \subset V \subset U$ and pV is open).

<u>Proof</u>. Assume p is pseudo-open. Let U be open in X containing g. Then pU is a neighborhood of g in $\mathcal{I}$. So pU contains an open set O in $\mathcal{I}$ containing g. $p^{-1}O$ is open by continuity. Let $V = p^{-1}O \cap U$. $g \in V$ open $\subset U$ and $pV = p(p^{-1}O \cap U) = O$ since $O \subset pU$.

Conversely, if U is open in X containing g, by hypothesis there is an open set V with $g \subset V \subset U$ and pV open. Since $pV \subset pU$ this makes pU a neighborhood of g in $\mathcal{I}$.

<u>Proposition 8</u>. p is pseudo-open $\Longleftrightarrow$ ($g \;\ell p\; A$ in $\mathcal{I}$ $\Longleftrightarrow$ there is a point $x \in g$ such that $x \;\ell p\; A^*$ in X).

<u>Proof</u>. Assume p is pseudo-open. If no point of g is a limit point of A^*, there is an open set $U \supset g$ such that U misses A^*. But pU is a neighborhood of g and misses A, contradicting $g \;\ell p\; A$. The converse of the implication just shown holds for continuous maps.

Conversely, suppose U is open in X containing g but $g \notin \operatorname{int} pU$. Then $g \in pU$ but $g \;\ell p\; (\mathcal{I} \setminus pU)$. So there is a point $x \in g$ such that $x \;\ell p\; p^{-1}(\mathcal{I} \setminus pU)$ by hypothesis. In particular, U contains x so U meets $p^{-1}(\mathcal{I} \setminus pU)$, which is impossible.

<u>Proposition 9</u>. p is pseudo-open $\Longleftrightarrow$ (A is closed in $B \subset \mathcal{I}$ $\Longleftrightarrow$ A^* is closed in $B^* \subset X$).

<u>Proof</u>. Assume p is pseudo-open. If A is closed in B, then A^* is closed in B^* by continuity. Suppose A^* is closed in B^*. If A is not closed in B there is an element $g \in B \setminus A$ with

$g\ \ell p\ A$. Now g misses A^* in X while there is a point $x \in g$ such that $x\ \ell p\ A^*$. But $x \in g \subset B^* \setminus A^*$ which contradicts the hypothesis.

Conversely, if p is not pseudo-open, then for some g and A, $g\ \ell p\ A$ with no point of g a limit point of A^*. We can assume $g \notin A$. Then A^* is closed in $A^* \cup g$. Hence, A is closed in $p(A^* \cup g)$ by hypothesis. But $p(A^* \cup g) = A \cup g$ which contradicts $g \in \overline{A} \setminus A$.

Corollary 9.1. p is pseudo-open $\Longleftrightarrow$ (A is open in $B \Longleftrightarrow A^*$ is open in B^*).

Proof. This follows from the proposition, since C is open in D iff $X \setminus C$ is closed in $X \setminus D$, and if C is an inverse set then $X \setminus C$ is an inverse set.

Definition. (Whyburn) A map f is hereditarily quasi-compact iff $f|Y$ is quasi-compact for each inverse set Y.

Proposition 10. p is pseudo-open $\Longleftrightarrow$ p is hereditarily quasi-compact.

Proof. p is hereditarily quasi-compact iff whenever O and Y are inverse sets with O open in Y, then pO is open in pY. Letting $A = pO$ and $B = pY$, this condition becomes: A^* is open in $B^* \Longrightarrow A$ is open in B. The converse of this implication is continuity so we have the characterization of a pseudo-open map in Corollary 9.1.

In particular, we have from these propositions that if p is pseudo-open then $p|\text{Ker } p$ is a homeomorphism. (Ker p =

$\{x : f^{-1}fx = x\} = X \setminus H_G^*\}$.

Another way of describing hereditarily quasi-compact (hereditarily quotient, pseudo-open) maps is that they preserve the subspace topology on inverse sets. That is, if G is a decomposition of X and Y is an inverse set in X, then $Y = G'^*$ for some $G' \subset G$. Y has the subspace topology from X and there is an induced quotient space Y/G'. As a set this is precisely the subset of X/G whose elements lie in Y, but as a subspace of X/G this set may have a different topology, i.e., strictly weaker. These two topologies are the same (for all inverse sets Y) if and only if p is pseudo-open.

Definition. A map $f: X \to Y$ is monotone iff $f^{-1}y$ is connected for each $y \in Y$.

Proposition 11. p is pseudo-open, monotone, C connected in $\mathcal{I} \Longrightarrow C^*$ is connected.

Proof. Suppose C is connected but $C^* = A \cup B$, a separation in X. Then A and B are non-empty and A is both open and closed in C^*. Since p is monotone, A and B are inverse sets. Because p is pseudo-open, pA is open and closed in $pC^* = C$, while pA and pB are also non-empty. This gives a separation of $C = pA \cup pB$.

Clearly, open $\Longrightarrow$ pseudo-open and closed $\Longrightarrow$ pseudo-open.

Proposition 12. $\mathcal{I}$ is M' $\Longrightarrow$ p is pseudo-open

Proof. Suppose $g \, \ell p \, A$ in $\mathcal{I}$ but $g \cap \overline{A^*} = \phi$. By M' there is a subset $A_1 \subset A$ such that $g \, \ell p \, A_1$ and $A_1 \cup g$ is closed in $\mathcal{I}$.

So $A_1^* \cup g$ is closed in X by continuity of p. Hence A_1^* is closed in X because g contains no limit points of $A_1^* \subset A^*$. Then A_1 is closed in I as p is quasi-compact. This contradicts $g \in \overline{A}_1 \setminus A_1$.

It is evident that it was not necessary in this proof to have $A_1 \cup g$ closed but only that $g \ell p A_1$ and $g \not\ell p (\overline{A}_1 \setminus A_1)$. This suggests the following definition.

<u>Definition</u>. X is <u>weak M'</u> iff $x \ell p A \Longrightarrow$ there is a subset B of A such that x is an isolated point of $\overline{B} \setminus B$.

Clearly M' $\Longrightarrow$ weak M' and Proposition 12 is immediately eclipsed by:

<u>Proposition 13</u>. I is weak M' $\Longrightarrow$ p is pseudo-open.

<u>Proof</u>. Suppose $g \ell p A$ in I and $g \cap \overline{A^*} = \phi$. Let $A_1 \subset A$ such that $g \ell p A_1$ but $g \not\ell p \overline{A}_1 \setminus A_1$. Then g contains no limit points of A_1^* and none of $(\overline{A}_1 \setminus A_1)^*$ by assumption and continuity of p. Hence g contains no limit points of $(\overline{A}_1 \setminus A_1)^* \cup A_1^* = (\overline{A}_1)^*$. But this is a closed inverse set and since p is quasi-compact, $g \not\ell p \overline{A}_1$.

In recent articles ([20] and [21]), Whyburn has introduced the notions of M' and weak M', calling spaces with these properties "accessibility spaces." He has proven a stronger statement than Proposition 13, showing that we cannot improve on weak M' as a topological condition on I to guarantee that p is pseudo-open, as a T_1 space which is not weak M' can be expressed as a quotient whose corresponding projection fails to be pseudo-open. The author's work with these concepts was done independently and prior to the appearance of Whyburn's publications.

Weak M' does not yield M' or the other separation properties mentioned here even with first countability which we will show makes those properties equivalent.

Example A. Let X be the subspace of the plane consisting of $(0,0) \cup (0,1) \cup \bigcup_{n=1}^{\infty} g_n$, where $g_n = \{(\frac{1}{n}, y):0 \leq y \leq 1\}$. Let $H_G = \{g_n\}$. Then $\mathcal{I}$ is first countable T_1 and weak M'. p is open. But $\mathcal{I}$ is not T_2, not M', not Mc, not M.

That $\mathcal{I}$ need not be weak M' in order for p to be pseudo-open even if X is metric can be seen by modifying Example A to include the other limit points of the lines, i.e., let

$$X = \{(0,y):0 \leq y \leq 1\} \cup \bigcup_{n=1}^{\infty} g_n$$

with $H_G = \{g_n\}$. Then p is still open but weak M' fails.

2.3. Some partial converses

Proposition 14. M, first countable $\Longrightarrow T_2$

Proof. Suppose $x \neq y$ and x and y do not have disjoint neighborhoods. Let $\{U_i\}$ and $\{V_i\}$ be countable neighborhood bases at x and y, respectively. Then $U_i \cap V_i \neq \phi$ for each i. Let $z_i \in U_i \cap V_i$. Then $z_i \to x$ and $z_i \to y$, contrary to M.

Definition. A space is E iff limit points are sequential limits, i.e., $x \,\ell p\, A \Longrightarrow$ there is a sequence $x_n \in A \setminus x$ such that $x_n \to x$.

This is called a Frechet space by some. McDougle dubbed it E [14].

Clearly, $T_1, E \Longrightarrow (y \; \ell p \cup x_n \Longrightarrow$ some subsequence $x_{n_i} \to y)$

$$M, E \Longrightarrow (x_n \to x, \; y \; \ell p \cup x_n \Longrightarrow x = y)$$

i.e., $M, E \Longrightarrow (x_n \to x \Longrightarrow \cup x_n \cup x$ is closed)

<u>Proposition 15</u>. $M, E \Longrightarrow KC$

<u>Proof</u>. Suppose K is compact and $x \in \overline{K} \setminus K$. By E, there is a sequence $x_n \in K \setminus x$ with $x_n \to x$. Since $M \Longrightarrow T_1$, $\cup x_n$ is infinite. So there is a point $k \in K$ with $k \; \ell p \cup x_n$ since K is compact. But $k \neq x$.

<u>Proposition 16</u>. $M, E \Longrightarrow M'$

<u>Proof</u>. Suppose $x \; \ell p \; A$. Then $x_n \to x$ for some sequence $x_n \in A \setminus x$. And $x \; \ell p \cup x_n$. Let $B = \cup x_n$. Then $x \; \ell p \; B$ but $B \cup x$ is closed by M, E.

So in an E space, M', Mc, M and KC are all equivalent and in a first countable space they are equivalent to T_2. But we do not get T_2 from these if only E is assumed.

<u>Proposition 17</u>. E is preserved by pseudo-open maps.

<u>Proof</u>. If $g \; \ell p \; A$ in $\mathcal{T}$ then there is a point $x \in g$ such that $x \; \ell p \; A^*$. Hence a sequence x_n in A^* converges to x. Then $p x_n \in A$ and $p x_n \to px = g$ by continuity.

<u>Example B</u>. A space which is M', E but not T_2.

This example is a well-known one in which a closed map (with non-compact point-inverses) does not preserve T_2. Let X be the subset of the plane consisting of $\{(x,y):y \geq 0\}$, with the topology in which neighborhoods of a point off the x-axis are the ordinary E^2 neighborhoods and those of a point x on the axis consist of the point x plus an open disk tangent to the axis at x. Let $H_G = \{Q,J\}$, where $Q = \{(x,0):x$ is rational$\}$ and $J = \{(x,0):x$ is irrational$\}$. Then p is closed, so $\mathcal{I}$ is M' and E, as X is. Hence $\mathcal{I}$ is also Mc, M and KC. But $\mathcal{I}$ is not T_2 as Q and J do not have disjoint neighborhoods.

__Definition.__ $f:X \to Y$ is __compact__ iff $f^{-1}(K)$ is compact for each compact subset K of Y. f is __point-compact__ iff $f^{-1}(y)$ is compact for each point $y \in Y$.

It is well-known that X is normal and p is closed $\Longrightarrow$ $\mathcal{I}$ is T_2 and that X is T_2 and p is closed and point-compact $\Longrightarrow$ $\mathcal{I}$ is T_2. (Also, closed and point-compact $\Longrightarrow$ compact.) But we can obtain the equivalence of T_2 with the separation properties being considered under the weaker condition of pseudo-open and point-compact if the underlying space X is first countable.

__Proposition 18.__ X is first countable, p is pseudo-open and point-compact $\Longrightarrow$ T_2, M, Mc and M' are equivalent on $\mathcal{I}$.

__Proof.__ Note that such a decomposition must be an E space, although it need not be first countable, and since M, Mc and M' are equivalent in an E space, it suffices to show that one of them implies T_2. So we suppose $\mathcal{I}$ is M.

Let $g \neq h$ in $\mathcal{I}$. Let $x \in g$, $y \in h$ with countable neighborhood bases $\{U_i\}$ and $\{V_i\}$, respectively. For each i, consider pU_i and pV_i. Suppose these intersect for each i. Then there is an element $g_i \in G$ meeting both U_i and V_i. Let $x_i \in U_i \cap g_i$ and $y_i \in V_i \cap g_i$. Then $x_i \to x$ and $y_i \to y$. So $g_i \to g$ and $g_i \to h$ by continuity. But this contradicts M. Thus there is an integer i such that $pU_i \cap pV_i = \phi$.

Since x and y were arbitrary points of g and h, respectively, for each $x \in g$, for each $y \in h$ there are neighborhoods $U_y(x)$ of x and $V(y)$ of y such that $p(U_y(x)) \cap p(V(y)) = \phi$. Covering h with a finite number of the sets $V(y)$, let $V_x = \bigcup_{i=1}^{k} V(y_i)$ so that V_x is open in X containing h. Let $U(x) = \bigcap_{i=1}^{k} U_{y_i}(x)$. Then $U(x)$ is open containing x and $p(V_x) \cap p(U(x)) = \phi$. Now, $\{U(x) : x \in g\}$ covers g. So there is a finite subcover. Let $U' = \bigcup_{j=1}^{\ell} U(x_j)$ with U' open in X containing g. Let $V' = \bigcap_{j=1}^{\ell} V_{x_j}$. Then V is open and contains h, while $pU' \cap pV' = \phi$. Now, since p is pseudo-open, the sets pU' and pV' have interiors containing g and h, respectively. So we have T_2.

<u>Corollary 18.1</u>. If X is first countable and p is point-compact, then $\mathcal{I}$ is $M' \implies \mathcal{I}$ is T_2.

The results so far suggest that T_2 is somehow "stronger" than M'. This is far from the case. The following example provides a space which is T_2 and not M' and also illustrates that the assumption of a pseudo-open map was crucial in the preceding proposition.

<u>Example C</u>. Let $X = E^2 \setminus \{(0,y) : y > 0\}$. Let G be the

decomposition of X such that $H_G = \{g_n\}_{n=1}^{\infty}$, where $g_n = \{(\frac{1}{n}, y): 0 \leq y \leq 1\}$. Then $\mathcal{I}$ is T_2 and hence Mc and M, but not M'. $\mathcal{I}$ is not first countable. ($\mathcal{I}$ is first countable at every point except $g = \{(0,0)\}$.) p is not pseudo-open (at g). But X is metric and p is point-compact and monotone.

We have observed that a monotone pseudo-open map assures that inverses of connected sets are connected. To see how it fails here without the pseudo-open condition but in the presence of other nice properties, let A be the projection of $\{(x,y): y > 1\}$. Then $g \not\in pA$ and $A \cup \{g\}$ is connected in $\mathcal{I}$ while $A^* \cup g$ is not connected in X. p is not pseudo-open at g since no point of g is a limit point of A^*. Any subset of A having g as a limit point must also have g_n as a limit point for infinitely many n. In this way, weak M' fails.

The fact that the space $\mathcal{I}$ in this example is not first countable is disconcerting in itself, as this is a point-compact decomposition of a (complete) metric space which is McAuley-usc (and monotone). This illustrates a difference between McAuley's definition of upper semi-continuity and that of Whyburn, as the latter would have to yield a metrizable decomposition space. As we will see, this cannot occur if X is locally compact, as it fails to be in this example at the point g. In fact, if X is a locally compact E space and p is monotone and point-compact, then $\mathcal{I}$ is Mc $\Longrightarrow$ p is closed.

Another example of a T_2 space which is not M' is the following, which corrects an assertion in [20] that locally compact T_2 yields M'.

<u>Example D.</u> A compact T_2 space which is not M'.

Let X be the space of ordinals $\leq \Omega$, where Ω is the first uncountable ordinal. X is compact, T_2. Let E be the set of limit ordinals in X, i.e., elements which have no (immediate) predecessors. Let $A = X \setminus E$. Then $\Omega \in E$ and $\Omega \, \ell p A$. But for any subset $B \subset A$ such that $\Omega \, \ell p B$ and any neighborhood U of Ω, there is a point $e \in U \cap E \setminus \{\Omega\}$ such that $e \, \ell p B$. Hence, X is not weak M' and so, in particular, X is not M'.

Weak M' must fail in a space which is compact T_2 and not M' because of the next proposition.

<u>Proposition 19</u>. Regular T_1, weak M' $\Longrightarrow$ M'

<u>Proof</u>. Let $x \in \bar{A} \setminus A$. For some $A_1 \subset A$, x is an isolated point of $\bar{A}_1 \setminus A_1$. So there is an open set U containing x such that U contains no other points of $\bar{A}_1 \setminus A_1$. By regularity, there is an open set V satisfying $x \in V \subset \bar{V} \subset U$. Let $B = \bar{V} \cap A_1$. Then $x \, \ell p B$ and $B \cup x$ is closed, since if $y \, \ell p B$ and $y \notin B$, $y \, \ell p A_1$ and $y \in \bar{V}$ so $y \notin A_1$. Hence $y \in \bar{A}_1 \setminus A_1$. But $\bar{V} \subset U$. So $y = x$.

<u>Corollary 19.1</u>. Compact (locally compact, locally peripherally compact) T_2, weak M' $\Longrightarrow$ M'.

2.4. <u>Other conditions weaker than first countability</u>.

<u>Definition</u>. (Christoph [2]). X is <u>semi-first countable</u> (semi-1^{st}) iff whenever A_i is a sequence of closed disjoint sets such that $\cup A_i$ is not closed then there exists a $t \in \overline{\cup A_i} \setminus \cup A_i$ and a subsequence A_{i_k} and $x_{i_k} \in A_{i_k}$ such that $x_{i_k} \to t$.

If we require such a sequence for each limit point of $\cup A_i$ which is not in $\cup A_i$, we get a stronger notion.

<u>Definition</u>. X is <u>strongly semi-1st</u> iff whenever A_i is a sequence of closed disjoint sets and $x \in \overline{\cup A_i} \setminus \cup A_i$ then there is a subsequence A_{i_k} and $x_{i_k} \in A_{i_k}$ such that $x_{i_k} \to x$.

We can state something like this for arbitrary rather than countable collections.

<u>Definition</u>. X has <u>Property P</u> iff whenever $\{A_\nu\}$ is a collection of closed disjoint sets and $x \in \overline{\cup A_\nu} \setminus \cup A_\nu$, then there is a set $P \subset \cup A_\nu$ such that no A_ν contains more than one point of P and $x \ell p P$.

In Property P limit points are required to be accessible not necessarily by sequences but by "selections" from the A_ν. We do not necessarily get convergent subsequences, however, even in case P is countable.

<u>Definition</u>. X is <u>countably E</u> (c-E) iff A is countable, $x \ell p A \implies$ there is a sequence $x_n \in A \setminus x$ such that $x_n \to x$. (E applied to countable sets.)

<u>Proposition 20</u>. E $\implies$ Prop P.

<u>Proof</u>. Suppose $x \in \overline{\cup A_\nu} \setminus \cup A_\nu$, where A_ν are closed and mutually disjoint. By E, there is a sequence $x_n \in \cup A_\nu$ such that $x_n \to x$. No A_ν contains infinitely many x_n, since $x \notin \cup A_\nu$. So there is a subsequence $x_{n_i} \in A_{\nu(i)}$, with $A_{\nu(i)}$ distinct. Let $P = \cup x_{n_i}$.

<u>Proposition 21</u>. Property P and c-E $\implies$ strongly semi-1st

<u>Proof.</u> If $x \in \overline{\cup A_i} \setminus \cup A_i$, property P gives $x_{n_i} \in A_{n_i}$ such that $x \, \ell p \cup x_{n_i}$ and by $c - E$ a subsequence $x_{n_{i_k}} \to x$.

Franklin [6] and Richel [15] have given the following definition.

<u>Definition.</u> X is a <u>c - space</u> iff the closure of each set is the union of the closures of its countable subsets.

Clearly, c - space is equivalent to: $x \, \ell p \, A \Longrightarrow$ there is a countable subset B of A such that $x \, \ell p \, B$. Of course, any countable space is a c - space.

<u>Proposition 22.</u> c - space, $c - E \Longrightarrow E$

<u>Proof.</u> If $x \, \ell p \, A$ then $x \, \ell p \, B$ for some countable subset B of A and $c - E$ gives a sequence $x_n \in B$ such that $x_n \to x$.

<u>Example E.</u> (Kelley [10; p. 77] originally Arens). Let $X = N \times N + x$, where X is discrete at each point of $N \times N$ while an (open) neighborhood of x is a set containing x and all but finitely many points of all but finitely many "columns" (i.e., sets having fixed first coordinates) of $N \times N$. This space is countable, normal T_1, with closed sets G_δ. The only compact sets are finite. No sequence from $X \setminus x$ converges to x. So it is not sequential, not $c - E$, not Property P and not even semi-1^{st}, though it is trivially a c - space.

To see that it is not semi-1^{st}, consider $A_i = \{i\} \times N$, the i^{th} column. The sets A_i are closed, disjoint and $\overline{\cup A_i} \setminus \cup A_i = x$, while if we select only one point from each A_i, the complement of the resulting set is a neighborhood of x.

Definition. (Arhangel'skii [1]) X is weak-first countable (weak 1^{st}) iff for each $x \in X$ there is a countable collection T_x of sets containing x such that $T, T' \in T_x \Longrightarrow T \cap T' \in T_x$ and a set A is open iff for each $x \in A$ there is a $T \in T_x$ satisfying $T \subset A$. (Or, equivalently, A is closed iff for each $x \notin A$ there is a $T \in T_x$ such that $T \cap A = \phi$).

In a weak 1^{st} space, $x \in \bar{A} \setminus A \Longrightarrow$ for each $T \in T_x$, $T \cap (\bar{A} \setminus x) \neq \phi$. Note that the family $T_x = \{t_n(x)\}_{n=1}^{\infty}$ can be assumed nested, i.e., $t_{n+1}(x) \subset t_n(x)$, since $t'_j = \bigcap_{i \leq j} t_i$ is also in T.

The definition of weak 1^{st} gives easily: If $T_x = \{t_n\}_{n=1}^{\infty}$ is a nested weak base at x, then $y_n \in t_n \Longrightarrow y_n \to x$.

Also, it is easy to verify:

In a weak 1^{st} space, (1) $\bar{A} \setminus A \neq \phi \Longrightarrow$ there is an $x \in \bar{A} \setminus A$ and a sequence $y_n \in A$ such that $y_n \to x$, (2) $x \in \bar{A} \setminus A \Longrightarrow$ there is a sequence $y_n \in \bar{A} \setminus x$ such that $y_n \to x$, and (3) $\bar{A} \setminus A = \{x\} \Longrightarrow$ there is a sequence $y_n \in A$ such that $y_n \to x$.

Note that (1) $\Longrightarrow$ (2) and (2) $\Longleftrightarrow$ (3). These conditions hold in any E space, as well. Condition (1) has been studied by Franklin who calls spaces satisfying this sequential.

Definition. (Franklin [4]) A set is sequentially open if no sequence outside the set converges to a point inside. A sequential space is one in which every sequentially open set is open.

Clearly, (1) above is a characterization of a sequential space, so weak 1^{st} $\Longrightarrow$ sequential and E $\Longrightarrow$ sequential.

<u>Proposition 23</u>. sequential, weak M' $\Longrightarrow$ E

<u>Proof</u>. If $x \in \bar{A} \setminus A$ then for some subset $B \subset A$, $x \, \ell p \, B$ and $x \not\!\ell p \, \bar{B} \setminus B$. By (2) above, a sequence y_n in $\bar{B} \setminus x$ converges to x. This sequence must ultimately be in B.

<u>Example F</u>. A space in which (2), and hence (3), holds, but not (1).

Let $X = D + N$, where D is an uncountable discrete set and for each $n \in N$, a neighborhood of n is n + all but countably many points of D + all but finitely many points of n. The sequence $x_n = n \in N$ converges to each of its points. This space is not sequential, since no sequence in D converges to any of the limit points of D. However, (2) is satisfied, since if $x \in \bar{A} \setminus A$, then $x \in N$ and $N \subset \bar{A}$. So the sequence $x_n = x + n$ is a sequence lying in $\bar{A} \setminus x$ and converging to x.

<u>Proposition 24</u>. sequential $\Longrightarrow$ semi-1^{st}

<u>Proof</u>. Let A_i be closed, disjoint such that $\cup A_i$ is not closed. There is an $x \in \overline{\cup A_i} \setminus \cup A_i$ and a sequence $y_n \in \cup A_i$ converging to x. No A_i can contain infinitely many y_n since each A_i is closed and $x \notin \cup A_i$. So there are subsequences y_{n_j} and A_{i_j} with $y_{n_j} \in A_{i_j}$.

In [1], Arhangel'skii introduces the notion of weak 1^{st} and asserts that weak 1^{st} and E $\Longleftrightarrow$ first countable. In that section he assumes all spaces completely regular. We can give a proof assuming M.

<u>Proposition 25</u>. Weak 1^{st}, weak M' $\Longrightarrow$ first countable

<u>Proof</u>. Let $\{t_n(x)\}$ be a nested weak base at x. If $x \in \text{int}\, t_n$ for infinitely many n, then we have a base. So we may assume for each n, $x \notin \text{int}\, t_n$. So $x\, \ell p\, X \setminus \cup t_n$. By weak M' there is a set $B \subset X \setminus \cup t_n = X \setminus t_1$ such that $x \in \overline{B} \setminus B$ and $x \not\ell p\, \overline{B} \setminus B$. Then there is an open set U containing x such that $U \cap (\overline{B} \setminus B) = \{x\}$. But $(X \setminus B) \cap U$ is open in the weak base topology, for if $z \in (X \setminus B) \cap U$ and $z \neq x$ then $z \notin \overline{B}$. So $(X \setminus \overline{B}) \cap U$ is an open set containing z and lying in $(X \setminus B) \cap U$. And as for x, $X \setminus B$ contains all of the weak neighborhoods of x while U contains them ultimately. So $(X \setminus B) \cap U$ contains a weak basic neighborhood of x. Then $(X \setminus B) \cap U$ is an open set containing x and missing B, which contradicts $x\, \ell p\, B$.

It is interesting to observe that not only does weak M' guarantee that weak 1^{st} $\Longrightarrow$ first countable but that the interiors of an arbitrary weak base must give a base.

<u>Corollary 25.1</u>. (Arhangel'skii) Weak 1^{st}, E and M $\Longrightarrow$ first countable.

<u>Proof</u>. This follows from Proposition 25 since M,E $\Longrightarrow$ M' $\Longrightarrow$ weak M'.

Also, note that in the corollary we get T_2 as well, since in a first countable space, M and T_2 are equivalent.

From the results so far, a T_2 sequential space is E iff it is M' and a T_2 weak 1^{st} space is first countable iff it is M' (also, iff it is weak M').

While first countable and E are hereditary properties, weak 1^{st}

is not. In fact,

<u>Proposition 26</u>. hereditarily weak 1^{st} $\Longrightarrow$ E

<u>Proof</u>. Suppose $x \in \bar{A} \setminus A$. Consider the subspace $B = A \cup x$. Then since B is weak 1^{st} and in B, $\bar{A} \setminus A = \{x\}$, there is a sequence $y_n \in A$ such that $y_n \to x$.

In Proposition 26, we have used only the sequential property of weak 1^{st} so we actually have no more than Franklin's result that hereditarily sequential $\Longrightarrow$ E .

<u>Corollary 26.1</u>. hereditarily weak 1^{st}, M $\Longrightarrow$ first countable

<u>Example G</u>. A space which is weak 1^{st}, E and T_1 but not first countable. (not hereditarily weak 1^{st})

Let $X = x + \{w_k\}_{k=1}^{\infty} + \bigcup_{n=1}^{\infty} \gamma_n$, where each γ_n is a sequence $\{x_j^n\}$, such that $w_k \to x$ and for each n, $\gamma_n = \{x_j^n\} \to x$ and $\gamma_n \to w_k$ for each $k \geq n$. To achieve this convergence, let the topology be defined as follows. X is discrete at each x_j^n. For each k, a neighborhood of w_k is w_k + the union of tails of each γ_n for $n \leq k$. A neighborhood of x is x + a tail of $\{w_k\}$ + the union of tails of each γ_n. (A <u>tail</u> of a sequence $\{a_i\}$ is $\{a_i : i \geq k\}$ for some k.)

X is first countable at each point except x. This is trivially so at each x_j^n. For w_k, each J let $W_J(w_k) = w_k + \bigcup_{n \leq k}$ (J^{th} - tail of γ_n), where the J^{th} - tail of γ_n is $\{x_j^n : j \geq J\}$. This is a base at w_k, since if U is open containing w_k then U contains some j_i - tail of γ_i for each $i \leq k$. Let

$J = \max\{j_1,\cdots,j_k\}$. Then U contains $W_J(w_k)$.

X is E, since if $x \ell p A$ then A has to contain some subsequence of $\{x_j^n\}$ for some n or of $\{w_k\}$. Either way this gives a sequence in A converging to x.

X is not first countable at x, for if we suppose that $\{V_n\}$ is a countable local base at x, then each V_n is open and must contain a tail of each γ_n. Let $x_n \in V_n \cap \gamma_n$. Then $X \setminus \{x_n\}_{n=1}^{\infty}$ still contains a tail of each γ_n and contains all $\{w_k\}$, hence is an open set containing x and not containing any V_n.

X is weak 1^{st}, since we have a countable open base at each point other than x and we may let $t_n(x) = x +$ the n^{th}-tail of $\{w_k\}$. Suppose A contains x and contains a weak basic neighborhood of each of its points. Then for each n, A contains some w_k with $k > n$ and hence must contain a tail of γ_n. So A contains a tail of each γ_n and a tail of w_n, i.e., a neighborhood of x. So A is open.

We can modify the space of Example G to be compact without altering the other properties by adding to the space a point z whose neighborhoods are of the form $z + \bigcup_{n \geq N} \gamma_N$.

So a compact, weak 1^{st}, E, T_1 space need not be first countable. However, this space would not be LW locally compact (see §2.6 for the definition of LW locally compact).

<u>Question 1</u>. LW locally compact, weak 1^{st}, E, $T_1 \Longrightarrow$ first countable?

Also the space of Example G is not hereditarily weak 1^{st}. (The subspace $X \setminus \{w_k\}$ is not weak 1^{st}, as it is discrete at all except the point x, where it is not first countable.) So another question can be raised:

<u>Question 2</u>. hereditarily weak $1^{st} \Longrightarrow$ first countable?

If the answer to Question 2 is affirmative, the proof will not be trivial since the hypothesis does not force an arbitrary weak base to provide a base (as in the case of M or weak M'), as illustrated next.

<u>Example H</u>. A space which is first countable, T_1 but a weak base may not be a base.

Let $X = x + \{w_n\}_{n=1}^{\infty} + \{x_j\}_{j=1}^{\infty}$ such that $w_n \to x$ and $x_j \to x$ and $x_j \to w_n$ for each n. i.e., X is discrete at each x_j. A neighborhood of w_n is w_n + a tail of $\{x_j\}$; a neighborhood of x is x + a tail of $\{x_j\}$ + a tail of $\{w_n\}$.

X is first countable, since a base at w_n is given by $V_k(w_n) = w_n$ + the k^{th} tail of $\{x_j\}$, and a base at x by $V_k(x) = x$ + the k^{th} tail of $\{x_j\}$ + the k^{th} tail of $\{w_n\}$. However, if we take the same base at each w_n, but at x take only x + the tails of w_n, we get a weak base which is not a base, i.e., x is not interior to its weak basic neighborhoods.

<u>Definition</u>. If G is a collection of subsets of X, X is <u>first countable with respect to G</u> (1^{st} countable wrt G) iff for each $g \in G$ there is a sequence $\{U_n\}$ of open sets containing g such that $g \subset R$ open $\Longrightarrow$ there is an n such that $U_n \subset R$.

<u>Proposition 27</u>. X is 1^{st} countable with respect to $G \Longrightarrow I$ is weak 1^{st}.

<u>Proof.</u> Let $\{U_n(g)\}$ be a sequence of open sets containing g as in the definition of 1st countable with respect to G. Then $\{p(U_n(g))\}$ is a weak base for $\mathcal{I}$ at g, i.e., R is open in $\mathcal{I}$ iff for each $g \in R$ there is an integer n such that $pU_n(g) \subset R$, or, equivalently, R^* is open in X iff for each $g \in G$ such that $g \subset R^*$ there is an integer n such that $p^{-1}pU_n(g) \subset R^*$. To see this, note that if R^* is open containing g then for some n, $U_n(g) \subset R^*$. So $pU_n(g) \subset R$. And conversely, if for each $g \in G$ such that $g \subset R^*$ there exists n such that $p^{-1}pU_n(g) \subset R^*$, then, as each point $x \in R^*$ belongs to some such g, we have for each $x \in R^*$, for some n, $x \in U_n(p(x)) \subset p^{-1}p(U_n(p(x))) \subset R^*$. But $U_n(p(x))$ is open so $x \in \text{int } R^*$. Hence R^* is open.

<u>Corollary 27.1.</u> A point-compact decomposition of a developable space is weak 1st.

<u>Proof.</u> The corollary is immediate from the lemma below.

The following lemma is surely known, but as we have not encountered its proof anywhere else, we include it here. For a discussion of developable spaces, see [24].

<u>Lemma 27.2.</u> X is developable $\Rightarrow$ X is 1st countable with respect to compact sets.

<u>Proof.</u> Let $\{G_n\}$ be a monotone development for X, i.e., $G_{n+1} \subset G_n$ for each n. (It is easy to show that for any developable space there exists such a development.) Let K be a compact subset of X. Let U_1 be a finite collection of elements of G_1 covering K. Let U_2 be a finite collection of elements of G_2 covering K and

such that each element u of U_2 contains a point $x_u \in U \cap K$ such that u is contained in each element of U_1 containing x_u, i.e., $u \subset \bigcap U_1(x_u)$. And in general, given U_i for $i \leq n$, let U_{n+1} be a finite collection of elements of G_{n+1} covering K and such that for each element $u \in U_{n+1}$ there is a point $x_u \in u \cap K$ such that u is contained in every element of $\bigcup_{i=1}^{n} U_i$ which contains x_u. (To obtain this, note that for each $x \in K$ the collection of all elements of $\bigcup_{i=1}^{n} U_i$ which contain x is finite and its intersection V is an open set containing x. For some $N \geq n+1$, $G_N^*(x)$ is contained in this open set V, so some $u(x) \in G_N$ contains x and lies in V. By monotonicity, $u(x)$ is also in G_{n+1}. So we select this $u(x)$ for each $x \in K$, producing a cover of K by elements of G_{n+1}. We take a finite subcover, $U_{n+1} = \{u(x_1), \cdots, u(x_\ell)\}$. Then if $u \in U_{n+1}$, $u = u(x_i)$ for some i and x_u in the notation above is x_i.)

Now, $\{U_n^*\}_{n=1}^{\infty}$ is a countable collection of open subsets of X containing K, and if R is open containing K then for some n, $U_n^* \subset R$. Otherwise, there is an open set $R \supset K$ such that for each n, $U_n^* \setminus R \neq \phi$ so $u_n \setminus R \neq \phi$ for some $u_n \in U_n$. Consider $x_{u_n} \in K$. Since K is compact there is a point $x \in K$ and a subsequence $x_{u_{n_i}} \to x$. For some integer N, $G_N^*(x) \subset R$. Now, some element $u \in U_N$ contains x and for some $I > N$, u contains $x_{u_{n_i}}$ for $i \geq I$ (since $x_{u_{n_i}} \to x$) and $u \subset R$. In particular, $x_{u_{n_I}} \in u$. But since $x_{u_{n_I}} \in u \in U_N$ and $n_I \geq I > N$, $u_{n_I} \subset u$ by the construction of U_{n_I}. Hence $u_{n_I} \subset R$ and we have a contradiction.

The notion here called 1st countable with respect to G was suggested by F. B. Jones. It was hoped that a semimetric space having this property with respect to compact sets would be developable. However, Heath gave an example of a semimetric nondevelopable space which is 1^{st} countable with respect to compact sets [9]. Of course, any point-compact decomposition of Heath's space would also be weak 1^{st}.

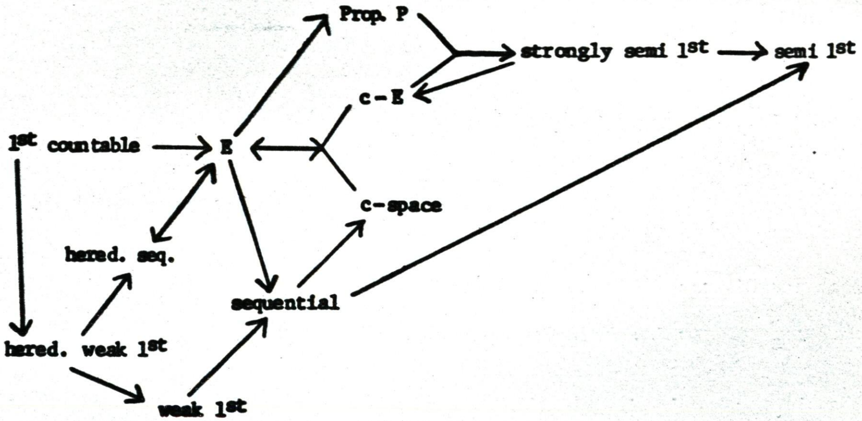
<u>Corollary 27.3</u>. X is 1^{st} countable with respect to G, p pseudo-open $\Longrightarrow \mathcal{I}$ is first countable.

<u>Proof</u>. This is corollary to the proof of Proposition 26, as the weak base for $\mathcal{I}$ at g, $\{p(U_n(g))\}$, must provide a base if p is pseudo-open, i.e., $g \in \text{int } p\,(U_n(g))$.

<u>Corollary 27.4</u>. A pseudo-open point-compact decomposition of a developable space is first-countable.

In Corollary 27.4 the condition that X be developable cannot be weakened to semimetric even if p is closed. (see Example R.)

To return to the consideration of the properties introduced in this section, we have for arbitrary spaces:



__Remark.__ A number of these properties can be associated in pairs in a natural way. Some are of the type (α): If A is not closed, then there exists a point $x \in \bar{A} \setminus A$ such that $P(x,A)$, where $P(x,A)$ is some property of x and A, e.g., some sequence in A converges to x, or x is a limit point of a countable subset of A, etc. For each definition of this type there is a potentially stronger form requiring the property hold for A and for __each__ $x \in \bar{A} \setminus A$, i.e., type (β): if $x \in \bar{A} \setminus A$ then $P(x,A)$. For instance, Franklin's "sequential" is the (α) form whose corresponding (β) form is the Frechet condition, E .

Whenever type (α) holds hereditarily and $P(x,A)$ is passed from subspaces to the whole space, then (β) holds. In most cases here considered, hereditarily (α) $\Longrightarrow$ (β). And whenever the (β) form is hereditary, we would also have hereditarily (α) $\Longleftrightarrow$ (β). For example, hereditarily sequential $\Longleftrightarrow$ E and hereditarily semi-1^{st} $\Longleftrightarrow$ strongly semi-1^{st}. We can also state the definitions of quotient and pseudo-open maps in such a way that a quotient map is of type (α) and a pseudo-open (hereditarily quotient) map is of type (β).

The (α) and (β) forms for the definition of c-space are equivalent. So sequential $\Longrightarrow$ c-space, though in general (α) forms are weaker than (β) forms.

This suggests a way of generalizing properties which have been introduced by a definition of type (β). The weaker form of M' is: A is not closed $\Longrightarrow$ there exists $x \in \bar{A} \setminus A$ and $B \subset A$ such that $x \notin p B$ and $B \cup x$ is closed. (This is not equivalent to what we have called weak M'.) This, however, is an instance in which the property

$P(x,A)$ does not extend from the subspace back to the space, as a set may be closed in the subspace without being closed in the space. For instance, the space $X = \{\text{ordinals} \leq \Omega\}$ does satisfy this (α) form of M' hereditarily but it is not M'.

However, M' can provide a link between these pairs, the essential difference between the (α) and (β) types being that the (α) form asserts a condition for some element of $\overline{A} \smallsetminus A$ and the (β) form for each element of $\overline{A} \smallsetminus A$. If $\overline{A} \smallsetminus A$ is a single point the two are equivalent. For an arbitrary element x of $\overline{A} \smallsetminus A$, M' provides a subset $B \subset A$ such that $\overline{B} \smallsetminus B$ is precisely the single point x. The (α) form yields $P(x,B)$ and for the properties considered here, this implies $P(x,A)$. As we have already noted, sequential $M' \Longrightarrow E$ and, in a less straightforward manner, semi-1^{st} $M' \Longrightarrow$ strongly semi-1^{st}.

<u>Example I.</u> A space which is weak 1^{st} (hence semi-1^{st}) but not $c-E$ (hence not strongly semi-1^{st}).

As in Example C, let $X = E^2 \smallsetminus \{(0,y):y > 0\}$, $H_G = \{g_n\}_{n=1}^{\infty}$, where $g_n = \{(\frac{1}{n}, y):0 \leq y \leq 1\}$, $g = p(0,0)$. Then I is a point-compact decomposition of a metric space, hence weak 1^{st} (see Corollary 27.1).

For each i,n let $x_i^n = (\frac{1}{n}, 1 + \frac{1}{i})$. Then $A = \{x_i^n\}_{i,n=1}^{\infty}$ is countable and $g \, \ell p \, A$ but no sequence in A converges to g, so $c-E$ fails.

<u>Example J.</u> A space which has Property P and $c-E$ (hence strongly semi-1^{st}) but not sequential (hence not E and not weak 1^{st}).

Let X be as in Example D, $X = \{\text{ordinals} \leq \Omega\}$. X is c-E since the space is first countable at every point but Ω while Ω is not a limit point of any countable set. Also, X has Property P: if $\Omega \, \ell p \bigcup A_\nu$ with A_ν closed, disjoint and $\Omega \notin \bigcup A_\nu$, then $\{A_\nu\}$ is uncountable (otherwise $y_i = \sup A_i$ has $\sup < \Omega$) so we can choose any $p_\nu \in A_\nu$ and $P = \{p_\nu\}$ is uncountable, whence $\Omega \, \ell p \, P$ (otherwise $\sup P < \Omega$ with P uncountable). X is not sequential since no sequence converges to Ω at all from $X \setminus \{\Omega\}$.

Semi-1^{st} is preserved by all quotient maps, while Example I illustrates that strongly semi-1^{st} is not (even if X is metric).

<u>Proposition 28</u>. (Christoph) X is semi-1^{st} $\Longrightarrow$ $\mathcal{I}$ is semi-1^{st}.

<u>Proof</u>. Suppose $\{A_i\}$ is a countable collection of closed disjoint sets in $\mathcal{I}$ while $\bigcup A_i$ is not closed. Then $\bigcup A_i^*$ is not closed in X, while $\{A_i^*\}$ are closed, disjoint. So there exists $t \notin \bigcup A_i^*$ and $x_{i_j} \in A_{i_j}^*$ such that $x_{i_j} \to t$. Then $p(x_{i_j}) \in A_{i_j}$ and $p(x_{i_j}) \to p(t)$.

Similarly, sequential and c-space are each preserved by quotient maps.

<u>Proposition 29</u>. Property P is preserved by pseudo-open maps.

<u>Proof</u>. Suppose $g \in \overline{\bigcup A_\nu} \setminus \bigcup A_\nu$, with A_ν closed, disjoint. Then there is a point $x \in g$ such that $x \, \ell p \bigcup A_\nu^*$ and hence a subset $P \subset \bigcup A_\nu^*$ such that $x \, \ell p \, P$ and no A_ν^* contains more than one point of P. Then $g \, \ell p \, p(P) \subset \bigcup A_\nu$ and no A_ν contains more than one point of $p(P)$.

Similarly, strongly semi-1^{st} is preserved by pseudo-open maps. The proof of this proceeds exactly like that of Proposition 29. Pursuing remarks made earlier on these definitions, a general principle operates in the case of propositions such as 27 above. When hereditarily type $(\alpha) \Longleftrightarrow$ type (β) and (α) is preserved by quotients, then (β) is preserved by pseudo-open maps.

Weak 1^{st} may not be preserved by closed maps (see Example B).

Many statements about decompositions can be culled from combinations of the results above, which we will not explicitly state here. For example, if X is a semi-1^{st} c-space then I is $M' \Longrightarrow I$ is E.

We might ask how these conditions further affect implications between T_2, M', etc. We have found that weak 1^{st}, $M' \Longrightarrow T_2$ but this is only an apparent improvement since it gives first countability anyway. Each of the Examples I and J is T_2 while neither is weak M' so it seems nothing in the list less than E will give $T_2 \Longrightarrow M'$.

<u>Proposition 30</u>. M, sequential $\Longrightarrow$ ($x_n \to x \Longrightarrow \cup x_n \cup x$ is closed).

<u>Proof</u>. Suppose $x_n \to x$ but $\cup x_n \cup x$ is not closed. Then there is a point $y \notin \cup x_n \cup x$ and $y_i \in \cup x_n \cup x$ with $y_i \to y$. Since the space is T_1 we may assume $\{y_i\}$ is a subsequence of $\{x_n\}$. So $y_i \to x$, which contradicts M, since $y \neq x$.

<u>Proposition 31</u>. M, sequential $\Longrightarrow$ KC

<u>Proof</u>. Suppose K is compact and not closed. Then there is a point $x \notin K$ and a sequence $x_n \in K$ such that $x_n \to x$. By T_1, $\cup x_n$ is infinite and since K is compact some point $k \in K$ is a

limit point of $\cup x_n$. We can assume $k \notin \cup x_n$. But $\cup x_n \cup x$ is closed by Proposition 30, which gives a contradiction.

We have seen that compact T_2 does not yield M'. Similarly, compactness does not make M' stronger than T_2. The following example also appears in [5].

<u>Example K.</u> A space which is compact M' (in fact, M, E) but not T_2.

Let $X = x + w + \bigcup_{n=1}^{\infty} \gamma_n$, where each γ_n is a sequence $\{x_j^n\}$ such that each $\gamma_n \to x$ and $\{\gamma_n\}_{n=1}^{\infty} \to w$. X is discrete at each x_j^n; a neighborhood of x is $x +$ the union of tails of each γ_n; a neighborhood of w is $w +$ the union of all γ_n for $n > N$. X is first countable at every point except x since $V_k(w) = w + \bigcup_{n \geq k} \gamma_n$ gives a base at w. X is E since if $x \, \ell p \, A$ then A must contain a subsequence of some γ_n. X is M: if a sequence $z_n \to x$ then $\{z_n\} \subset \bigcup_{n \leq N} \gamma_n$, for some N, so it can't converge to w. Otherwise, there is a subsequence γ_{n_i} of the sequence of sets $\{\gamma_n\}_{n=1}^{\infty}$ such that $\{z_n\}$ meets each γ_{n_i}. Let $z_i' \in \{z_n\} \cap \gamma_{n_i}$. Then $X \setminus \{z_i'\} \setminus w$ is an open set containing x and missing the sequence $\{z_i'\}$. This contradicts $z_n \to x$. X is compact since a neighborhood of w covers all but a finite number of the sets γ_n, while a neighborhood of x covers all but a finite number of points of these. X is not T_2 since every neighborhood of x must contain a tail of each γ_n and hence meets every neighborhood of w.

Since the space of Example K is M,E it is also KC. We have not exhibited a space which is M' and not KC so we pose the following question.

<u>Question 3</u>. Does $M' \implies KC$?

So far we have only the following partial answers to this question.

<u>Proposition 32</u>. If points are G_δ, $M' \implies KC$.

<u>Proof</u>. Suppose K is compact and $x \in \overline{K} \setminus K$. Then $x = \bigcap_{n=1}^{\infty} G_n$, where for each n, G_n is open and $G_n \supset G_{n+1}$. By M' there is a subset $K_1 \subset G_1 \cap K$ such that $x \ell p K_1$ and $K_1 + x$ is closed. And in general, given $K_n \subset G_n \cap K_{n-1}$ with $x \ell p K_n$ and $K_n + x$ closed there is a subset $K_{n+1} \subset G_{n+1} \cap K_n$ such that $x \ell p K_{n+1}$ and $K_{n+1} + x$ is closed. Now, $\{X \setminus (K_n + x)\}_{n=1}^{\infty}$ is an open cover of K, as it covers $X \setminus x$. So by the compactness of K, for some N, $K \subset \bigcup_{n=1}^{N} X \setminus (K_n + x)$. But this set is $X \setminus (K_N + x)$, which gives a contradiction since $K_N \subset K$ and $K_N \neq \phi$.

The hypothesis of Proposition 32 is a relatively mild restriction (the space of Example E has points G_δ but it is not even semi-1^{st}), but we believe an unnecessary one.

<u>Corollary 32.1</u>. X is countable, $M' \implies X$ is KC.

<u>Proof</u>. Any countable T_1 space has points G_δ.

<u>Corollary 32.2</u>. M', c-space $\implies KC$

<u>Proof</u>. Suppose K is compact and not closed. Then there is a point $x \in \overline{K} \setminus K$ and a countable subset B of K such that $x \ell p B$.

By M', for some subset $B' \subset B$, $x \ell p B'$ and $B' + x$ is closed. Hence B' is closed in K and thus B' is compact. So we have B' compact, $B' + x$ is countable and, since M' is hereditary, $B' + x$ has M'. So by Corollary 32.1, B' is closed in $B' + x$ which contradicts $x \ell p B'$.

The following example shows that the weaker Mc does not imply KC.

Example L. A space which is compact Mc but not KC.

Let $X = I + x$, where $I = [0,1]$ with its usual topology and an open neighborhood of x consists of x plus the complement of a countable closed subset of I. So x is a limit point of any uncountable subset of I, making I a compact non-closed subset of X. The subspace I is T_2 so Mc holds for pairs of points in I. Now suppose $x,y \ell p A$ and $x \neq y$. Since X is first countable at $y \in I$, some sequence in A converges to y and the complement of this convergent sequence is a neighborhood of x. Also $\bar{A}$ is uncountable, so for some n, $\bar{A} \setminus N_{\frac{1}{n}}(y)$ is uncountable and x is a limit point of this set while clearly y is not. Hence $x \ell p B$, where $B = A \setminus N_{\frac{1}{n}}(y)$, a subset of A, and $y \not\ell p B$.

So Mc and KC are independent (the interval with the topology in which open sets are complements of countable sets is KC and not Mc).

Christoph [2] introduced the following notion related to Hausdorff-like properties of a decomposition space.

Definition. (Christoph) G is semi-Hausdorff (semi-H) iff whenever $x_i \to x$ and $px_i \to y$ then $y = px$.

$\mathcal{I}$ is M $\Longrightarrow$ G is semi-H, but the converse does not hold, as seen from Example M below. G is semi-H $\Longrightarrow$ $\mathcal{I}$ is T_1.

__Definition.__ (McDougle) A map $f: X \to Y$ is __semi-closed__ iff $f(K)$ is closed for each compact subset K of X.

p is semi-closed $\Longrightarrow$ $\mathcal{I}$ is T_1

__Proposition 33.__ p is semi-closed $\Longrightarrow$ G is semi-H.

__Proof.__ Suppose $x_i \to x$ and $px_i \to y$. By continuity, $px_i \to px$ and we can assume $y \notin \{px_i\}$ since $\mathcal{I}$ is T_1. So $y \; \ell p \bigcup px_i$. Since $\bigcup x_i \cup x$ is compact, $\bigcup px_i \cup px$ is closed. Hence $y = px$.

The converse of Proposition 33 is easily seen to be false by taking X to be any space which is M but not KC, $\mathcal{I} = X$ and $p = id_X$. Of course, $\mathcal{I}$ is KC $\Longrightarrow$ p is semi-closed. So if X is sequential and $\mathcal{I}$ is M then p is semi-closed.

__Proposition 34.__ X is semi-1[st], G is semi-H $\Longrightarrow$ $\mathcal{I}$ is M.

__Proof.__ Let $g_n \to g$ and $g_n \to g'$ with $g \neq g'$ in $\mathcal{I}$. We can assume the elements g_n are all different and none is g or g'. So $g \; \ell p \cup g_n$ and $g' \; \ell p \cup g_n$. Then $\cup g_n{}^*$ is not closed in X. So there exists x and a subsequence $x_{n_i} \in g_{n_i}{}^*$ such that $x_{n_i} \to x$. Then $px_{n_i} \to px$. But $px_{n_i} = g_{n_i}$. So by semi-H, $px = g = g'$.

__Corollary 34.1.__ X is semi-1[st], p is semi-closed $\Longrightarrow$ $\mathcal{I}$ is M.

<u>Example M.</u> p is semi-closed (hence G is semi-H), but I is not M.

Let S be the space of Example E, $S = N \times N + x$, $A_n = \{n\} \times N \subset S$. Let $X = S^1 + S^2$, where $S^1 = S^2 = S$. Let $H_G = \{g_n\}_{n=1}^{\infty}$, where $g_n = A_n^{\ 1} + A_n^{\ 2}$. Then $g_n \to x^1$ and $g_n \to x^2$ so I is not M. p is semi-closed since compact subsets of X are the only limit points of any set A in I and these are always limit points of A^* in X. X is M' (any set in S^1 containing x^1 is closed, while x^1 is not a limit point of subsets of S^2), though I is not.

2.5. <u>k-spaces</u>

In [8], Halfar gives the following definition.

<u>Definition.</u> X is a <u>K</u> space iff $x \ell p A \implies$ there is a compact set $K \subset A + x$ such that $x \ell p K$.

K is hereditary and in a KC space, $K \implies M'$. Also, compact $M' \implies K$.

The (α) form (see §2.4) of Property K may be stated:

<u>Definition.</u> X is <u>weak-K</u> iff A is not closed $\implies$ there exists $x \in \overline{A} \setminus A$ and a subset $K \subset A$ such that $x \ell p K$ and $K + x$ is compact.

Clearly, hereditarily weak-K $\iff$ K, and M', weak-K $\implies$ K. This notion of Halfar's is related to that of a k-space. The definition of k-space commonly appears as one of a variety of conditions which are equivalent in a T_2 space, as a k-space is frequently assumed to be. As we do not wish to impose this restriction, we state these conditions separately.

Definition. X is a k_1 - space iff

A is closed iff (1) for each closed compact set C, $A \cap C$ is closed.

X is a k_2 - space iff

A is closed iff (2) for each compact set C, $A \cap C$ is closed in C.

It is clear that (2) $\Rightarrow$ (1), so $k_1 \Rightarrow k_2$. Of course, in a KC space k_1 and k_2 are equivalent, and in that case we refer to the space as a k - space.

Whyburn defines a k - space as k_2, attributing it to Hurewicz. Kelley's definition of k - space corresponds to k_1.

Any compact space is trivially a k_i - space for $i = 1,2$. It is known that first countable T_2 or locally compact $T_2 \Rightarrow$ k - space. While we are not assuming a k_i - space to be T_2 we can make stronger statements than these. Commonly used definitions of locally compact spaces are equivalent in the presence of T_2. The definition of locally compact we use in section 2.6 will yield our k_i - space, $i = 1,2$, without assuming T_2.

Proposition 35. weak - K $\Rightarrow$ k_2

Proof. Suppose A is not closed but $A \cap C$ is closed in C for each compact set C. There exists $x \in \bar{A} \setminus A$ and a subset $K \subset A$ such that $x \, \ell p \, K$ and $K + x$ is compact. Let $C = K + x$. Then

$A \cap C = K$ is not closed in C.

Corollary 35.1. sequential $\Rightarrow k_2$

Proof. sequential $\Rightarrow$ weak-K trivially

Note that by Proposition 31, sequential $M \Rightarrow k_1$.

Corollary 35.2. $K \Rightarrow$ hereditarily k_2

Proposition 36. hereditarily $k_2 \Rightarrow K$

Proof. Suppose $x \in \overline{A} \setminus A$. Then A is not closed in $A + x$, so there is a compact subset K of $A + x$ such that $K \cap A$ is not closed in K, i.e., there exists $k \in K$ with $k \, \ell p \, (K \cap A)$ but $k \notin K \cap A$. So $k \notin A$ and hence $k = x$ and $x \, \ell p \, K$.

We now have the equivalence of K and hereditarily k_2-space asserted in [21].

Proposition 37. $M', k_2 \Rightarrow k_1$

Proof. Suppose A is not closed. Then there is a compact set C such that $A \cap C$ is not closed in C. Let $x \in C \setminus A$ such that $x \, \ell p \, A \cap C$. By M' there exists $B \subset A \cap C$ such that $x \, \ell p \, B$ and $B + x$ is closed. Then $B + x$ is a closed subset of C, hence compact. So we have $B + x$ a closed compact set while $A \cap (B + x) = B$ is not closed.

Proposition 38. $M', k_1 \Rightarrow K$.

Proof. Suppose $x \in \overline{A} \setminus A$. By M' there exists $B \subset A$ such that $\overline{B} \setminus B = x$. Now B is not closed so there is a closed compact set C such that $B \cap C$ is not closed. But if $y \ell p B \cap C$ and $y \notin B \cap C$ then $y \in C \sim B$. But $y \in \overline{B}$, hence $y = x$. So $x \in C$ and $(B \cap C) + x$ is closed. Furthermore, since it is a subset of C, $(B \cap C) + x$ is compact. As $B \cap C \subset A$, this completes the proof.

Corollary 38.1. $M', k_2 \Longrightarrow$ hereditarily k_1

Proof. By Proposition 37, $M', k_2 \Longrightarrow M', k_1$ and by the above proposition this gives M', K which in turn, by Corollary 35.2, gives M', hereditarily k_2. Since M' is also hereditary, applying Proposition 37 to an arbitrary subspace, we have k_1.

So M' makes each of k_1 and k_2 hereditary as well as rendering them equivalent. It should be noted that hereditarily k_2 (K) by itself does not yield k_1. In fact there exist E spaces, and hence K, which are not k_1. Such a space must be not M. See, for instance, Example N of section 2.6.

Since $E \Longrightarrow K$ and $K, KC \Longrightarrow M'$, it appears that we have found something weaker than E which makes $T_2 \Longrightarrow M'$. However, a T_2 K-space is necessarily an E space. This has been proved independently by Arhangel'skii and M. E. Rudin, as noted in [21].

This author has recently seen the unpublished manuscript of E. D. Shirley, titled "Pseudo-open maps," in which the notion of accessibility by closed sets, which is equivalent to M' in T_1 spaces, is discussed. His results overlap or extend some of those included here, though these are obtained independently and by different arguments. A question Shirley raises at the conclusion of his paper may be related to that of whether an M' space must be KC. He asks whether there is a k-space (meaning our k_2) which is M' but not E. If there is an M' space which is not KC, then there is an M', k_1-space which is not a c-space. (Recall that M', c-space $\Longrightarrow$ KC). For if K is a compact non-closed subset of an M' space then the subspace consisting of $K + x$, where $x \in \overline{K} \setminus K$, is a compact M', hence hereditarily k_1, space which is not KC.

Each of the properties k_1 and k_2 has been defined here by a definition of type (α). There are (β) forms of these, which have been given separate attention by other authors. However, any compact space also satisfies each of the (β) forms and so these are not hereditary, though they are implied whenever their (α) forms hold hereditarily. The (β) form of k_2, namely: $x \,\ell p\, A \Longrightarrow$ there exists a compact set K such that $x \,\ell p\, (A \cap K)$, was discussed, along with k_2 and K, by R. V. Fuller in [7]. Whyburn denoted this property <u>k'</u> in [21]. Fuller mentioned that these two concepts, k_2 and k' here (he calls them k_3 and k_2, respectively), may not be equivalent. We can point out that the decomposition space in Example I (also C) is a k_2 space but it fails to have the stronger k' property at the element g. (g is not a limit point of the intersection of A with any compact set).

k_2 is preserved by all quotient maps. Example N of section 2.6 shows that k_1 is not preserved by open maps.

<u>Proposition 39</u>. X is k_2 $\Longrightarrow$ $\mathcal{I}$ is k_2.

<u>Proof</u>. Suppose A is not closed in $\mathcal{I}$. Then A^* is not closed so there exists a compact set $K \subset X$ such that $A^* \cap K$ is not closed in K. Then there exists $k \in K \setminus A^*$ such that $k \,\ell p\, (A^* \cap K)$. Then $p(k) \in pK \setminus A$ and $p(k) \,\ell p\, p(A^* \cap K)$. Since

$p(A^* \cap K) \subset p(A^*) \cap pK = A \cap pK$, $A \cap pK$ is not closed in the compact set pK.

Corollary 39.1. K is preserved by pseudo-open maps.

An argument analogous to that of Proposition 39 will establish that k_1 is preserved by any closed or semi-closed quotient map.

2.6. Local compactness

Definition. We call a space locally compact iff each point has a neighborhood whose closure is compact.

We call a space weak locally compact iff each point has a compact neighborhood.

These are equivalent in any space in which compact sets have compact closure (e.g., T_2 or M,E) and they both hold in any compact space, but the second condition is strictly weaker, in general.

Example N. Let X be the subspace of E^2 consisting of $I_0 + \bigcup_{n=1}^{\infty} I_n$, where $I_0 = \{(0,y): 0 \leq y < 1\}$ and $I_n = \{(\frac{1}{n}, y): 0 \leq y \leq 1\}$. Let $H_G = \{I_n\}_{n=1}^{\infty}$. Then I is not locally compact at any point of I_0. The sequence I_n converges to each point of I_0. Each neighborhood of a point $g \in I_0$ contains I_n n-ultimately, hence its closure contains all of I_0 and is not compact. But I is weak locally compact. For $g \in I_0$, choose $[a,b]$ such that $g \in \text{int}\,[a,b] \subset [a,b] \subset I_0$. Then $\bigcup I_n \cup [a,b]$ is a compact neighborhood of g. I is E but not M. I is not KC.

In the above example, X is locally compact while I is not. In Stone's Theorem that X is locally compact, I is T_2, first countable and ∂g is σ-compact for each $g \in G \Longrightarrow \mathcal{I}$ is locally compact, we have violated T_2.

<u>Definition</u>. (Arhangel'skii) A map $f: X \to Y$ is <u>almost open</u> iff for each $y \in Y$ there exists $x_y \in f^{-1}(y)$ such that if U is open containing x_y, then $y \in \text{int } fU$.

Clearly, open $\Longrightarrow$ almost open $\Longrightarrow$ pseudo-open

<u>Proposition 40</u>. Almost open maps preserve weak locally compact spaces.

<u>Proof</u>. For $g \in \mathcal{I}$ choose $x_g \in g$ at which p is almost open. Since X is weak locally compact there exist an open set 0 and a compact set K such that $x \in 0 \subset K$. Then $p0$ is a neighborhood of g while pK is compact containing $p0$.

<u>Proposition 41</u>. Pseudo-open point-compact maps preserve weak locally compact spaces.

<u>Proof</u>. For $g \in \mathcal{I}$ and each $x \in g$ there exist an open set 0_x containing x and a compact set $K_x \supset 0_x$. g is covered by a finite number $0_{x_1}, \cdots, 0_{x_n}$. So $p(\bigcup_{i=1}^{n} 0_{x_i})$ is a neighborhood of g in $\mathcal{I}$. And $p(\bigcup_{i=1}^{n} 0_{x_i}) \subset p(\bigcup_{i=1}^{n} K_{x_i}) = \bigcup_{i=1}^{n} pK_{x_i}$, a finite union of compact sets, hence compact.

<u>Corollary 41.1</u>. X is weak locally compact, p is pseudo-open and point-compact, $\mathcal{I}$ has the property that K compact $\Longrightarrow \overline{K}$ compact, $\Longrightarrow \mathcal{I}$ is locally compact.

<u>Proposition 42</u>. Pseudo-open, point-compact, semi-closed maps preserve locally compact spaces.

Proof. For $g \in \mathcal{I}$ and each $x \in g$ there exists O_x open containing x such that $\overline{O}_x$ is compact. So

$g \subset \bigcup_{i=1}^{n} O_{x_i} \subset \bigcup_{i=1}^{n} \overline{O}_{x_i}$, for some finite subset $\{x_i\}_{i=1}^{n}$ of g. Then

$g \in R = \text{int } p(\bigcup_{i=1}^{n} O_{x_i}) \subset p(\bigcup_{i=1}^{n} \overline{O}_{x_i}) = \bigcup p\overline{O}_{x_i}$ which is closed since p is semi-closed and compact as it is a finite union of compact sets. Hence $\overline{R}$ lies in a compact subset of $\mathcal{I}$ and so is compact.

Proposition 43. Closed point-compact maps preserve locally compact spaces.

Proof. The proof of this proposition proceeds exactly like that of Proposition 42 since a closed map is pseudo-open and the semi-closed property was invoked to apply to closed compact subsets of X.

(Note: Proposition 43 is corollary to the proof of Proposition 42 but not directly of the proposition. A closed map may fail to be semi-closed if the domain is not KC. The property K compact $\Longrightarrow$ $\overline{K}$ compact is preserved by closed compact maps so on domains having this property, Proposition 43 is corollary to Proposition 41.)

Definition. X is locally peripherally compact iff each neighborhood of a point $x \in X$ contains an open neighborhood of x whose boundary is compact.

The semi-closed condition gives the following result on monotone decompositions.

Proposition 44. If p is monotone, point-compact and X is locally peripherally compact, then p is semi-closed $\Longrightarrow$ p is closed.

Proof. Let $g \subset U$ open in X. For each $x \in g$ there exists an open set O_x containing x such that $O_x \subset U$ and ∂O_x is compact. A finite number of these covers g, say, $g \subset V = \bigcup_{i=1}^{n} O_{x_i} \subset U$. Now, $\partial V \subset \bigcup \partial O_{x_i}$ which is compact and hence ∂V is compact. Since p is semi-closed, $p(\partial V)$ is closed in $\mathcal{I}$ and hence $p(\partial V)^*$ is closed in X. Let $Q = V \setminus p(\partial V)^*$. Then Q is open in X containing g. Q is an inverse set, since if $h \in G$ meets V and h does not meet ∂V then, since h is connected, $h \subset V$. So $h \subset V \setminus p(\partial V)^*$. So $g \subset Q$, an open inverse set contained in U.

Corollary 44.1. If X is locally peripherally compact, p is monotone and point-compact, then p is closed under any of the conditions:

(a) $\mathcal{I}$ is T_2

(b) $\mathcal{I}$ is KC

(c) X is sequential, $\mathcal{I}$ is Mc or M

(d) $\mathcal{I}$ is M,E

Proof. Each of the conditions guarantees p is semi-closed.

Since we may be as interested to know that a decomposition is upper semicontinuous as that it preserve local compactness, these last results are especially useful.

That the local compactness condition on X cannot be eliminated in the case of (a), (b) or (c) is seen from Example C. All of the conditions except local compactness of X hold in the following:

Example O. Let $X = E^2 \setminus A$, where A is the relative complement of a point g in the boundary of a circular disk D.

$(A = \partial D \setminus \{g\})$. Let H_G be the collection of concentric circles filling up D. Then X is E, p is monotone, point-compact, pseudo-open, (in fact open), semi-closed, but not closed. $\mathcal{I}$ is T_2,E and hence M', Mc, M. ($\mathcal{I}$ is separable metric). We can retain all these properties without an open map by replacing g by an arc on the boundary of D and including g in H_G. The decomposition space is the same, but p is neither open nor closed.

That monotone is needed is illustrated by an example of Arhangel'skii:

Example P. $X = E^1$, $H_G = \{g_n\}_{n=2}^{\infty}$, where $g_n = \{\frac{1}{n}, n\}$. X is E, locally compact. $\mathcal{I}$ is T_2 and hence Mc and M. p is point-compact, semi-closed but not closed (not pseudo-open). $\mathcal{I}$ is not first countable. $\mathcal{I}$ is weak 1st but not E.

To summarize conditions under which a decomposition into compact elements preserves local compactness:

If X is locally compact and p is point-compact, then $\mathcal{I}$ is locally compact if:

1. $\mathcal{I}$ is T_2, first countable. (Stone)
2. p is pseudo-open, $\mathcal{I}$ has K compact $\Longrightarrow \overline{K}$ compact.
3. $\mathcal{I}$ is E and M (or M', Mc, T_2).
4. p is pseudo-open and semi-closed.
5. X is sequential, $\mathcal{I}$ is M and p is pseudo-open.
6. p is closed.
7. p is monotone and semi-closed.
8. p is monotone and $\mathcal{I}$ is T_2. (Whyburn)

or $\mathcal{I}$ is KC.

or X is sequential, $\mathcal{I}$ is M.

In [23], Wilansky deals with a "local compactness" which may fail to hold in a compact space.

Definition. We call a space locally weak locally compact (LW locally compact) iff $p \in U$ open $\implies$ there exists a compact set $V \subset U$ such that $p \in \text{int } V$. (Each neighborhood of a point contains a compact neighborhood of that point.)

Clearly, LW locally compact $\implies$ weak locally compact, so if compact sets have compact closure, then LW locally compact $\implies$ locally compact.

Proposition 45. KC, LW locally compact $\implies$ Regular T_2 (and locally compact).

Proof. If $p \in U$ open, there exists V such that $p \in \text{int } V$ and V is compact, $V \subset U$. Since V is closed, $\overline{\text{int } V} \subset V \subset U$, hence the space is regular.

Corollary 45.1. (Wilansky) LW locally compact $\implies$ [KC = T_2 = regular T_2].

Proposition 46. Regular, locally compact $\implies$ LW locally compact.

Proof. Suppose $p \in U$ open. Then there exists V open such that $p \in V \subset \overline{V} \subset U$. By locally compact, $p \in W$ open such that $\overline{W}$ is compact. So $p \in V \cap W$ open and $\overline{V \cap W} \subset \overline{W}$ so $\overline{V \cap W}$ is compact and $\overline{V \cap W} \subset \overline{V} \subset U$.

Corollary 46.1. locally compact T_2 $\Longrightarrow$ LW locally compact

So in any T_2 space, the three sorts of local compactness discussed here are equivalent.

Just as for weak local compactness, almost open maps and pseudo-open point-compact maps preserve LW locally compact spaces.

In [23], Wilansky asked whether a LW locally compact M space must be T_2. Many negative answers have been given.

Example Q. A space which is compact, LW locally compact and M but not Mc.

Let $X = \{\text{ordinals} \leq \Omega\} + \Omega'$, where neighborhoods of Ω' are precisely those of Ω, with Ω replaced by Ω', (or take two copies of the ordinal set and identify in pairs the corresponding points except at Ω's).

We revise Wilansky's question:

Question 4. Does LW locally compact, Mc or M' $\Longrightarrow$ T_2?

(An answer to Question 3 may resolve this.)

We now have a corollary to Proposition 32:

Corollary 45.2. If points are G_δ, LW locally compact M' $\Longrightarrow$ Regular T_2.

2.7. Bases

Proposition 47. If p is pseudo-open and point-compact and B is a base for X, then $B' = \{\text{int}\, pU : U$ is a finite union of elements of $\mathcal{B}\}$ is a base for $\mathcal{I}$.

__Proof.__ Let $g \in R$ open in $\mathcal{I}$. Then $g \subset R^*$ open in X. For each $x \in g$ there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset R^*$. Since g is compact, a finite number of these B_x covers g, say $g \subset U = \bigcup_{i=1}^{n} B_{x_i}$. Then int $pU \in \mathcal{B}'$ and contains g since p is pseudo-open. $pU \subset R$ so int $pU \subset R$.

__Corollary 47.1.__ X is second countable, p is pseudo-open and point-compact $\Longrightarrow \mathcal{I}$ is second countable.

__Proposition 48.__ p is pseudo-open and point-compact, for each $x \in X$, $\mathcal{A}_x$ is a neighborhood base for X at x, for $g \in G$, $\mathcal{A}_g = \bigcup_{x \in g} \mathcal{A}_x \Longrightarrow \mathcal{A}'_g = \{$int $pU : U$ is a finite union of elements of $\mathcal{A}_g\}$ is a neighborhood base for $\mathcal{I}$ at g.

__Proof.__ The proof for this is exactly like that of Proposition 47 with $B_x \in \mathcal{A}_x$.

__Corollary 48.1.__ p is pseudo-open, each element of G is a compact, countable set, X is first countable $\Longrightarrow \mathcal{I}$ is first countable.

__Proof.__ For $g \in G$, $g = \{x_i\}_{i=1}^{\infty}$. For each $x_i \in g$, let $\{A_{ij}\}_{j=1}^{\infty}$ be a countable neighborhood base for X at x_i. Then $\mathcal{A}_g = \{A_{ij}\}_{i,j=1}^{\infty}$ is countable. Hence $\mathcal{A}'_g$ is countable and gives a neighborhood base for $\mathcal{I}$ at g by Proposition 48.

This corollary is false without requiring p to be pseudo-open even if X is metric and the elements of g are finite. (see Example P.)

The corollary is false for arbitrary compact elements even if p

is closed, as the following example illustrates.

<u>Example R.</u> X is semimetric, p is closed and point-compact but I is not first countable.

Let X be the space, first described by McAuley, consisting of the points of the plane, where neighborhoods of points off the x-axis are the ordinary E^2 neighborhoods and those of points on the x-axis are "bow-tie" regions. To describe these regions explicitly, for neither of p and q on the x-axis, define the semimetric $d(p,q) = |p-q|$, the E^2 distance. If either of p or q is on the x-axis, then $d(p,q) = |p-q| + \alpha(p,q)$, where α is the radian measure of the least non-negative angle between the segment $\overline{pq}$ and the x-axis. X is a regular paracompact semimetric, nondevelopable, space. The interval $g = \{(x,0):x \in [0,1]\}$ is compact. (The subspace topology on the x-axis is the usual real topology.)

Let $H_G = \{g\}$. Then p is closed and point-compact. I is T_2 and M' but not first countable. Any countable collection $\{V_n\}$ of open sets containing g would have to intersect some single vertical line in a sequence $x_n \in V_n$, but such a sequence $\{x_n\}$ is closed.

An open map preserves both first and second countability, but not developability even with compact elements as does a closed map [24]. Indeed, an open point-compact image of a developable space may fail to be semimetric [22].

<u>Definition.</u> If G is a family of subsets of X, call a development $\{G_n\}$ for X <u>uniform with respect to G</u> iff for each $g \in G$, if $g \subset U$ open, then there exists an integer n such that $G_n^*(g) \subset U$.

Any self-refining development is uniform with respect to finite sets and any metric space has a development uniform with respect to compact sets. In fact a T_1 space has a development uniform with respect to compact sets iff it is metrizable.

<u>Proposition 49</u>. X has a development uniform with respect to G, p is almost open $\Rightarrow \mathcal{I}$ is developable.

<u>Proof</u>. Let $\{G_n\}$ be a development for X uniform with respect to G. Since p is almost open, for each $g \in G$ there exists $x_g \in g$ such that $g \in \text{int } pU$ for each open set U containing x_g. For each $g \in G$ and each n, choose $g_n(g)$ such that $x_g \in g_n(g) \in G_n$. Then $\{H_n\} = \{\text{int } p(g_n(g)) : g \in G\}$ is a development for $\mathcal{I}$: Each H_n is an open cover of $\mathcal{I}$ by the choice of x_g. Now suppose $g \in R$ open in $\mathcal{I}$. Then $g \subset R^*$ open in X. There exists N such that $G_N{}^*(g) \subset R^*$ by the uniformity of $\{G_n\}$. If $g \in h_N \in H_N$, $h_N = \text{int } p(g_N(g'))$ for some $g' \in G$. But $g_N(g') \in G_N$ and g meets $g_N(g')$. Hence $g_N(g') \subset R^*$. So $p\, g_N(g') \subset R$ and hence $h_N \subset R$.

<u>Corollary 49.1</u>. X is developable, p is almost open, each element of G is finite $\Rightarrow \mathcal{I}$ is developable.

<u>Corollary 49.2</u>. X is metric, p is almost-open and point-compact $\Rightarrow \mathcal{I}$ is developable.

<u>Proposition 50</u>. X is semimetric T_1, p is pseudo-open, each element of G is finite $\Rightarrow \mathcal{I}$ is semimetric.

<u>Proof</u>. Using Heath's characterization of T_1 semimetric spaces

[9], let $\{g_n(x)\}_{\substack{x \in X \\ n \in N}}$ be such that $g_{n+1} \subset g_n$ and for each x, $\{g_n(x)\}_{n \in N}$ is a local base at x, and $y \in g_n(x_n) \Longrightarrow x_n \to y$. For each $g \in G$ and each n, let $G_n(g) = \text{int}\, p\left(\bigcup_{x \in g} g_n(x)\right)$. This is open and contains g since p is pseudo-open. Then for each g, $\{G_n(g)\}$ is a base for $\mathcal{I}$ at g, with $G_{n+1} \subset G_n$: if $g \in R$ open in $\mathcal{I}$, $g = \{x_1, \cdots, x_k\}$, there exists $g_{n_i}(x_i)$ such that $\bigcup_{i=1}^{k} g_{n_i}(x_i) \subset R^*$. Let $N = \max_{i \le k} n_i$. Then for each $i, g_N(x_i) \subset R^*$. So $G_N(g) \subset R$. If $g \in G_n(h_n)$ then $g\, \ell p\, \{h_n\}$, for suppose $g \in G_n(h_n) = \text{int}\, p\left(\bigcup_{x \in h_n} g_n(x)\right)$. Then g meets $\bigcup_{x \in h_n} g_n(x)$ for each n. Some point x of g lies in $\bigcup_{x \in h_n} g_n(x)$ for infinitely many n, since g is finite. So there exists $x_{n_i} \in h_{n_i}$ such that $x \in g_{n_i}(x_{n_i}) \subset g_i(x_{n_i})$. Hence $x_{n_i} \to x$, which gives $h_{n_i} \to g$ and thus $g\, \ell p\, \{h_n\}$. This suffices to give Heath's characterization, making $\mathcal{I}$ a semimetric space.

<u>Proposition 51</u>. X is metric, p is pseudo-open and point-compact $\Longrightarrow \mathcal{I}$ is semimetric.

<u>Proof</u>. Let $\{G_n\}$ be a monotone development for X uniform with respect to G. For each $g \in G$ and each n, let $H_n(g) = \text{int}\, p\, G_n{}^*(g)$. $\{H_n(g)\}_{n \in N}$ is a base for $\mathcal{I}$ at g, since $g \in R$ open in $\mathcal{I} \Longrightarrow$ for some N, $G_N{}^*(g) \subset R^*$. Furthermore, $g \in H_n(h_n) \Longrightarrow h_n \to g$: If $g \in R$ open in $\mathcal{I}$, there exists N such that $G_n{}^*(g) \subset R^*$ for $n > N$. But g meets $G_n{}^*(h_n)$ for $n > N$ so $h_n \in G_n{}^*(g)$. Hence $h_n \subset R^*$ and $h_n \in R$ for $n > N$.

2.8. <u>Duda's reflexive-compact mappings</u>

<u>Definition</u>. (Duda [3]) $f: X \to Y$ is <u>reflexive compact</u> iff

$f^{-1}fK$ is compact for each compact $K \subset X$.

Trivially, compact $\Longrightarrow$ reflexive compact $\Longrightarrow$ point-compact.

If p is reflexive compact, then p is compact iff each compact set K in $\mathcal{I}$ has compact section, i.e., a compact $A \subset X$ such that $pA = K$. If X is KC, then p is reflexive compact $\Longrightarrow$ p is semi-closed.

For any spaces, closed and point-compact $\Longrightarrow$ compact. Duda has proven a sort of converse for the weaker reflexive compact, namely, if X is a k-space, then p is reflexive compact $\Longrightarrow$ p is closed, (hence, compact). Duda deals only with T_2 spaces. It is not necessary to assume $\mathcal{I}$ is T_2 but something like it is needed for X. For if X is a compact T_1 space which is not KC, say $X = B + x$ with B compact and $x \in pB$ (for instance, X may be a sequence converging to two distinct limit points), let $H_G = \{B\}$. Then $\mathcal{I}$ consists of two points, one of which is a limit point of the other. $\mathcal{I}$ is not T_1 so p is not closed. p is compact, however, and X is a k_1-space.

To prove Duda's Theorem, it suffices to assume X is KC.

<u>Proposition 52</u>. (Duda) X is k-space, KC, p is reflexive compact $\Longrightarrow$ p is closed.

<u>Proof</u>. Let F be closed in X. If $p^{-1}pF$ is not closed, then there exists a compact closed set C such that $p^{-1}pF \cap C$ is not closed. Now, $p^{-1}pF \cap C = p^{-1}p(p^{-1}pC \cap F) \cap C$. Since p is reflexive compact, $p^{-1}pC$ is compact. In any space, if H is closed and K compact then $H \cap K$ is closed in K and hence compact. So $p^{-1}pC \cap F$ is compact. Hence $p^{-1}p(p^{-1}pC \cap F)$ is also compact by the reflexive compactness of p. And $p^{-1}p(p^{-1}pC \cap F) \cap C$ is compact,

as C is closed. So if X is KC this set is closed and we have a contradiction.

<u>Corollary 52.1</u>. X is sequential M, p is reflexive compact $\Rightarrow$ p is closed.

<u>Proof</u>. sequential M $\Rightarrow$ both k - space and KC

These theorems give full compactness of the map as a dividend by way of closedness of the map. They also suggest that reflexive compact is not very much weaker than compact and of course these properties may be equivalent under conditions which do not force the map to be closed.

We call p <u>countably compact</u> iff K countably compact $\Rightarrow$ $p^{-1}K$ is countably compact.

<u>Proposition 53</u>. X is strongly semi-1^{st}, $\mathcal{I}$ is T_1, p is pseudo-open and reflexive countably compact $\Rightarrow$ p is countably compact.

<u>Proof</u>. Let K be countably compact in $\mathcal{I}$ and suppose there exists an infinite set $\{x_n\}_{n=1}^{\infty} \subset p^{-1}K$ with no limit point in $p^{-1}K$. Each $g \in G$ is countably compact so we may assume $\{px_n\}$ are all different. Each $px_n \in K$ so there exists $g \in K$ such that $g\, \ell p\, \{px_n\}$ and we may assume $g \notin \{px_n\}$. Since p is pseudo-open there exists $x \in g$ such that $x\, \ell p\, p^{-1}(\{px_n\})$. Since X is strongly semi-1^{st} and $\mathcal{I}$ is T_1 there exists a subsequence $y_{n_i} \in p^{-1}px_{n_i}$ with $y_{n_i} \to x$. Now, $\{y_{n_i}\} \cup x$ is compact, hence $p^{-1}p(\{y_{n_i}\} \cup x) = \{p^{-1}px_{n_i}\}^* \cup g$ is a countably compact subset of $p^{-1}K$. As it contains $\{x_{n_i}\}$, this means $\{x_{n_i}\}$, and hence $\{x_n\}$, has a limit point in $p^{-1}K$, which contradicts our assumption.

__Corollary 53.1.__ Proposition 53 with countably compact replaced by sequentially compact.

__Proof.__ sequentially compact $\Longrightarrow$ countably compact and countably compact, c - E $\Longrightarrow$ sequentially compact

__Corollary 53.2.__ X is developable or strongly semi-1st, Lindelof, Y is T_1, p pseudo-open and reflexive compact $\Longrightarrow$ p is compact.

CHAPTER III

Shrinkable Decompositions

Definition. (McAuley) A subset K of a metric space M is locally shrinkable iff for each open $U \supset K$ and each $\varepsilon > 0$ there exists a homeomorphism $h: M \approx M$ such that $h = id$ off U and $\operatorname{diam} hK < \varepsilon$.

A compact locally shrinkable subset is connected.

As originally stated in [12], the theorem: If G is a McAuley-usc decomposition of a complete metric space M such that H_G is countable and G_δ, each element $g \in H$ is a locally shrinkable continuum and lies in an open set with compact closure, then $\mathcal{I} \approx M$, is false, as illustrated by Example C of section 2.3, where $\mathcal{I}$ is not first countable. The theorem fails when there exists a point which is a degenerate limit of elements having diameters bounded away from zero. This cannot happen if p is closed, but, as the example shows, it is not a violation of Mc. The hypotheses of the theorem and the condition that there be no such "bad" points guarantee the map p is closed. The theorem is true if McAuley - usc is replaced by Whyburn - usc (p closed) and we will obtain this from a more general proposition which restates another of McAuley's theorems.

If G is a decomposition of X, we call a subset of X p-open if it is an open inverse set (for p).

<u>Definition</u>. If G is a decomposition of a metric space M, H is <u>tightly shrinkable in M</u> (tsh) iff given any p-open cover U of H^*, $\varepsilon > 0$, and $h:M \approx M$, there exists a p-open (refinement of U) V covering H^* and a homeomorphism $f:M \approx M$ such that 1) $f = h$ off V^*, 2) for each $g \in H$, diam $f(g) < \varepsilon$ and 3) for each $v \in V$ there exists $u \in U$ such that $h(v) \cup f(v) \subset h(u)$.

H is <u>weakly tsh</u> if the above holds for the special case of $h = id_M$.

We will make use of the following theorem of McAuley, slightly revised.

<u>Convergence Theorem</u>. (McAuley) If M is a metric space, $\sum \varepsilon_n < \infty, (\varepsilon_n > 0)$ for each n, $f_n:M \approx M$, $f_0 = id$, for each $n \geq 1$, V_n is a collection of open sets with compact closure and $V_n^* \supset V_{n+1}^*$, for each $n \geq 0$, $f_{n+1} = f_n$ off V_{n+1}^*, $D \in V_{n+1} \Longrightarrow$ diam $f_nD < \varepsilon_n$ and $x \in V_{n+1}^* \Longrightarrow$ there exists $D \in V_{n+1}$ such that $f_nD \supset f_nx \cup f_{n+1}x$, then $\{f_n\}$ are uniformly Cauchy and if $\{f_n(x)\}_{n=1}^{\infty}$ converges for $x \in \Delta = \bigcap V_n^*$ then $f_n \to f$ [unif], $f:M \to M$ is continuous and onto, and f is 1-1 off Δ. Furthermore, if M is locally compact on $\overline{V_1^*}$, then f is closed.

<u>Proof</u>. First, we show that $\{f_n\}$ are uniformly Cauchy. Let $\varepsilon > 0$. For some N, $\sum_{n=N}^{\infty} \varepsilon_n < \varepsilon$. Let $x \in M$. For each n, if $x \notin V_{n+1}^*$ then $f_{n+1}x = f_nx$. If $x \in V_{n+1}^*$ then there exists $D \in V_{n+1}$ such that $f_nD \supset f_nx \cup f_{n+1}x$, but diam $f_nD < \varepsilon_n$. So, in either case, $d(f_nx, f_{n+1}x) < \varepsilon_n$. So for $m > N$,

$$d(f_Nx, f_mx) < \sum_{i=N}^{m} \varepsilon_i < \sum_{i=N}^{\infty} \varepsilon_i < \varepsilon .$$

$\{f_n x\}$ converges for $x \notin \Delta = \bigcap V_n^*$, for if $x \notin V_{J+1}^*$ then $f_n x = f_J x$ for $n > J$, i.e., $\{f_n x\}$ is ultimately constant. So if $\{f_n x\}$ converges for $x \in \Delta$ then we have pointwise convergence everywhere. And since $\{f_n\}$ are uniformly Cauchy, $f_n \to f = \lim f_n$ [unif], and f is continuous.

To show f is onto, let $p \in M$. Let $z_n = f_n^{-1} p$. It suffices to show $\{z_n\}$ has a convergent subsequence, since if $z_{n_i} \to x$ then continuity gives $fz_{n_i} \to fx$ while $d\left(f_{n_i} z_{n_i}, fz_{n_i}\right) < \varepsilon$ for large i by uniform convergence. So $fz_{n_i} \to p$ and hence $p = fx$. Now, if $p \notin V_1^*$ then for each n, $f_n p = p$. Thus $\bigcup f_n^{-1} p = \{p\}$. If $p \in V_1^*$, $p \in D \in V_1$ with $\overline{D}$ compact. Choose $\delta > 0$ such that $N_\delta(p) \subset D$. By the uniform convergence there exists N such that $n > N \implies f_N z \in N_\delta f_n z$ for all $z \in M$. So $f_N z_n \in N_\delta f_n z_n = N_\delta(p) \subset D$. Hence $\{f_N z_n\}_{n=1}^\infty \subset D$ and $\{z_n\}_{n=1}^\infty \subset f_N^{-1} D$. Since f_N is a homeomorphism, $\overline{f_N^{-1} D}$ is compact and so $\{z_n\}$ has a convergent subsequence.

Now we suppose that M is locally compact at each point of $\overline{V_1^*}$. To show f is closed, let D be a closed subset of M and $y_n \to y$ with $y_n \in fD$. We must show $y \in fD$. There exists $x_n \in D$ with $y_n = fx_n$. If $\{x_n\}$ has a convergent subsequence, we are done, since if $x_{n_i} \to x$ then $x \in D$ and $fx_{n_i} = y_{n_i} \to fx$ by continuity. Hence $fx = y$. Furthermore, if M is locally compact at y, we can choose $\varepsilon > 0$ so that $\overline{N_\varepsilon y}$ is compact. By uniform convergence there exists I so that for every $x \in M$, $f_I x \in N_{\frac{\varepsilon}{2}} fx$. In particular, for each n $f_I x_n \in N_{\frac{\varepsilon}{2}} fx_n$. But for $n > N$ $fx_n \in N_{\frac{\varepsilon}{2}} y$. So $f_I x_n \in N_{\frac{\varepsilon}{2}} fx_n \subset N_\varepsilon y$, which has compact closure. So $\{f_I x_n\}$ has a convergent subsequence and thus $\{x_n\}$ does also, as f_I is a homeomorphism.

We may suppose then that $y \notin \overline{V_1^*}$. Now $f_j(V_1^*) = V_1^*$ for each

j since f_j is a homeomorphism which is the identity off V_1^*. For some $\varepsilon > 0$, $N_\varepsilon y$ misses $\overline{V_1^*}$ and for large n, $y_n \in N_{\frac{\varepsilon}{2}} y$. For large i, $f_i x_n \in N_{\frac{\varepsilon}{2}} y_n \subset N_\varepsilon y$ so $f_i x_n \notin V_1^*$ and thus $x_n \notin V_1^*$. So $fx_n = x_n$ and since $fx_n \to y$, we have $x_n \to y$.

<u>Theorem T</u>. If M is a metric space, G a decomposition of M such that p is closed and point-compact, H is tightly shrinkable in M, and M is locally compact at H^*, then $I \approx M$.

<u>Proof</u>. For each $g \in H$, let $w_1(g)$ be a p-open set containing g such that $\overline{w_1(g)}$ is compact $\subset N_{\frac{1}{2}}(g)$. Let $W_1 = \{w_1(g) : g \in H\}$. Let U_1 be a star refinement of W_1 by p-open sets. (I is metrizable, hence paracompact, by Stone's Theorem [16].) By tsh, there exists $f_1 : M \approx M$ and V_1 a p-open refinement of U_1 covering H^* such that:

$$f_1 = \text{id off } V_1^*$$

$$g \in H \Longrightarrow \operatorname{diam} f_1 g < \frac{1}{2}$$

$$v \in V_1 \Longrightarrow \text{there exists } u \in U_1 \text{ such that } v \cup f_1 v \subset u.$$

For each $g \in H$, choose $v_1(g) \in V_1$ containing g and let $w_2(g)$ be p-open containing g so that $\overline{w_2(g)}$ compact $\subset N_{\frac{1}{2^2}}(g) \cap v_1(g) \cap f_1^{-1}(N_{\frac{1}{2^2}} f_1 g)$. Let $W_2 = \{w_2(g) : g \in H\}$. Let U_2 be a star refinement of W_2 by p-open sets. By tsh there exists $f_2 : M \approx M$ and V_2 a p-open refinement of U_2 covering H^*, satisfying

$$f_2 = f_1 \text{ off } V_2^*$$

$$g \in H \Longrightarrow \operatorname{diam} f_2 g < \frac{1}{2^2}$$

$$v \in V_2 \Longrightarrow \text{there exists } u \in U_2 \text{ such that } f_1 v \cup f_2 v \subset f_1 u.$$

Inductively, given $f_{n-1}:M \cong M$, V_{n-1} p- open refinement of U_{n-1} covering H^* with

$$f_{n-1} = f_{n-2} \text{ off } V_{n-1}^*$$

$$g \in H \Longrightarrow \text{diam } f_{n-1}g < \frac{1}{2^{n-1}}$$

$v \in V_{n-1} \Longrightarrow$ there exists $u \in U_{n-1}$ with $f_{n-2}v \cup f_{n-1}v \subset f_{n-2}u$,

for each $g \in H$, choose $v_{n-1}(g) \in V_{n-1}$ containing g and let $w_n(g)$ be p- open containing g so that $\overline{w_n(g)}$ is compact $\subset$

$N_{\frac{1}{2^n}}(g) \cap v_{n-1}(g) \cap f_{n-1}^{-1}(N_{\frac{1}{2^n}} f_{n-1}g)$. Let $W_n = \{w_n(g): g \in H\}$ and U_n a star-refinement of W_n by p- open sets. By tsh there exists $f_n:M \cong M$ and V_n a p- open refinement of U_n covering H^*, satisfying:

$$f_n = f_{n-1} \text{ off } V_n^*$$

$$g \in H \Longrightarrow \text{diam } f_ng < \frac{1}{2^n}$$

$v \in V_n \Longrightarrow$ there exists $u \in U_n$ such that $f_{n-1}v \cup f_nv \subset f_{n-1}u$.

It is clear that this construction gives for each n, $g \in G \Longrightarrow f_{n-1}V_n^*(g) \cup f_nV_n^*(g) \subset f_{n-1}U_n^*(g) \subset f_{n-1}w_n(g')$, some $g' \in H \subset f_{n-1}v_{n-1}(g') \cap N_{\frac{1}{2^n}} f_{n-1}(g')$, this last set having diameter $< \frac{1}{2^{n-2}}$.

Also, we have for each $g \in G$, for each $k \geq 1$ and $n \geq k$, $f_k(V_n^*(g)) \cup f_n(V_n^*(g)) \subset f_k(V_k^*(g))$. To see this, let $k \geq 1$ and induct on n: For $n = k$ the statement is trivial. Suppose it holds for some $n \geq k$. Now, $f_n(V_{n+1}^*(g)) \cup f_{n+1}(V_{n+1}^*(g)) \subset f_n(U_{n+1}^*(g))$ by construction and this is a subset of $f_n(w_{n+1}(g'))$, for some

$g' \in H$, which in turn lies in $f_n(v_n(g'))$. We assume $g \in p(V_{n+1}^*)$ since otherwise the statement is trivial. So $g \subset V_{n+1}^*(g) \subset v_n(g')$. Hence $v_n(g') \in V_n(g)$ and $v_n(g') \subset V_n^*(g)$. So $f_n(v_n(g')) \subset f_n(V_n^*(g)) \subset f_k(V_k^*(g))$ by the inductive hypothesis. Also, since $V_{n+1}^*(g) \subset V_n^*(g)$, $f_k(V_{n+1}^*(g)) \subset f_k(V_k^*(g))$ also by the inductive hypothesis and this establishes the corresponding statement for the case of $n+1$.

We may restate the last result: for each $g \in G$, for each $k \geq 1$ and $n \geq k$, $\bigcup_{n=k}^{\infty} f_n(V_n^*(g)) \subset f_k(V_k^*(g))$. In particular, for each $g \in \bigcap p(V_n^*)$ (where $g \subset V_n^*(g)$ for each n),
$\bigcup_{n=k}^{\infty} f_n(g) \subset \bigcup_{n=k}^{\infty} f_n V_n^*(g) \subset f_k V_k^*(g)$.

The result is a sequence $f_n: M \to M$ and $\{U_n\}$ such that each U_n is a collection of p- open sets with compact closure, $f_{n+1} = f_n$ off U_{n+1}^* (actually off $V_{n+1}^* \subset U_{n+1}^*$). Furthermore, $x \in U_{n+1}^* \Longrightarrow$ there exists $u \in U_{n+1}$ with $f_n u \supset f_n x \cup f_{n+1} x$, since if $x \notin V_{n+1}^*$, $f_{n+1} x = f_n x$, which is in the image under f_n of whichever element of U_{n+1} contains x. And if $x \in V_{n+1}^*$, $x \in$ some $v \in V_{n+1}$ but $f_n v \cup f_{n+1} v \subset f_n u$ for some $u \in U_{n+1}$.

For each $u \in U_{n+1}$, diam $f_n u < \frac{1}{2^{n-1}}$. And since $\sum \frac{1}{2^{n-1}} < \infty$, we have verified all of the conditions we need of the Convergence Theorem except convergence itself at points of $\bigcap U_n^*$. But suppose $x \in \bigcap U_n^* = \bigcap V_n^*$. $p(x) = g \subset \bigcap V_n^*$ so $g \subset V_n^*(g)$ for each n, while $\bigcup_{n=1}^{\infty} f_n(V_n^*(g)) \subset f_1(V_1^*(g)) \subset U_1^*(g)$, which has compact closure. So $\{f_n x\}_{n=1}^{\infty}$ lies in a compact set. Thus it has a convergent subsequence. But the sequence $\{f_n x\}$ is Cauchy and hence converges.

So by the Convergence Theorem, $f_n \to f:M \to M$ [unif], f is continuous, onto and f is 1-1 off $\Delta = \bigcap U_n^*$.

We now establish that for each $g \in H$, $f(g)$ is a point. For each k and $n \geq k$, $f_n(g) \subset f_k(V_k^*(g))$. So for each k, $f(g) \subset \overline{f_k(V_k^*(g))}$. Thus $f(g) \subset \bigcap_{k=1}^{\infty} \overline{f_k(V_k^*(g))}$, while the sets in this intersection have diameters tending to zero as k increases. So $f(g) = \bigcap_{k=1}^{\infty} \overline{f_k V_k^*(g)}$ = a point.

We claim also: $g \neq g' \in G \implies$ for some N, $\overline{V_N^*(g)} \cap \overline{V_N^*(g')} = \phi$. To prove this, note that since g and g' are compact, there exists $\varepsilon_1 > 0$ such that $\overline{N_{2\varepsilon_1}}(g) \cap \overline{N_{2\varepsilon_1}}(g') = \phi$. Let U and V be p-open with $g \subset U \subset N_{\varepsilon_1} g$ and $g' \subset V \subset N_{\varepsilon_1} g'$. So $N_{\varepsilon_1} U \subset N_{2\varepsilon_1} g$ and $N_{\varepsilon_1} V \subset N_{2\varepsilon_1} g'$ and $\overline{N_{\varepsilon_1} U} \cap \overline{N_{\varepsilon_1} V} = \phi$. Choose $\varepsilon > 0$ so that $\varepsilon < \varepsilon_1$ and $N_\varepsilon g \subset U$, $N_\varepsilon g' \subset V$. Choose N so $\frac{1}{2^N} < \varepsilon$. Then $\overline{W_N^*(g)} \cap \overline{W_N^*(g')} = \phi$. For if $w \in W_N(g)$, $g \subset w = w_N(g_0)$, some $g_0 \in H, \subset N_{\frac{1}{2^N}}(g_0) \subset N_\varepsilon(g_0)$. So g_0 meets $N_\varepsilon g$ and thus $g_0 \subset U$. So $w \subset N_\varepsilon g_0 \subset N_\varepsilon U \subset N_{\varepsilon_1} U$. Thus $W_N^*(g) \subset N_{\varepsilon_1} U$. Similarly, if $w' \in W_N(g')$, $w' \subset N_{\varepsilon_1} V$. So $W_N^*(g') \subset N_{\varepsilon_1} V$. So $\overline{W_N^*(g)}$ and $\overline{W_N^*(g')}$ are disjoint and as V_N refines W_N, $\overline{V_N^*(g)} \cap \overline{V_N^*(g')} = \phi$.

We can now show that $fx = fy$ iff $px = py$. If $px = py = g$ then since $f(g)$ is a single point, $fx = fy$. Now suppose $fx = fy$ and $px = g \neq py = g'$. Since f is 1-1 off $\bigcap V_n^*$ we may assume at least one of g and g' is in $\bigcap pV_n^*$. In case both g and g' are in $\bigcap pV_n^*$, choose N so that $\overline{V_N^*(g)} \cap \overline{V_N^*(g')} = \phi$. Then

$f_N \overline{V_N^*}(g) \cap f_N \overline{V_N^*}(g') = \phi$, while the first of these sets contains $f(g)$ and the second contains $f(g')$, contradicting $f(g) = f(g')$. Now assume that $g \notin \bigcap pV_n^*$ while $g' \in \bigcap pV_n^*$. For some M, $g \notin pV_M^*$ and $f(g) = f_M(g) = f_k(g)$ for $k \geq M$. There exists $N > M$ such that $g \notin \overline{V_N^*}(g')$ so $f_N(g) \notin f_N V_N^*(g')$ but $f_N(g) = f(g)$ while $f(g') \in f_N V_N^*(g')$.

So fp^{-1} is a homeomorphism of I onto M iff f is quasi-compact.

We will show f is closed but first we will prove: if $y \notin U_1^*$ (so $fy = f_j y = y$ for each j) and if $fz_n \to y$ with each $z_n \in \bigcap V_n^*$ then $z_n \to y$. Let $p(z_n) = g_n$. So $f(z_n) = f(g_n)$. Since each $g_n \in \bigcap pV_n^*$, $f(g_n) = \bigcap_{k=1}^{\infty} f_k(\overline{V_k^*}(g_n))$. So for each k,n $f(g_n) \in f_k\overline{V_k^*}(g_n)$. But $f(g_n) \to y$. So $y \,\ell p \bigcup_{n=1}^{\infty} f_k \overline{V_k^*}(g_n)$ and since f_k is a homeomorphism, $f_k^{-1} y \,\ell p \bigcup_{n=1}^{\infty} V_k^*(g_n)$. i.e., for each k, $y \,\ell p \bigcup_{n=1}^{\infty} \overline{V_k^*}(g_n)$. Now, $y \,\ell p \cup g_n$. For suppose not. Then there exists $\varepsilon > 0$ such that $N_\varepsilon y$ misses $\cup g_n$. There exists $\varepsilon_1 > 0$ such that if $g \in G$ meets $N_{\varepsilon_1} y$ then $g \subset N_{\frac{\varepsilon}{2}} y$. Choose K so that $\frac{1}{2^K} < \frac{\varepsilon_1}{2}$. Since $y \,\ell p \bigcup_{n=1}^{\infty} \overline{V_K^*}(g_n)$ there is a point $x \in \bigcup_{n=1}^{\infty} \overline{V_K^*}(g_n) \cap N_{\frac{\varepsilon_1}{2}} y$, say $x \in \overline{V_K^*}(g_N) \cap N_{\frac{\varepsilon_1}{2}} y$. But by construction, $\overline{V_K^*}(g_N) \subset N_{\frac{1}{2^K}}(g_N')$ some $g_N' \in H \subset N_{\frac{\varepsilon_1}{2}}(g_N')$. So there exists $z \in g_N'$ such that $d(x,z) < \frac{\varepsilon_1}{2}$, while $d(x,y) < \frac{\varepsilon_1}{2}$ so $d(x,z) < \varepsilon_1$. Thus g_N' meets $N_{\varepsilon_1} y$ and $g_N' \subset N_{\frac{\varepsilon}{2}} y$. Meanwhile

$g_N \subset N_{\frac{1}{2^K}}(g_N') \subset N_{\varepsilon_1}(g_N') \subset N_{\frac{\varepsilon}{2}}(g_N') \subset N_\varepsilon y$, which contradicts the choice of $N_\varepsilon y$.

So $\{y\}$ ℓp $\{g_n\}$ in I by continuity of p. Hence $z_n \to y$ since p is closed and $g_n = p(z_n)$ and the argument applies as well to any subsequence z_{n_i}.

To show f is closed, let D be closed $\subset M$ and suppose $y_n \to y$ with $y_n \in fD$. Let $x_n \in D$ such that $y_n = f(x_n)$. As in the proof of the last part of the convergence theorem, it suffices to have M locally compact at y or that $\{x_n\}$ has a convergent subsequence. So we may assume $y \notin U_1{}^*$ since U_1 is a collection of open sets which have compact closure. Then for each j, $f_j y = y = fy$. If for some J, $\{x_n\}$ is frequently not in $U_J{}^*$, then for a subsequence $\{x_{n_i}\} \subset M \setminus U_J{}^*$, $f(x_{n_i}) = f_J x_{n_i}$ for each i. So $f_J(x_{n_i}) \to y$ hence $x_{n_i} \to f_J^{-1} y = y$. So we may suppose $\{x_n\}$ is ultimately in each $U_J{}^*$. There is a subsequence $\{x_{n_i}\}$ with $x_{n_i} \in U_i{}^*$. Since it is only subsequences we are interested in, let us assume $x_n \in U_{n+1}{}^*$. Now, since U_{n+1} refines W_{n+1}, there exists $g_n \in H$ such that $x_n \in w_{n+1}(g_n) \subset N_{\frac{1}{2^{n+1}}}(g_n) \cap v_n(g_n)$. So $g_n \in H$, $d(x_n, g_n) < \frac{1}{2^{n+1}}$ and $x_n \in V_n{}^*(g_n)$. Thus for each j, $f_j x_n \in f_j V_n{}^*(g_n)$.

Let $\varepsilon > 0$. Choose N so that $n > N \Longrightarrow fx_n \in N_{\frac{\varepsilon}{4}} y$, since $fx_n \to y$. By uniform convergence there exists J such that $j > J \Longrightarrow f_j x \in N_{\frac{\varepsilon}{4}} fx$ for $x \in M$. So $n > N$, $j > J \Longrightarrow f_j x_n \in N_{\frac{\varepsilon}{2}} y$. But for each $g \in H$ and each k, $\operatorname{diam} f_k V_k{}^*(g) < \frac{1}{2^{k-2}}$. So there exists K such that $k > K \Longrightarrow \operatorname{diam} f_k V_k{}^*(g) < \frac{\varepsilon}{4}$ and since

$V_\ell^*(g) \subset V_k^*(g)$ for $\ell \geq k$, for each $\ell \geq k$, diam $f_k U_\ell^*(g) < \frac{\varepsilon}{4}$.
Choose $I > J,K$ then for $n > I,N$, $f_I x_n \in N_{\frac{\varepsilon}{2}} y$ and
diam $f_I V_n^* g < \frac{\varepsilon}{4}$ for $g \in H$. But $f_I x_n \in f_I V_n^*(g_n)$. So
$f_I V_n^*(g_n) \subset N_{\frac{3\varepsilon}{4}} y$, and since $I > J$,
$f(g_n) \in f(V_n^*(g_n)) \subset N_{\frac{\varepsilon}{4}} f_I V_n^*(g_n) \subset N_\varepsilon y$. We have shown: given $\varepsilon > 0$ there exists M such that $n > M \Longrightarrow f(g_n) \in N_\varepsilon y$. So $f(g_n) \to y$.
But $d(x_n, g_n) < \frac{1}{2^{n+1}}$. Choose $z_n \in g_n$ such that $d(x_n, z_n) < \frac{1}{2^{n+1}}$.
Now $f(z_n) \to y$ and $z_n \in H^*$. So $z_n \to y$, as we have already proved. But $d(x_n, z_n) \to 0$ so $x_n \to y$ also. This completes the proof of Theorem T.

We will use Theorem T to establish McAuley's Theorem in case p is closed. Some further observations will be useful.

First, if G is a decomposition of a metric space M, then H_G is tsh iff for each homeomorphism $h:M \approx M$, $H_{h(G)}$ is weakly tsh. This is an immediate consequence of the definitions and the fact that under a homeomorphism $h:M \approx M$, $h(H_G) = H_{h(G)}$ and if $p':M \to M/h(G)$ is the quotient map and u a p-open set then $h(u)$ is p'-open. This enables us to carry maps and coverings back and forth via the given homeomorphism. The details are straightforward and omitted here.

Consequently, if we find a set of purely topological conditions on a decomposition G (preserved under homeomorphisms on M) which yield H_G is weakly tsh, then H_G is tsh also.

We also note that local shrinkability of continua is topological, i.e., if M and M' are metric, h a homeomorphism of M onto M'

and C a locally shrinkable continuum in M, then $h(C)$ is a locally shrinkable continuum in M'.

__Proof.__ Trivially, hC is a continuum. Since C is locally shrinkable in M, for each positive integer k there exists $f_k:M \cong M$ such that $f_k = id$ off $N_{\frac{1}{k}} C$ and $\operatorname{diam} f_k C < \frac{1}{k}$. $C_k = f_k C \subset N_{\frac{1}{k}} C$. Each open set containing C contains C_k ultimately as C is compact. There exists $x \in C$ such that each neighborhood of x meets C_k for infinitely many k, again by compactness of C. Since M is metric a subsequence $C_{k_i} \to x$, i.e., each neighborhood of x meets C_{k_i} ultimately. And since $\operatorname{diam} C_{k_i} \to 0$ each neighborhood of x contains C_{k_i} ultimately. Now, since h is a homeomorphism $hC_{k_i} \to hx \in hC$. Also $\operatorname{diam} hC_{k_i} \to 0$ since if V is any neighborhood of $h(x)$, $h^{-1}V$ is a neighborhood of x and contains C_{k_i} ultimately. Then V ultimately contains hC_{k_i}. Since we may choose neighborhoods V of $h(x)$ with arbitrarily small diameter, $\operatorname{diam} hC_{k_i}$ must tend to zero. Now let U open $\supset hC$, $\varepsilon > 0$. Then $h^{-1}U$ is open $\supset C$. Choose I so that $\operatorname{diam} hC_{k_I} < \varepsilon$ and $N_{\frac{1}{k_I}} C \subset h^{-1}U$. Then $f_{k_I}:M \cong M$, $f_{k_I} = \text{if}$ off $h^{-1}U$, $f_{k_I}C = C_{k_I}$. Let $h' = hf_{k_I}h^{-1}$: $M' \cong M'$ so $h' = id$ off U and $h'(hC) = hf_{k_I}C = hC_{k_I}$ has diameter $< \varepsilon$, which means hC is locally shrinkable.

We need the following theorem of McAuley:

__Theorem H.__ (McAuley) If M is a metric space, $\{f_i\}:M \cong M$, $\{U_i\}$ a sequence of open subsets of M such that $U_i \supset \overline{U}_{i+1}$, $\bigcap U_i = \phi$, $f_i = f_{i-1}$ off U_i, $f_0 = id$, and for each $p \in M$, $\bigcup_{i=1}^{\infty} f_i^{-1}p$ has compact closure then $\{f_i\} \to f:M \cong M$.

Remark. Excluding the last hypothesis of Theorem H yields $f = \lim f_i$ continuous, 1-1 and open. This last condition provides that f is onto.

Theorem H'. (McAuley, revised) If G is a decomposition of a metric space M satisfying

1) p is closed and point-compact,

2) each element of H is locally shrinkable,

3) H is countable and G_δ,

4) M is locally compact at H^*,

then H is weakly tsh in M.

Proof. In this proof the notation $\langle O,D \rangle$ is used to replace the sequence of symbols: O p-open $\subset \overline{O} \subset D$ p-open $\subset \overline{D}$ compact. By hypothesis, $H = \{C_j\}_{j=1}^{\infty}$, $H^* = \bigcap_{i=1}^{\infty} G_i$, G_i open $\supset G_{i+1}$. Let A be a p-open cover of H^*, $\varepsilon > 0$. For each j, choose $A_j \in A$ with $C_j \subset A_j$. Let $h_0 = \text{id}$.

Let $H_1 = \{C \in H: \text{diam } C \geq \varepsilon\}$. By usc, $H_1{}^*$ is closed. If $H_1 \neq \phi$, let k_1 be least such that $C_{k_1} \in H_1$. So $C_j \notin H_1$ for $j < k_1$. $H_1^* \subset W_1$ open such that W_1 misses C_j for $j < k_1$. $H_1^* \subset U_1$ open such that $\overline{U}_1 \subset W_1 \cap G_1$. Let $C_{k_1} \subset \langle O_1,D_1 \rangle \subset U_1 \cap A_{k_1}$ and let $h_1 : M \approx M$ such that $h_1 = \text{id}$ off O_1 and $\text{diam } h_1 C_{k_1} < \varepsilon$.

Let $H_2 = \{C \in H: \text{diam } h_1 C \geq \varepsilon\}$. H_2^* is closed $\subset U_1$. If $H_2 \neq \phi$, let k_2 be least such that $C_{k_2} \in H_2$. Then $k_2 > k_1$. $H_2^* \subset W_2$ open such that W_2 misses C_j for $j < k_2$. $H_2^* \subset U_2$ open such that $\overline{U}_2 \subset U_1 \cap W_2 \cap G_2$. Let $C_{k_2} \subset \langle O_2,D_2 \rangle \subset U_2 \cap A_{k_2}$ and such that if $C_{k_2} \cap \overline{O}_1 = \phi$, we select D_2 so that $\overline{D}_2 \cap \overline{O}_1 = \phi$, while if $C_{k_2} \cap \overline{O}_1 \neq \phi$, then choose D_2 so that $\overline{D}_2 \subset D_1$. Let $h_2: M \approx M$ such that $h_2 = h_1$ off O_2 and h_2 shrinks C_{k_2} to diameter $< \varepsilon$, (hence C_j for $j \leq k_2$).

Inductively, given $h_\ell: M \approx M$ for $0 \leq \ell \leq i$ such that for $1 \leq \ell \leq i$ $h_\ell = h_{\ell-1}$ off O_ℓ, W_ℓ is open missing C_j for $j < k_\ell$, $C_{k_\ell} \subset \langle O_\ell, D_\ell \rangle \subset U_\ell \cap A_{k_\ell} \subset U_\ell$ open $\subset \overline{U}_\ell \subset U_{\ell-1} \cap G_\ell \cap W_\ell$ and $\overline{D}_\ell \cap \overline{O}_j = \phi$ or $\overline{D}_\ell \subset D_j$ (and $\overline{O}_\ell \cap \overline{O}_j \neq \phi$) for all $j < \ell$, and, h_ℓ shrinks C_j for $j \leq k_\ell$.

Let $H_{i+1} = \{C \in H : \text{diam } h_i C \geq \varepsilon\}$. Then H_{i+1}^* is closed $\subset U_i$. If $H_{i+1} \neq \phi$ let k_{i+1} be least such that $C_{k_{i+1}} \in H_i$. Then $k_{i+1} > k_i$ and $C_j \notin H_i$ for $j < k_{i+1}$. $H_{i+1}^* \subset W_{i+1}$ open such that W_{i+1} misses C_j for $j < k_{i+1}$. $H_{i+1}^* \subset U_{i+1}$ open $\subset$
$\overline{U}_{i+1} \subset U_i \cap W_{i+1} \cap G_{i+1}$. Let $C_{k_{i+1}} \subset \langle O_{i+1}, D_{i+1} \rangle \subset U_{i+1} \cap A_{k_{i+1}}$ and such that for each ℓ, $1 \leq \ell \leq i$, if $C_{k_{i+1}} \cap \overline{O}_\ell \neq \phi$, choose $\overline{D}_{i+1} \subset D_\ell$ and if $C_{k_{i+1}} \cap \overline{O}_\ell = \phi$, choose D_{i+1} so that $\overline{D}_{i+1} \cap \overline{O}_\ell = \phi$ also. (so we have $\overline{D}_j \cap \overline{O}_\ell = \phi$ or $\overline{D}_j \subset D_\ell$ and $\overline{O}_j \cap \overline{O}_\ell \neq \phi$ for each $j \leq i + 1$ and $\ell < j$.) Let $h_{i+1}: M \approx M$ such that $h_{i+1} = h_i$ off O_{i+1} and h_{i+1} shrinks $C_{k_{i+1}}$ to diameter $< \varepsilon$ (hence C_j for $j \leq k_{i+1}$).

If $H_i = \phi$ for some i, let $h = h_{i-1}$. This gives a homeomorphism $h: M \approx M$, without appeal to Theorem H, which shrinks each element of G to diameter $< \varepsilon$. And we can construct a p-open refinement V of A as required for weakly tsh in the same way as for the case that $\{H_i\}$ is infinite, which follows.

If $H_i \neq \phi$ for each i, then we have a sequence of homeomorphisms h_i of M onto M and open sets U_i such that
$U_i \supset \overline{U}_{i+1}$, $h_i = h_{i-1}$ off U_i (actually off O_i), $\bigcap U_i = \phi$ (since

$\bigcap U_i \subset \bigcap G_i = H^* = \bigcup C_j$, but each j, U_{j+1} misses C_j, so $H^* \cap (\bigcap U_i) = \phi$). So we have verified conditions of Theorem H which give $h_i \to h: M \to M$, with h 1-1, continuous and open.

We must show h is onto. Prior to this, we list some properties of the construction:

<u>Lemma 1</u>. For each i, $h_i A = h_{i-1} A$ for any set A containing O_i. In particular, $h_i \bar{O}_i = h_{i-1} \bar{O}_i \subset h_{i-1} D_i = h_i D_i$.

<u>Lemma 2.1</u>. For each $i < j$ if $x \notin O_\ell$ for $1 < \ell \leq j$ then $h_j x = h_i x$.

<u>Lemma 2</u>. For each i there exists $L(i) \leq i$ such that $\bigcup_{\ell=0}^{i} h_\ell (D_i) \subset D_{L(i)}$.

<u>Proof</u>. The statement holds for $i = 1$ since $h_1 D_1 = h_0 D_1 = D_1$. Let $L(1) = 1$. Assume for each $j < i$ that there exists $L(j) \leq j$ such that $\bigcup_{\ell=0}^{j} h_\ell D_j \subset D_{L(j)}$. If D_i misses $\bar{O}_j$ for each $j < i$ then $h_\ell D_i = D_i$ for $\ell < i$ by Lemma 2.1. But $h_i D_i = h_{i-1} D_i$ by Lemma 1 so $h_i D_i = D_i$ also. And $\bigcup_{\ell=0}^{i} h_\ell D_i = D_i$. Let $L(i) = i$. If D_i meets some $\bar{O}_j$ for $j < i$, let J be the largest such j. Then by construction $D_i \subset D_J$ and by our inductive assumption, there exists $L(J) \leq J$ such that $\bigcup_{\ell=0}^{J} h_\ell D_J \subset D_{L(J)}$. But $\bigcup_{\ell=0}^{J} h_\ell D_i \subset \bigcup_{\ell=0}^{J} h_\ell D_J$ and since D_i misses $\bar{O}_j$ for $J < j < i$, $h_\ell D_i = h_J D_i$ pointwise for $J < \ell \leq i-1$ by Lemma 2.1. So we also have $\bigcup_{\ell=0}^{i-1} h_\ell D_i \subset D_{L(J)}$. And by Lemma 1, $h_{i-1} D_i = h_i D_i$. Hence $\bigcup_{\ell=0}^{i} h_\ell D_i \subset D_{L(J)}$. So we let $L(i) = L(J) \leq J < i$.

Now it is easy to show h is onto. Let p be any point of M. If $p \notin \bigcup O_i$ then $h_i p = p$ for each i and $hp = p$. So suppose $p \in \bigcup O_i$ and let I be least such that $p \in O_I$. We will show that $\{h_i^{-1} p\}_{i \geq I} \subset \bigcup_{i=1}^{I} D_i$. Otherwise, there exists a least J such that $h_J^{-1}p \notin \bigcup_{i=1}^{I} D_i$. Let $z = h_J^{-1}p$. If $z \notin O_J$ then $p = h_J z = h_{J-1} z$ so $z = h_{J-1}^{-1}p$ contrary to the choice of J. So $z \in O_J$. But $z \cup p = h_0 z \cup h_J z \subset D_{L(J)}$ for some $L(J)$ by Lemma 2. So $D_{L(J)}$ meets $\overline{O}_I$ in p. If $L(J) > I$ then by construction $D_{L(J)} \subset D_I$. If $L(J) \leq I$, we still have $z \in \bigcup_{i=1}^{I} D_i$ which is a contradiction. So $\{h_i^{-1}p\}_{i \geq I} \subset \bigcup_{i=1}^{I} D_i$, which is a finite union of sets having compact closures. So we have confirmed the last hypothesis of Theorem H and we have $h_i \to h: M \cong M$.

<u>Lemma 3.1.</u> For each i and j with $i < j$ if $\overline{O}_i$ and $\overline{O}_j$ are disjoint then no $\overline{O}_\ell$ can meet them both for $\ell \geq j$.

<u>Proof.</u> If $\overline{O}_\ell$ meets both $\overline{O}_i$ and $\overline{O}_j$ with $\ell \geq j > i$ then $\overline{O}_\ell \subset D_\ell$ is chosen so that $D_\ell \subset D_i \cap D_j$. But D_j was chosen to miss $\overline{O}_i$.

<u>Lemma 3.2.</u> If A is any set which contains each $\overline{O}_i$ for $I \leq i \leq J$ which A intersects, then $h_I A = h_J A$.

<u>Proof.</u> Suppose not. Let L be least such that $h_L A \neq h_I A$ with $I < L \leq J$. Then $h_{L-1} A = h_I A$. But if $h_L A \neq h_{L-1} A$ then A meets $\overline{O}_L$ so $\overline{O}_L \subset A$. Hence $h_L A = h_{L-1} A$ by Lemma 1.

<u>Lemma 3.</u> For each I and $J \geq I$, $h_J \overline{O}_I \subset h_I D_I$.

<u>Proof.</u> For $J = I$ the statement is trivial. Given $J > I$, let $Q = \{\overline{O}_i : I \leq i \leq J\}$. Let $A = \{O \in Q$: there exists a (finite) sequence of elements of Q, consecutively intersecting and of increasing index from $\overline{O}_I$ to $O\}$. Clearly, $\overline{O}_I \in A$, and $A^* \subset D_I$, for

otherwise if there exists an element $\overline{O}_i \in A$ with $\overline{O}_i \not\subset D_I$ then $D_i \not\subset D_I$. Let K be least such that $\overline{O}_K \in A$ and $D_K \not\subset D_I$. There is a sequence from $\overline{O}_I$ to $\overline{O}_K$, as described above. An element $\overline{O}_j$ of this sequence meets $\overline{O}_K$ with $j < K$. So $D_j \subset D_I$ but also by construction $D_K \subset D_j$. Hence $D_K \subset D_I$. Furthermore, A^* contains each element of Q which A^* intersects. For if $\overline{O}_i \in Q$ and $\overline{O}_i$ meets A^*, let J be least such that $\overline{O}_J \in A$ and $\overline{O}_i$ meets $\overline{O}_J$. Now if $J < i$, augmenting the sequence from $\overline{O}_I$ to $\overline{O}_J$ by $\overline{O}_i$ gives a sequence from $\overline{O}_I$ to $\overline{O}_i$, placing $\overline{O}_i \in A$. So suppose $J > i$. Let $\overline{O}_k$ be the element of the sequence from $\overline{O}_I$ to $\overline{O}_J$ which meets $\overline{O}_J$. Then $k < J$. So $\overline{O}_i$ does not meet $\overline{O}_k$. But $\overline{O}_J$ cannot meet both of the disjoint sets $\overline{O}_i$ and $\overline{O}_k$ by Lemma 3.1. Now by Lemma 3.2 $h_I(A^*) = h_J(A^*)$. And since $\overline{O}_I \subset A^*$, $h_J(\overline{O}_I) \subset h_J(A^*) = h_I(A^*) \subset h_I D_I$, and Lemma 3 is proved.

Now, $\{\overline{U}_i\}$ is a locally finite collection since $U_i \supset \overline{U}_{i+1}$ and $\bigcap U_i = \phi$. $\{\overline{O}_i\}$ is locally finite, as $\overline{O}_i \subset U_i$. Since each $\overline{O}_j$ is compact, it meets at most a finite number of elements of $\{O_i\}$. So for each j there exists $N(j) \geq j$ such that $\overline{O}_j \subset M \setminus \bigcup_{i \geq N(j)} \overline{O}_i$. Then $h\overline{O}_j = h_{N(j)}\overline{O}_j \subset h_j D_j$ by Lemma 3, while $D_j \cup h_j D_j \subset D_{L(j)}$ for some $L(j) \leq j$ by Lemma 2. Thus $\overline{O}_j \cup h\overline{O}_j \subset D_{L(j)} \subset A_{L(j)}$. For each $C \in H \setminus \bigcup pO_i$, $hC = C$ and $\operatorname{diam} C < \varepsilon$. Again by the local finiteness of $\{O_i\}$ some neighborhood of the compact C misses $\bigcup O_i$ and hence there exists a p-open set $N(C)$ containing C and missing $\bigcup O_i$ and such that if $C = C_j$, $N(C_j) \subset A_j$. $hN(C) = N(C)$.

Let $V = \{O_j\}_{j=1}^{\infty} \bigcup \{N(C) : C \in H \setminus \bigcup p\, O_i\}$. Then V is a p-open refinement of $\mathcal{A}$, $h = \mathrm{id}$ off V^*, h shrinks each element of H to diameter $< \varepsilon$, and $v \in V \Longrightarrow$ there exists $A \in \mathcal{A}$ with

$A \supset v \cup hv$. Thus, H is weakly tsh.

Since the hypotheses of Theorem H' are topological, we have immediately that H is tsh. Hence, by Theorem T,

Corollary H'. (McAuley) Under the hypotheses of Theorem H', $I = M$.

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