1974

Properties of ultraproduct spaces

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PROPERTIES OF ULTRAPRODUCT SPACES

BY

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Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in State University of New York
at Binghamton
1974

Accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in State University of New York at Binghamton

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ACKNOWLEDGEMENT

I wish to express my sincere gratitude to Professor William R. Transue for his valuable advice and unfailing encouragement, without which this work could never have been completed. I also wish to express my thanks to Dr. Charles Stegall, Dr. John Walsh and Mr. David Thomas for the many valuable discussions we had.

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INTRODUCTION

There seems to be general agreement that in this stage of the development of the theory of Banach spaces a greater wealth of examples would be useful in pointing the way for future research. There has not been a great variety of procedures for the construction of Banach spaces, but one was introduced in [3] by Bretagnolle, Dacunha-Castelle and Krivine. In [4] their procedure was presented more systematically, the space constructed was termed the ultraproduct, and various applications were made. There is also a brief discussion of ultraproducts in [7].

To our knowledge the process has not been subjected to a detailed and systematic study, and that is the purpose of this dissertation. While the definition does not require an order structure, in the applications which have been made the spaces involved have been Banach lattices and the lattice structure extends to the ultraproduct in a natural way. Hence the present study has included these order properties.

The ultraproduct procedure has an apparent defect in that it requires free ultrafilters which cannot be explicitly constructed, but this is not as serious as might be supposed. Many properties of the ultraproduct may be obtained with merely the knowledge that the ultra-filter used contains a given filter, and the utility of the construction is well established by the applications in [3] and [4]. The property (which has been called countable intersection property) of a free ultrafilter of possessing a countable sequence of elements of the filter whose intersection is empty has shown itself to be useful. We do not know whether every free ultrafilter has this property.

In the references on ultraproducts cited above the statement of the definition of an ultraproduct of Banach spaces differs slightly from the one we have given in requiring a completion of the space as we have defined it. Since we show that the space we have defined is already complete, our definition is not in fact different from that already given.

1. PRELIMINARIES

In the rest of this paper we will let A denote an index set which will be of arbitrary cardinality (although assumed infinite) unless otherwise stated.

Definition 1.1 A filter on a set A is a collection Λ of subsets of A having the properties:

(i) $\emptyset \notin \Lambda$

(ii) $E \cap F \in \Lambda$ whenever $E, F \in \Lambda$

(iii) If $F \in \Lambda$ and $F \subseteq E$, then $E \in \Lambda$

An ultrafilter is a filter which is properly contained in no other filter.

Using Zorn's lemma it is easy to see that ultrafilters exist and moreover, that any filter is contained in an ultrafilter (although not unique unless the filter is itself an ultrafilter).

Example 1.2 Let $a_o \in A$. Define $\Omega = \{X \subseteq A | a_o \in X\}$ Then Ω is a filter on A which is in fact an ultrafilter. This ultrafilter is said to be fixed at the point a_o. Notice that $\{a_o\} \in \Omega$ and furthermore, if Λ is a filter such that $\{a_o\} \in \Lambda$, then $\Lambda = \Omega$.

Definition 1.3 A filter Λ such that $\bigcap\{X | X \in \Lambda\} \neq \emptyset$ is a fixed filter. Otherwise it is said to be free. A fixed ultrafilter is of the type given in example 1.2.

Example 1.4 Let ω denote the positive integers and let $\Lambda = \{X \subseteq \omega \mid X^c$ is finite$\}$. Then Λ is a filter on ω called the Frèchet filter. If we let Ω be any ultrafilter containing Λ, then since the sets $[n,\infty)$ all belong to Λ, Λ must be free and hence Ω will be free also.

Definition 1.5 A collection Ψ of subsets of A satisfying the conditions

(i) $\emptyset \notin \Psi$

(ii) If $E, F \in \Psi$, then there is a $G \in \Psi$ such that $G \subseteq E \cap F$

can be extended to a filter on A by adding all supersets. Such a collection Ψ is called a base for a filter.

Example 1.6 Let B be a Banach space, let A be the collection of finite dimensional subspaces of B. For each $F \in A$ let

$$X(F) = \{G \in A \mid F \subseteq G\}.$$

Then $\{X(F) \mid F \in A\}$ is a base for a free filter on A.

The following proposition giving some well-known and useful results on ultrafilters is included here for completeness.

Proposition 1.7 Let Ω be an ultrafilter on a set A.

(i) For every $X \subseteq A$, either $X \in \Omega$ or $X^c \in \Omega$ (where X^c denotes the complement of X relative to A).

(ii) If the disjoint union $A \dot{\cup} B \in \Omega$, then either $A \in \Omega$ or $B \in \Omega$ (clearly not both).

(iii) If the disjoint finite union $X_1 \dot{\cup} X_2 \dot{\cup} \ldots \dot{\cup} X_n \in \Omega$ then $X_i \in \Omega$ for precisely one i.

Proof: We will indicate a proof for statement (i). The remaining statements follow from (i) in a straightforward manner.

If there were an $E \in \Omega$ such that $E \subseteq X$, then $X \in \Omega$. If there were no such E then for every $E \in \Omega$, $E \cap X^c \neq \emptyset$ so the collection $\Omega \cup X^c$ is a base for a filter containing Ω, contradicting the maximality of Ω unless $X^c \in \Omega$. Q.E.D.

Definition 1.8 An ultrafilter Ω is said to have the countable intersection property if there is a sequence $\{X_i\}_{i=1}^{\infty}$ of elements of Ω with $\bigcap_{i=1}^{\infty} X_i = \emptyset$.

Since every intersection of elements of a fixed ultrafilter is non-empty the only ultrafilters which may have the countable intersection property are the free ultrafilters.

Proposition 1.9 Given an ultrafilter Ω on a set A the following are equivalent.

1° Ω has the countable intersection property.

2° There is a sequence $\{X_i\}_{i=1}^{\infty}$ of elements of Ω such that each $a \in A$ belongs to at most a finite number of the X_i.

3° There is a sequence $\{X_i\}_{i=1}^{\infty}$ of elements of Ω such that $\bigcap_{i=1}^{\infty} X_i \notin \Omega$.

4° There is a sequence $\{X_i\}_{i=1}^{\infty}$, with $X_i \notin \Omega$ for any i, such that $\bigcup_{i=1}^{\infty} X_i \in \Omega$.

Proof: 1° => 2°

Since Ω has the countable intersection property choose a sequence $\{Y_i\}_{i=1}^{\infty}$ of elements of Ω with $\bigcap_{i=1}^{\infty} Y_i = \emptyset$. For each $n \in \omega$ define $X_n = \bigcap_{i=1}^{n} Y_i$. Then $X_n \in \Omega$ for all $n \in \omega$ and

$$\bigcap_{n=1}^{\infty} X_n = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{n} Y_i = \bigcap_{i=1}^{\infty} Y_i = \emptyset$$

Furthermore, $X_{n+1} \subseteq X_n$ for all $n \varepsilon \omega$. Let $a \varepsilon A$. Then a belongs to at most a finite number of the X_n, for since the $\{X_n\}$ are nested, if $a \varepsilon \bigcap_{j=1}^{\infty} X_{n_j}$ for some subsequence (n_j) of ω then $a \varepsilon \bigcap_{n=1}^{\infty} X_n$ which is impossible.

2° => 3°

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence such that each $a \varepsilon A$ belongs to at most a finite number of the $\{X_i\}$. Then $\bigcap_{i=1}^{\infty} X_i = \emptyset \not\varepsilon \Omega$.

3° <=> 4°

Since $X_i \varepsilon \Omega$ implies $X_i^c \not\varepsilon \Omega$ and $\bigcap_{i=1}^{\infty} X_i \not\varepsilon \Omega$ implies $(\bigcap_{i=1}^{\infty} X_i)^c = \bigcup_{i=1}^{\infty} X_i^c \varepsilon \Omega$, 4° is just a restatement of 3° in terms of complements.

3° => 1°

If $\bigcap_{i=1}^{\infty} X_i \not\varepsilon \Omega$ then since $X_i \varepsilon \Omega$ we have $\overline{X}_i = X_i \setminus \bigcap_{i=1}^{\infty} X_i \varepsilon \Omega$ and $\bigcap_{i=1}^{\infty} \overline{X}_i = \emptyset$. Q.E.D.

A natural question to ask is, whether ultrafilters with the countable intersection property exist.

<u>Proposition 1.10</u> Every free ultrafilter on a set of cardinality $\aleph_0$ has the countable intersection property.

Proof: We may consider Ω to be an ultrafilter on the set ω of positive integers.

If M is a finite set, say $M = \{a_1, \ldots, a_n\}$, and if $M \varepsilon \Omega$, then M can be written as the disjoint finite union of its points so for some i, $\{a_i\} \varepsilon \Omega$. But then Ω would be fixed at a_i.

Thus, since Ω is free, Ω can contain no finite sets and hence must contain the Frèchet filter. Then the sets $X_i = [i,\infty)$ have the desired properties. Q.E.D.

<u>Proposition 1.11</u> Every free ultrafilter on a set of cardinality c has the countable intersection property.

Proof: We may consider Ω to be an ultrafilter on the set $[0,1)$. Consider the sets $[0,1/2)$ and $[1/2,1)$. One of these sets belongs to Ω, call it X_1. Take X_1 and split it up into two equal parts in the same way. One of these belongs to Ω, call it X_2. Continuing this a countable number of times we get a sequence $\{X_i\}_{i=1}^{\infty}$ with the property that $\bigcap_{i=1}^{\infty} X_i$ is either $\emptyset$ or a single point p. If $\bigcap_{i=1}^{\infty} X_i = \emptyset$ then Ω has the countable intersection property. If $\bigcap_{i=1}^{\infty} X_i = \{p\}$ then $\{p\} \notin \Omega$ since Ω is free so the sets $X_i' = X_i \setminus \{p\} \in \Omega$ and $\bigcap_{i=1}^{\infty} X_i' = \emptyset$ so Ω has the countable intersection property. Q.E.D.

<u>Example 1.12</u> An ultrafilter on a set of cardinality greater than c which has the countable intersection property.

Let $B = L_1(\Gamma)$ where the cardinality of Γ is 2^c. Then B is a Banach space with a Hamel basis of cardinality at least as large as 2^c so there are at least 2^c finite dimensional subspaces of B. Construct the filter and ultrafilter of example 1.6. Let $\{b_i\}_{i=1}^{\infty}$ be a basic sequence in B and for each i define $X_i = \{F \in A \mid b_i \in F\}$. Since each finite dimensional subspace F contains only finitely many b_i, F belongs to at most finitely many X_i. Thus Ω has the countable intersection property by proposition 1.9 (2°).

Let us recall now from Bourbaki [2] (Ch. 1), that if Ω is an ultrafilter on a set A and f is a mapping of A into a set A' then $f(\Omega)$ is a base for an ultrafilter on A'. Also, a filter base B on a topological space X converges to a point $x \varepsilon X$ if every set of a fundamental system of neighborhoods of x contains a set of B. Now if A is an index set and Ω is an ultrafilter on A let $(x_\alpha)_{\alpha \varepsilon A}$ be a collection of real numbers. This defines a function $T: A \to R$ by $T(\alpha) = x_\alpha$. Then $T(\Omega)$ is a base for an ultrafilter on R. If this filterbase converges to a number L we write $L = \lim_\Omega x_\alpha$. Notice that $L = \lim_\Omega x_\alpha$ means that for every $\varepsilon > o$, $\{\alpha \varepsilon A \mid |x_\alpha - L| < \varepsilon\} \varepsilon \Omega$.

Now if $(x_\alpha)_{\alpha \varepsilon A}$ is a bounded collection then the filterbase $T(\Omega)$ lies in a compact set and hence converges which means that $\lim_\Omega x_\alpha$ exists. Furthermore, since the filter Ω' induced on a set $X \varepsilon \Omega$ by the ultrafilter Ω is itself an ultrafilter, $T(\Omega')$ will be a base for an ultrafilter on R so $\lim_\Omega x_\alpha$ exists provided only that the collection $(x_\alpha)_{\alpha \varepsilon X}$, $X \varepsilon \Omega$ is bounded. The converse of this statement is clearly true, namely, if $\lim_\Omega x_\alpha$ exists then for some $X \varepsilon \Omega$ $(x_\alpha)_{\alpha \varepsilon X}$ is bounded.

Since $\lim_\Omega x_\alpha$ is defined in terms of the direct image of an ultrafilter, all the usual theorems about convergence of filters on R hold. In particular

(i) $\lim_\Omega x_\alpha$ is unique

(ii) $k = \lim_\Omega k$, k a constant

(iii) $\lim_{\Omega}(x_{\alpha} + y_{\alpha}) = \lim_{\Omega}x_{\alpha} + \lim_{\Omega}y_{\alpha}$

(iv) $\lim_{\Omega}(x_{\alpha}y_{\alpha}) = (\lim_{\Omega}x_{\alpha})(\lim_{\Omega}y_{\alpha})$

(v) $\lim_{\Omega}(x_{\alpha} \vee y_{\alpha}) = (\lim_{\Omega}x_{\alpha}) \vee (\lim_{\Omega}y_{\alpha})$

Since the following result will be used many times, we will sketch a proof of it.

Proposition 1.13 Let $X \varepsilon \Omega$ and suppose $x_{\alpha} \leq y_{\alpha}$ for all $\alpha \varepsilon X$. If $L_1 = \lim_{\Omega}x_{\alpha}$ and $L_2 = \lim_{\Omega}y_{\alpha}$ then $L_1 \leq L_2$.

Proof: Suppose $L_1 > L_2$. Let $\varepsilon < \frac{L_1 - L_2}{2}$. Then

$$X_1 = \{\alpha \varepsilon A \mid |x_{\alpha} - L_1| < \varepsilon\} \varepsilon \Omega \text{ and}$$

$$X_2 = \{\alpha \varepsilon A \mid |y_{\alpha} - L_2| < \varepsilon\} \varepsilon \Omega.$$

Let $\alpha \varepsilon X \cap X_1 \cap X_2$. Then $x_{\alpha} \leq y_{\alpha}$ but

$y_{\alpha} < L_2 + \varepsilon < L_1 - \varepsilon < x_{\alpha}$ which implies that

$X \cap X_1 \cap X_2 = \emptyset$ which is not possible since

$X \cap X_1 \cap X_2 \varepsilon \Omega$. Thus $L_1 \leq L_2$. Q.E.D.

The following corollary follows immediately from this proposition together with the preceeding remarks.

Corollary 1.14 Let $X \varepsilon \Omega$ and suppose $|x_{\alpha}| \leq M$ for all $\alpha \varepsilon X$. Then $\lim_{\Omega}x_{\alpha}$ exists and

$$|\lim_{\Omega}x_{\alpha}| \leq M.$$

For the sake of completeness we include some basic definitions and theorems on vector lattices. For a more complete treatment see M. M. Day [5].

Definition 1.15 Let V be a real vector space with order relation $\geq$. V is a vector lattice if

(i) $\geq$ is reflexive and transitive.

(ii) translation and multiplication by positive numbers preserve order; multiplication by negative numbers reverses order.

(iii) $x \geq y$ and $y \geq x$ implies $x = y$ and each pair of elements x and y of V has a least upper bound, denoted $x \vee y$.

In the preceeding definition we assume that the concepts of upper bound and least upper bound are defined in a manner analogous to their definitions on the real line.

Definition 1.16 If V is a vector lattice than we may define the greatest lower bound of two elements x and y by

$$x \wedge y = -((-x) \vee (-y))$$

Definition 1.17 Let V and V' be vector lattices. A function $f: V \to V'$ is a lattice homomorphism if $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in V$.

Definition 1.18 A vector lattice V is boundedly [σ-] complete if each [countable] set A in V which has an upper bound has a least upper bound. V is conditionally σ-complete if each increasing sequence of non-negative elements of V which has an upper bound has a least upper bound.

Let us remark that conditionally σ-complete is clearly equivalent to the condition that every decreasing sequence of non-negative elements of V has a greatest lower bound.

For a proof of the following proposition refer to Jameson [6].

Proposition 1.19 If V is a conditionally σ-complete vector lattice, then V is σ-complete.

Definition 1.20 A vector lattice V is Archimedean if $x \leq o$ whenever for some y, $nx \leq y$ for all $n \varepsilon \omega$.

We remark that to show V is Archimedean it suffices to show that $x=o$ whenever $x,y \geq o$ and $nx \leq y$ for all $n \varepsilon \omega$.

The following proposition is immediate.

Proposition 1.21 Every σ-complete vector lattice is Archimedean.

Definition 1.22 A normed [Banach] lattice is a normed linear [Banach] space which is also a lattice in which $\wedge$ and $\vee$ are continuous functions of both their variables.

Definition 1.23 An (AB)-lattice is a normed linear space and vector lattice in which order and norm are related by

(A) $|| \, |x| \, || = ||x||$

(B) If $o \leq x \leq y$, then $o \leq ||x|| \leq ||y||$

Proposition 1.24 An (AB)-lattice is a normed lattice.

Definition 1.25 A vector lattice V has semicontinuous norm if for any decreasing sequence of non-negative elements $\{x_n\}$ with greatest lower bound x we have

$$||x|| = \inf_n ||x_n||$$

Definition 1.26 An element s of a vector lattice V is said to be positive complete if $s > o$ and if $s \wedge |x| = 0$ with $x \varepsilon V$ implies $x = o$.

<u>Definition 1.27</u> A Banach lattice is an <u>(AM)-space</u> if norm and order are related by (A) and

(M) if $x,y \geq o$, then $||x \vee y|| = ||x|| \vee ||y||$

A Banach lattice is an <u>(AL)-space</u> if norm and order are related by (A) and

(L) if $x,y \geq o$, then $||x+y|| = ||x||+||y||$

Notice that since (M) or (L) clearly implies (B) an (AM)-space or (AL)-space is an (AB)-lattice.

2. TOPOLOGICAL PROPERTIES

Definition 2.1 Let A be an index set and Ω an ultrafilter on A. Associate with each $\alpha \varepsilon A$ a normed linear space L_α. Form the product space $\prod_{\alpha \varepsilon A} L_\alpha$ and denote by L_o the linear subspace

$$L_o = \{\{x_\alpha\} \varepsilon \prod_{\alpha \varepsilon A} L_\alpha \mid \text{there is a real number } N>o \text{ such that } ||x_\alpha|| \leq N \text{ for all } \alpha \varepsilon A\}$$

Define a semi-norm on L_o by putting

$$||\{x_\alpha\}|| = \lim_\Omega ||x_\alpha||$$

Letting N be the subspace of L_o consisting of the elements of semi-norm zero, form the associated normed space

$$L = L_o/N.$$

We will call a normed space L constructed in this manner an ultraproduct space.

Let us remark that this definition does in fact make sense. L_o is clearly a linear space and the properties of limits imply that $||\cdot||_{L_o}$ is in fact a semi-norm. Notice also that since every element of N has semi-norm zero,

$||\{x_\alpha\}+N||_L = ||\{x_\alpha\}||_{L_o}$ where $\{x_\alpha\}$ is an arbitrary representative of $\{x_\alpha\}+N$.

Recall that if E and F are normed spaces then the distance from E to F is

$$d(E,F) = \inf ||T||\cdot||T^{-1}||$$

where the infimum is taken over all isomorphisms T and E onto F.

Proposition 2.2 Suppose L and L' are ultraproduct spaces indexed by the same set A and using the same ultrafilter Ω. Let $K\varepsilon\omega$. If $d(L_\alpha,L'_\alpha)\leq K$ for all $\alpha\varepsilon A$ then $d(L,L')\leq K$.

Proof: Choose $\varepsilon>o$. Since $d(L_\alpha,L'_\alpha)\leq K$ we can find an isomorphism $T_\alpha:L_\alpha\to L'_\alpha$ with $||T_\alpha||\cdot||T_\alpha^{-1}||<K+\varepsilon$, for each $\alpha\varepsilon A$. Furthermore we may assume that $||T_\alpha|| = ||T_\alpha^{-1}||<\sqrt{K+\varepsilon}$.

Define a mapping $T:L \to L'$ by

$$T(\{x_\alpha\}+N) = \{T_\alpha(x_\alpha)\}+N'.$$

T is clearly linear and onto since each T_α is. Furthermore, the inverse of T is the mapping $T^{-1}:L' \to L$ given by $T^{-1}(\{x'_\alpha\}+N')= \{T_\alpha^{-1}(x'_\alpha)\}+N$.

To show T is an isomorphism we observe that

$$||T_\alpha(x_\alpha)|| \leq ||T_\alpha|| \cdot ||x_\alpha|| < \sqrt{K+\varepsilon}\ ||x_\alpha||$$

so again by proposition 1.13

$$||\{T_\alpha(x_\alpha)\}|| \le \sqrt{K+\varepsilon}\ ||\{x_\alpha\}||$$

which shows T is continuous and $||T|| \le \sqrt{K+\varepsilon}$. The continuity of T^{-1} follows from the continuity of each T_α^{-1} in the same way. Thus T is an isomorphism and furthermore the proof shows that $d(L,L') \le ||T|| \cdot ||T^{-1}|| \le K+\varepsilon$. Since this holds for every $\varepsilon > o$,

$$d(L,L') \le K. \quad \text{Q.E.D.}$$

<u>Theorem 2.3</u> If each L_α is a Banach space so is the ultraproduct space L.

Proof: Let $<\{x_\alpha^n\}+N>_n$ be a Cauchy sequence in L, where the representatives $\{x_\alpha^n\} \varepsilon L_o$ are chosen arbitrarily. Since

$$||\{x_\alpha^n\} + N - \{x_\alpha^m\}+N|| = ||\{x_\alpha^n\} - \{x_\alpha^m\}||$$

the sequence $<\{x_\alpha^n\}>_n$ is Cauchy in L_o. We will show that $<\{x_\alpha^n\}>_n$ converges in L_o.

For every $p\varepsilon\omega$, the positive integers, there is an integer N_p such that for $n,m \ge N_p$,

$$||\{x_\alpha^n\} - \{x_\alpha^m\}|| < \frac{1}{2^p} .$$

Without loss of generality we may assume the integers N_p are chosen strictly increasing. Furthermore, if $n,m > N_p$ and $\varepsilon < \frac{1}{2^p} - ||\{x_\alpha^n\}-\{x_\alpha^m\}||$ then $X = \{\alpha\varepsilon A \mid \,|\, ||x_\alpha^n - x_\alpha^m|| - ||\{x_\alpha^n\} - \{x_\alpha^m\}||\,| < \varepsilon\} \varepsilon \Omega$ implies $||x_\alpha^n - x_\alpha^m|| < \frac{1}{2^p}$ for all $\alpha\varepsilon X$ which means the set

$$A_p^{n,m} = \{\alpha \varepsilon A \mid ||x_\alpha^n - x_\alpha^m|| < \frac{1}{2^p}\} \varepsilon \Omega$$

Let $A_p = A_p^{N_p, N_{p+1}}$ for every $p \varepsilon \omega$ and let $B_p = \bigcap_{i=1}^{p} A_i$. Since each $A_p \varepsilon \Omega$ and B_p is a finite intersection, $B_p \varepsilon \Omega$ also. Furthermore, by construction, $B_p \subseteq B_{p-1}$ for all $p>1$.

Define an element $\{Z_\alpha\}$ of the product space as follows:

$$Z_\alpha = o \qquad \text{for } \alpha \varepsilon B_1^c$$

$$Z_\alpha = x_\alpha^{N_p} \qquad \text{for } \alpha \varepsilon B_{p-1} \setminus B_p$$

$$Z_\alpha = \lim_{p\to\infty} x_\alpha^{N_p} \qquad \text{for } \alpha \varepsilon \bigcap_{p=1}^{\infty} B_p$$

Let us first remark that this definition does in fact make sense since in the first place $B_p \subseteq B_{p-1}$ for all $p>1$ implies the second condition is meaningful and also that the three given conditions exhaust all choices for α. In the second place, for any $\alpha \varepsilon \bigcap_{p=1}^{\infty} B_p$ we have $\alpha \varepsilon A_p$ for all $p \varepsilon \omega$ so that $||x_\alpha^{N_p} - x_\alpha^{N_{p+1}}|| < \frac{1}{2^p}$ for all $p \varepsilon \omega$. This implies the sequence $\langle x_\alpha^{N_p} \rangle p$ is Cauchy in L_α since for any $\varepsilon > o$ we may choose p so that $\sum_{t=p}^{\infty} \frac{1}{2^t} < \varepsilon$. Then for $u, v > p$, and without loss of generality with $u>v$, we have

$$||x_\alpha^{N_u}-x_\alpha^{N_v}|| \le ||x_\alpha^{N_u}-x_\alpha^{N_{u-1}}|| + ||x_\alpha^{N_{u-1}}-x_\alpha^{N_{u-2}}||$$

$$+ \dots + ||x_\alpha^{N_{v+1}}-x_\alpha^{N_v}||$$

$$\le \sum_{t=v}^{u-1} \frac{1}{2^t} < \varepsilon$$

Now by hypothesis each L_α is complete so the Cauchy sequence $< x_\alpha^{N_p}>p$ converges to a limit which we will denote by Z_α.

Having shown that the definition of $\{Z_\alpha\}$ makes good sense we will next show that $\{Z_\alpha\}\varepsilon\ L_o$. To do this we must show that there is a number M such that

$$||Z_\alpha|| \le M \text{ for all } \alpha\varepsilon A.$$

Since $\{x_\alpha^{N_1}\}\varepsilon L_o$ we know there exists a number M_1 such that $||x_\alpha^{N_1}||\le M_1$ for all $\alpha\varepsilon A$. Choose any $M>M_1+1$. For $\alpha\varepsilon B_1^c$, $Z_\alpha = o$ so $||z_\alpha|| = o<M$. Suppose $\alpha\varepsilon B_{p-1}\setminus B_p$ for some p. Then $z_\alpha = x_\alpha^{N_p}$ so $||z_\alpha|| = ||x_\alpha^{N_p}||\le||x_\alpha^{N_p}-x_\alpha^{N_1}||+||x_\alpha^{N_1}||$. But $\alpha\varepsilon B_{p-1}$ implies $\alpha\varepsilon B_s$ for all $s\le p-1$ so we have

$$||x_\alpha^{N_p}-x_\alpha^{N_1}||\le||x_\alpha^{N_p}-x_\alpha^{N_{p-1}}||+||x_\alpha^{N_{p-1}}-x_\alpha^{N_{p-2}}||$$

$$+ \dots + ||x_\alpha^{N_2}-x_\alpha^{N_1}||$$

$$\le \sum_{t=1}^{p-1} \frac{1}{2^t} < 1$$

and thus $||z_\alpha||\le 1+||x_\alpha^{N_1}||\le 1+M_1<M$. Finally, suppose

$\alpha \varepsilon \bigcap_{p=1}^{\infty} B_p$. Then $z_\alpha = \lim_{p\to\infty} x_\alpha^{N_p}$. Choose $\varepsilon < M-(M_1+1)$ and choose p so that $||z_\alpha - x_\alpha^{N_p}|| < \varepsilon$. Since $\alpha \varepsilon \bigcap_{p=1}^{\infty} B_p$ implies that $\alpha \varepsilon B_{p-1}$ we can use the same construction as in the preceeding case to get $||x_\alpha^{N_p}|| \leq M_1+1$ so we have

$$||z_\alpha|| \leq ||z_\alpha - x_\alpha^{N_p}|| + ||x_\alpha^{N_p}||$$

$$\leq \varepsilon + (M_1+1) < M.$$

Thus $||z_\alpha|| < M$ for all $\alpha \varepsilon A$ so $\{z_\alpha\} \varepsilon L_o$.

We claim now, of course, that $<\{x_\alpha^n\}>_n$ converges to $\{z_\alpha\}$ in L_o. Since L_o is a semi-normed space it suffices to show that some subsequence converges to $\{z_\alpha\}$. We shall show that the subsequence $<\{x_\alpha^{N_p}\}>_p$ converges to $\{z_\alpha\}$ in L_o. This means we must show that for every $\varepsilon > o$ there is a $p \varepsilon \omega$ such that for $s \geq p$, $||\{x_\alpha^{N_s}\} - \{z_\alpha\}|| < \varepsilon$. By corollary 1.14 it suffices to show that for every $\varepsilon > o$ there is a $p \varepsilon \omega$ such that for each $s \geq p$ there is a set $C_s \varepsilon \Omega$ such that $||x_\alpha^{N_s} - z_\alpha|| < \varepsilon$ for all $\alpha \varepsilon C_s$.

Let $\varepsilon > o$ and choose p so that $\sum_{t=p}^{\infty} \frac{1}{2^t} < \frac{\varepsilon}{2}$. Let $s \geq p$ and let $C_s = B_s$. Let $\alpha \varepsilon C_s$. Then either $\alpha \varepsilon B_{q-1} \setminus B_q$ for some $q > s$ or $\alpha \varepsilon \bigcap_{p=1}^{\infty} B_p$. Suppose $\alpha \varepsilon B_{q-1} \setminus B_q$ for some $q > s$. Then $z_\alpha = x_\alpha^{N_q}$. Now $\alpha \varepsilon B_{q-1}$ implies $\alpha \varepsilon B_r$ for $s \leq r \leq q-1$ so we have

$$||z_\alpha - x_\alpha^{N_s}|| = ||x_\alpha^{N_q} - x_\alpha^{N_s}||$$

$$\leq ||x_\alpha^{N_q} - x_\alpha^{N_{q-1}}|| + ||x_\alpha^{N_{q-1}} - x_\alpha^{N_{q-2}}||$$

$$+ \ldots + ||x_\alpha^{N_{s+1}} - x_\alpha^{N_s}||$$

$$\leq \sum_{t=s}^{q-1} \frac{1}{2^t} < \sum_{t=p}^{\infty} \frac{1}{2^t} < \frac{\varepsilon}{2} < \varepsilon$$

On the other hand, if $\alpha \varepsilon \bigcap_{p=1}^{\infty} B_p$, then $z_\alpha = \lim_{p\to\infty} x_\alpha^{N_p}$ so for some integer $q \geq s$

$$||\, z_\alpha - x_\alpha^{N_q}|| < \frac{\varepsilon}{2} \, .$$

By the same construction as in the preceding case we have $||x_\alpha^{N_q} - x_\alpha^{N_s}|| < \frac{\varepsilon}{2}$ so we get $||z_\alpha - x_\alpha^{N_s}|| < \varepsilon$. Thus, $||z_\alpha - x_\alpha^{N_s}|| < \varepsilon$ for all $\alpha \varepsilon C_s$ so $||\{z_\alpha\} - \{x_\alpha^{N_s}\}|| < \varepsilon$ for all $s \geq p$ and hence $\{z_\alpha\} = \lim_{p\to\infty} \{x_\alpha^{N_p}\}$.

To complete the proof we need only notice that

$$||\{x_\alpha^n\} - \{z_\alpha\}||_{L_o} = ||\{x_\alpha^n\} + N - \{z_\alpha\} + N||_L$$

so $\langle \{x_\alpha^n\} + N \rangle_n$ converges to $\{z_\alpha\} + N$ in L. This L is complete and therefore a Banach space. Q.E.D.

Things would be particularly nice if there were some relationship between convergence of a sequence in an ultra-product semi-normed space L_o and coordinatewise convergence, that is, convergence in each coordinate. However, as the next three examples show there is no relationship between them even in the simplest of cases.

Let Ω be a free ultrafilter on the set ω of positive integers and let L be the ultraproduct space formed by taking a countable number of copies of the real numbers R.

<u>Example 2.4</u> A sequence which converges coordinatewise but does not converge in L_o.

Let $\{x_i\}\varepsilon L_o$ be the element with $x_i=1$ for all $i\varepsilon\omega$. Define a sequence $<\{y_i^n\}>_n$ in L_o by

$$y_i^n = 1 \quad \text{for } i\leq n$$
$$y_i^n = n \quad \text{for } i>n$$

Then for each $i\varepsilon\omega$, $n\geq i$ implies $y_i^n=1$ so the sequence $<\{y_i^n\}>_n$ converges coordinatewise to $\{x_i\}$. But for any pair of integers m,n, since $||\{y_i^n\} - \{y_i^m\}|| = |n-m|$ the sequence $<\{y_i^n\}>_n$ is not even Cauchy in L_o and hence does not converge.

<u>Example 2.5</u> A sequence which converges in L_o but does not converge coordinatewise.

Let $\{x_i\}$ be $x_i=o$ for all $i\varepsilon\omega$.

Define $<\{y_i^n\}>_n$ by

$$y_i^n = 1+(-1)^n \quad \text{for } i\leq n$$

$$y_i^n = \frac{1}{n} \quad \text{for } i>n.$$

If $\varepsilon>o$ choose n so that $\frac{1}{n}<\varepsilon$. Since $\{i\varepsilon\omega|\, i>n\}\varepsilon\Omega$ we have the set

$$\{i\varepsilon\omega|\ |y_i^n-x_i| = \frac{1}{n} <\varepsilon\}\varepsilon\Omega$$

so $<\{y_i^n\}>_n$ converges to $\{x_i\}$ in L_o. On the other hand, for any $i\epsilon\omega$ and $n>i$ y_i^n alternates between 0 and 2 so does not converge.

Example 2.6 A sequence which converges both coordinatewise and in L_o but to different limits.

Let $\{x_i\}$ be $x_i=1$ for all $i\epsilon\omega$ and $\{z_i\}$ be $z_i=o$ for all $i\epsilon\omega$. Define

$<\{y_i^n\}>_n$ by $y_i^n=1$ for $i\leq n$

$y_i^n=\frac{1}{n}$ for $i>n$

As the preceding two examples show, $<\{y_i^n\}>_n$ converges coordinatewise to $\{x_i\}$ and in L_o to $\{z_i\}$. But clearly $\{x_i\}\neq\{z_i\}$, even modulo N.

Let A be an arbitrary index set, Ω an ultrafilter on A, $X\epsilon\Omega$ and L the ultraproduct space formed by taking for all L_α the same normed linear space for all $\alpha\epsilon X$ and arbitrary L_α for $\alpha\epsilon A\backslash X$. Consider the collection D_o of elements of L_o of the form $\{x_\alpha\}$ where $x_\alpha=x_\beta$ for all $\alpha,\beta\epsilon X$ and $x_\alpha=o$ for all $\alpha\epsilon A\backslash X$. D_o is a linear subspace of L_o. Furthermore, since $||\{x_\alpha\}|| = \lim_\Omega ||x_\alpha|| = ||x_\alpha||$ where $\alpha\epsilon X$ we see the semi-norm on L_o is a norm when restricted to D_o. This means the quotient map of $L_o\to L_o/N$ is a one-to-one linear isometry of D_o into L. Let D be the image of D_o in L. Then D is isometrically isomorphic to D_o.

Definition 2.7 We call the subspace D of L constructed above the subspace of diagonal elements.

<u>Proposition 2.8</u> The subspace D of diagonal elements is isometrically isomorphic to L_α.

Proof: Since we have already observed that D is isometrically isomorphic to D_o we need only show D_o is isometrically isomorphic to L_α. Let $T: L_\alpha \to L_o$ be defined by $T(x) = \{x_\alpha\}$ where $x_\alpha = x$ for all $\alpha \varepsilon X$ and $x_\alpha = o$ for all $\alpha \varepsilon A \setminus X$. Then T is clearly linear and

$$||T(x)|| = ||\{x_\alpha\}|| = \lim_\Omega ||x_\alpha|| = ||x_\alpha|| = ||x||$$

so T is an isometry. Q.E.D.

Notice that although D is defined relative to a set $X \varepsilon \Omega$ proposition 2.8 shows that different choices of X with the same L_α give isometrically isomorphic spaces D.

<u>Example 2.9</u> Let A be an index set, Ω an ultrafilter on A and L_α a real inner product space for each $\alpha \varepsilon A$. The ultraproduct space L is also a real inner product space.

Denote the inner product in L_α by $\langle x_\alpha, y_\alpha \rangle$. We shall assume that in each L_α the inner product is positive and non-degenerate, that is, it satisfies

$$\langle x_\alpha + y_\alpha, z_\alpha \rangle = \langle x_\alpha, z_\alpha \rangle + \langle y_\alpha, z_\alpha \rangle$$

$$\langle x_\alpha, y_\alpha + z_\alpha \rangle = \langle x_\alpha, y_\alpha \rangle + \langle x_\alpha, z_\alpha \rangle$$

$$\langle \lambda x_\alpha, y_\alpha \rangle = \lambda \langle x_\alpha, y_\alpha \rangle \text{ , } \lambda \text{ a real number}$$

$$\langle x_\alpha, \lambda y_\alpha \rangle = \lambda \langle x_\alpha, y_\alpha \rangle$$

$$\langle x_\alpha, y_\alpha \rangle = \langle y_\alpha, x_\alpha \rangle$$

$$\langle x_\alpha, x_\alpha \rangle \geq o \quad \text{for all } x_\alpha \varepsilon L_\alpha$$

$$\langle x_\alpha, x_\alpha \rangle = o \quad \text{implies } x_\alpha = o.$$

This inner product defines a norm in L_α by

$$||x_\alpha||^2 = \langle x_\alpha, x_\alpha \rangle$$

and in addition, the Cauchy-Schwartz inequality

$$|\langle x_\alpha, y_\alpha \rangle| \leq ||x_\alpha|| \cdot ||y_\alpha||$$

is consequently satisfied.

Define an inner product in L_o by

$$\langle \{x_\alpha\}, \{y_\alpha\} \rangle = \lim_\Omega \langle x_\alpha, y_\alpha \rangle.$$

We shall show that this is in fact a positive inner product and the semi-norm it generates is the same as the usual semi-norm of L_o. We will then use this inner product to define a positive, non-degenerate inner product in L in a natural way.

To begin with we notice that since $\{x_\alpha\}, \{y_\alpha\} \varepsilon L_o$ there exist constants M,M' such that $||x_\alpha|| \leq M$ and $||y_\alpha|| \leq M'$ for all $\alpha \varepsilon A$. By the Cauchy-Schwartz inequality in L_α

$$|\langle x_\alpha, y_\alpha \rangle| \leq ||x_\alpha|| \cdot ||y_\alpha|| \leq M \cdot M'$$

so the numbers $\{\langle x_\alpha, y_\alpha \rangle\}_{\alpha \varepsilon A}$ lie in the compact set $[-MM', MM']$ which means the limit $\lim_\Omega \langle x_\alpha, y_\alpha \rangle$ exists. Next notice that from the properties of limits we immediately get the bilinearity and homogeneity since, for example,

$$\begin{aligned}
\langle \{x_\alpha\} + \{y_\alpha\}, \{z_\alpha\} \rangle &= \lim_\Omega \langle x_\alpha + y_\alpha, z_\alpha \rangle \\
&= \lim_\Omega (\langle x_\alpha, z_\alpha \rangle + \langle y_\alpha, z_\alpha \rangle) \\
&= \lim_\Omega \langle x_\alpha, z_\alpha \rangle + \lim_\Omega \langle y_\alpha, z_\alpha \rangle \\
&= \langle \{x_\alpha\}, \{z_\alpha\} \rangle + \langle \{y_\alpha\}, \{z_\alpha\} \rangle
\end{aligned}$$

and also

$$\langle\lambda\{x_\alpha\},\{y_\alpha\}\rangle = \lim_\Omega \langle\lambda x_\alpha, y_\alpha\rangle$$
$$= \lim_\Omega \lambda\langle x_\alpha, y_\alpha\rangle$$
$$= \lambda \lim_\Omega \langle x_\alpha, y_\alpha\rangle$$
$$= \lambda\langle\{x_\alpha\},\{y_\alpha\}\rangle$$

Since $\langle x_\alpha, y_\alpha\rangle = \langle y_\alpha, x_\alpha\rangle$ for all $\alpha\varepsilon A$ we have $\langle\{x_\alpha\},\{y_\alpha\}\rangle = \langle\{y_\alpha\},\{x_\alpha\}\rangle$. Furthermore, since each $\langle x_\alpha, x_\alpha\rangle \geq o$ we have $\langle\{x_\alpha\},\{x_\alpha\}\rangle = \lim_\Omega \langle x_\alpha, x_\alpha\rangle \geq o$ also by proposition 1.13. Thus we have indeed defined a positive inner product on L_o. Since

$$\langle\{x_\alpha\},\{x_\alpha\}\rangle = \lim_\Omega \langle x_\alpha, x_\alpha\rangle$$
$$= \lim_\Omega ||x_\alpha||^2$$
$$= (\lim_\Omega ||x_\alpha||)^2$$
$$= (||\{x_\alpha\}||)^2$$

we see that the semi-norm defined by the inner product is the same as the usual norm in L_o.

Since N consists precisely of those elements $\{x_\alpha\}$ with $(||\{x_\alpha\}||)^2 = \langle\{x_\alpha\},\{x_\alpha\}\rangle = o$ we may define an inner product in L by

$$\langle\{x_\alpha\}+N,\{y_\alpha\}+N\rangle = \langle\{x_\alpha\},\{y_\alpha\}\rangle.$$

Provided this inner product is well-defined it will be positive and non-degenerate making L a real inner product space. To show this operation is well-defined in L suppose $\{x_\alpha\},\{x'_\alpha\},\{y_\alpha\},\{y'_\alpha\}\varepsilon L_o$ with

$$||\{x_\alpha\}-\{x'_\alpha\}|| = o \text{ and } ||\{y_\alpha\}-\{y'_\alpha\}|| = o.$$

Then $<\{x_\alpha\},\{y_\alpha\}> - <\{x'_\alpha\},\{y'_\alpha\}>$

$$= <\{x_\alpha\},\{y_\alpha\}> - <\{x'_\alpha\},\{y_\alpha\}> + <\{x'_\alpha\},\{y_\alpha\}> - <\{x'_\alpha\},\{y'_\alpha\}>$$

$$= <\{x_\alpha\}-\{x'_\alpha\},\{y_\alpha\}> + <\{x'_\alpha\},\{y_\alpha\}-\{y'_\alpha\}>$$

But by the Cauchy-Schwartz inequality

$$|<\{x_\alpha\}-\{x'_\alpha\},\{y_\alpha\}>| \leq ||\{x_\alpha\}-\{x'_\alpha\}|| \cdot ||\{y_\alpha\}|| = o$$

and $|<\{x'_\alpha\},\{y_\alpha\}-\{y'_\alpha\}>| \leq ||\{x'_\alpha\}|| \cdot ||\{y_\alpha\}-\{y'_\alpha\}|| = o$

so $<\{x_\alpha\},\{y_\alpha\}> = <\{x'_\alpha\},\{y'_\alpha\}>$.

Notice that this result together with theorem 2.3 yield the following corollary.

Corollary 2.10 An ultraproduct of Hilbert spaces is a Hilbert space.

Example 2.11 Let A be an index set, Ω an ultrafilter on A, $X\varepsilon\Omega$ and for each $\alpha\varepsilon X$ let L_α be R^n and for each $\alpha\varepsilon A\setminus X$ let L_α be any normed linear space. The resulting ultraproduct space L is isometrically isomorphic to R^n.

We shall show that every element of L belongs to the closure of the subspace D of diagonal elements. As D is closed and isometrically isomorphic to R^n we will have the desired result.

Define for each i=1, 2, ..., n the element of L_o $\{e^i_\alpha\}$ by

$$e^i_\alpha = (\underbrace{\underbrace{o, \ldots, o, 1}_{i}, o, \ldots o}_{n}) \quad \text{if } \alpha\varepsilon X$$

$$e^i_\alpha = o \quad \text{if } \alpha\varepsilon A\setminus X.$$

Then for each i, $\{e^i_\alpha\} \varepsilon D_o$. Let $\{z_\alpha\}$ be an arbitrary element of L_o and choose $\varepsilon > o$. There is a constant M such that $||z_\alpha|| \leq M$ for all $\alpha \varepsilon A$. In particular, for $\alpha \varepsilon X$ we can denote z_α by $z_\alpha = (a^1_\alpha, a^2_\alpha, \ldots, a^n_\alpha)$ which means that $|a^i_\alpha| \leq M$ for $i=1, \ldots, n$ and for all $\alpha \varepsilon X$. Choose a $\delta < \frac{\varepsilon}{\sqrt{n}}$ and let p be an integer such that $p > \frac{2M}{\delta}$. For each $k = o, 1, \ldots, p$ and each $i = 1, 2, \ldots, n$ define sets

$$A^i_k = \{\alpha \varepsilon X \mid -M+k\delta \leq a^i_\alpha < -M+(k+1)\delta\}.$$

Then for each $i = 1, \ldots, n$ the collection $\{A^i_k\}^p_{k=o}$ is a collection of disjoint subsets of X such that

$$X = \bigcup_{k=o}^{p} A^1_k = \bigcup_{k=o}^{p} A^2_k = \ldots = \bigcup_{k=o}^{p} A^n_k .$$

For each $k=o, 1, \ldots, p$ and each $i=1, \ldots, n$ define elements of L_o, $\{x^{k,i}_\alpha\}$ by

$$\begin{aligned} x^{k,i}_\alpha &= (o, \ldots, o) \text{ if } \alpha \varepsilon X \backslash A^i_k \\ &= (\underbrace{o, \ldots, o, 1}_{i}, o, \ldots o) \text{ if } \alpha \varepsilon A^i_k \\ &= o \qquad \text{if } \alpha \varepsilon A \backslash X. \end{aligned}$$

Now for each $i = 1, \ldots, n$, precisely one of the sets $\{A^i_k\}^p_{k=o}$ belongs to Ω. If $A^i_k \varepsilon \Omega$ then $\{x^{k,i}_\alpha\} = \{e^i_\alpha\}$ (mod N) and if $A^i_k \not\varepsilon \Omega$ then $X \backslash A^i_k \varepsilon \Omega$ so that $\{x^{k,i}_\alpha\} = \{o\}$ (mod N). For each $k = o, 1, \ldots, p$ define a scalar by $t_k = -M+k\delta$.

Consider the element of L_o given by

$$\{z_\alpha\} - [\sum_{i=1}^{n} \sum_{k=o}^{p} t_k \{x_\alpha^{k,i}\}]$$

For each i = 1, 2, ..., n, given any coordinate $\alpha \varepsilon X$ there is precisely one index k_i such that $\alpha \varepsilon A_{k_i}^{i}$. We thus have for any $\alpha \varepsilon X$

$$z_\alpha - [\sum_{i=1}^{n} \sum_{k=o}^{p} t_k x_\alpha^{k,i}] = (a_\alpha^1, \ldots, a_\alpha^n) - \sum_{i=1}^{n} \sum_{k=o}^{p} t_k x_\alpha^{k,i}$$

$$= (a_\alpha^1, \ldots, a_\alpha^n) - \sum_{i=1}^{n} t_{k_i} x_\alpha^{k_i,i}$$

$$= (a_\alpha^1 - (-M+k_1\delta), \ldots, a_\alpha^n - (-M+k_n\delta))$$

But for each i, $\alpha \varepsilon A_{k_i}^{i}$ implies $o \leq a_\alpha^i - (-M+k_i\delta) < \delta$ so we must have for any $\alpha \varepsilon X$

$$||z_\alpha - [\sum_{i=1}^{n} \sum_{k=o}^{p} t_k x_\alpha^{k,i}]||$$

$$= ||(a_\alpha^1 - (-M+k_1\delta), \ldots, a_\alpha^n - (-M+k_n\delta))||$$

$$< ||(\delta, \ldots, \delta)||$$

$$= \sqrt{n}\, \delta < \varepsilon$$

Since this inequality holds for all $\alpha \varepsilon X \varepsilon \Omega$ we have

$$||\{z_\alpha\} - [\sum_{i=1}^{n} \sum_{k=o}^{p} t_k \{x_\alpha^{k,i}\}]|| \leq \varepsilon$$

But now we notice that

$$[\sum_{i=1}^{n}\sum_{k=o}^{p} t_k\{x_\alpha^{k,i}\}]+N = \sum_{i=1}^{n}(\sum_{k=o}^{p} t_k\{x_\alpha^{k,i}\}+N)$$

$$= \sum_{i=1}^{n} b_i(\{e_\alpha^i\}+N)$$

where for each $i = 1, \ldots, n$, b_i is the sum of those t_k for k such that $\{x_\alpha^{k,i}\} \neq o \pmod N$. Therefore, since

$$||\{z_\alpha\} - \sum_{i=1}^{n}\sum_{k=o}^{p} t_k\{x_\alpha^{k,i}\}||$$

$$= ||\{z_\alpha\}+N-[\sum_{i=1}^{n}\sum_{k=o}^{p} t_k\{x_\alpha^{k,i}\}]+N||$$

we have shown that $\{z_\alpha\}+N$ lies in the closed linear span of the $\langle\{e_\alpha^i\}+N\rangle_{i\varepsilon\omega}$. Since all of the $\{e_\alpha^i\}+N$ lie in D we are done.

<u>Example 2.12</u> Let A be an index set, Ω an ultrafilter on A and $X\varepsilon\Omega$. For each $\alpha\varepsilon X$ let L_α be an n-dimensional Banach space and for each $\alpha\varepsilon A\setminus X$ let L_α be any normed linear space. The resulting ultraproduct space L is isomorphic to R^n. Moreover, $d(L,R^n) \leq n^2$.

We will first sketch a short proof of the known result that if E and F are n-dimensional Banach spaces, then $d(E,F) \leq n^2$. Since E and F are finite dimensional we may choose normal bases (e_i) and (f_i) in E and F respectively with coefficient functionals (e_i^*) and (f_i^*). Define a linear bijection $T: E \to F$ by

$$T\left(\sum_{i=1}^{n} \alpha_i e_i\right) = \sum_{i=1}^{n} \alpha_i f_i \quad \left(\sum_{i=1}^{n} \alpha_i e_i \varepsilon E\right)$$

Then for each element $\sum_{i=1}^{n} \alpha_i e_i \varepsilon E$ and any k, $1 \leq k \leq n$, we have

$$|\alpha_k| = ||\alpha_k e_k|| = ||e_k^*\left(\sum_{i=1}^{n} \alpha_i e_i\right)||$$

$$\leq ||e_k^*|| \; ||\sum_{i=1}^{n} \alpha_i e_i||$$

$$= ||\sum_{i=1}^{n} \alpha_i e_i||$$

which yields

$$||T\left(\sum_{i=1}^{n} \alpha_i e_i\right)|| = ||\sum_{i=1}^{n} \alpha_i f_i|| \leq \sum_{i=1}^{n} ||\alpha_i f_i||$$

$$= \sum_{i=1}^{n} |\alpha_i| \leq n||\sum_{i=1}^{n} \alpha_i e_i||$$

so we have $||T|| \leq n$. A similar argument shows that $||T^{-1}|| \leq n$ also, so we have $d(E,F) \leq n^2$. Now since L_α for $\alpha \varepsilon X$ is an n-dimensional Banach space this implies that $d(L_\alpha, R^n) \leq n^2$.

To complete our proof, denote by L' the ultraproduct space we get by taking for each $\alpha \varepsilon A \setminus X$ the same Banach space L_α as in L and for each $\alpha \varepsilon X$ taking $L_\alpha = R^n$. By example 2.11 L' is isometrically isomorphic to R^n. Then for each $\alpha \varepsilon X$, $d(L_\alpha, L'_\alpha) \leq n^2$ and for each $\alpha \varepsilon A \setminus X$, $d(L_\alpha, L'_\alpha) = 1$ so by proposition 2.2, $d(L, L') \leq n^2$. Since $d(L', R^n) = 1$ we have the desired result.

Theorem 2.13 Let A be an index set and Ω an ultrafilter on A having the countable intersection property. For each $\alpha\varepsilon A$ let L_α be an Banach space (either finite or infinite dimensional). The ultraproduct space L is finite dimensional if and only if for some integer n the set $\{\alpha\varepsilon A | \dim L_\alpha = n\}$ belongs to Ω. Otherwise L is non-separable.

Proof: If for some n, $\{\alpha\varepsilon A| \dim L_\alpha = n\}\varepsilon\Omega$ then the preceding example 2.12 shows that L is isomorphic to R^n so $\dim L = n$. To show the converse, suppose that for no integer n does the set $\{\alpha\varepsilon A| \dim L_\alpha = n\}$ belong to Ω. Since Ω has the countable intersection property we can find a sequence $\{D_i\}_{i=1}^{\infty}$ of elements of Ω such that $\bigcap_{i=1}^{\infty} D_i = \emptyset$. Furthermore, we may assume that $D_{i+1}\subseteq D_i$ for all $i\varepsilon\omega$. Since $A\varepsilon\Omega$ we can write $A = A_1 \cup A_2$ where $A_1 = \{\alpha\varepsilon A| \dim L_\alpha = \infty\}$ and $A_2 = \{\alpha\varepsilon A| \dim L_\alpha\} < \infty$ and have either $A_1\varepsilon\Omega$ or $A_2\varepsilon\Omega$.

Suppose $A_1\varepsilon\Omega$. Define for each $i\varepsilon\omega$ the set $X_i = D_i \cap A_1$. Then $X_i\varepsilon\Omega$, $X_{i+1} \subseteq X_i$ for all $i\varepsilon\omega$ and $\bigcap_{i=1}^{\infty} X_i = \emptyset$. Furthermore, for each $\alpha\varepsilon X_i \setminus X_{i+1}$, $\dim L_\alpha = \infty$. On the other hand, suppose $A_2\varepsilon\Omega$. For each $n\varepsilon\omega$ write

$$B_n = \{\alpha\varepsilon A| \dim L_\alpha = n\}.$$

By hypothesis, no B_n belongs to Ω. Furthermore, for each $M\varepsilon\omega$, $\bigcup_{i=1}^{M} B_i \notin\Omega$ either for if $\bigcup_{i=1}^{M} B_i\varepsilon\Omega$ then since this is a finite disjoint union we must have $B_i\varepsilon\Omega$ for some i, $1 \leq i \leq M$ contrary to hypothesis. Then since $\bigcup_{i=1}^{M} B_i\notin\Omega$

we must have $\bigcup_{i=M}^{\infty} B_i \varepsilon \Omega$ for all $M \varepsilon \omega$. For each $n \varepsilon \omega$ we define the set

$$X_n = D_n \cap \bigcup_{i=n+1}^{\infty} B_i .$$

Then $X_n \varepsilon \Omega$, $X_{n+1} \subseteq X_n$ for all $n \varepsilon \omega$ and $\bigcap_{n=1}^{\infty} X_n = \emptyset$. Furthermore, for each $\alpha \varepsilon X_n \setminus X_{n+1}$, $\dim L_\alpha \geq n+1$.

For each $\alpha \varepsilon A$ let $\langle x_\alpha^n \rangle_n$ be a normal basic sequence in L_α (we understand that if L_α is finite dimensional this sequence will have only a finite number of terms). Then $||x_\alpha^n|| = 1$ for all n and for any m,n with $m \neq n$

$$||x_\alpha^n - x_\alpha^m|| \geq 1.$$

Form in L_o the set S of all elements of the form $\{z_\alpha\}$ where for each $\alpha \varepsilon A$, $z_\alpha \varepsilon L_\alpha$ is one of the x_α^n for some n. Consider two such elements $\{z_\alpha\} = \{x_\alpha^{n_\alpha}\}$ and $\{z'_\alpha\} = \{x_\alpha^{m_\alpha}\}$. Then $\{z_\alpha - z'_\alpha\} = \{x_\alpha^{n_\alpha} - x_\alpha^{m_\alpha}\}$. Consider the set $C = \{\alpha \varepsilon A | x_\alpha^{n_\alpha} - x_\alpha^{m_\alpha} = o\}$. Either $C \varepsilon \Omega$ or $C^c \varepsilon \Omega$. If $C \varepsilon \Omega$ then

$||\{z_\alpha - z'_\alpha\}|| = o$. If $C^c \varepsilon \Omega$ then $||x_\alpha^{n_\alpha} - x_\alpha^{m_\alpha}|| \geq 1$ for all $\alpha \varepsilon C^c$ so $||\{z_\alpha - z'_\alpha\}|| \geq 1$. Thus $C^c \varepsilon \Omega$ implies $\{z_\alpha\} \neq \{z'_\alpha\}$ (mod N). Let Λ denote the collection of all those subsets F of S with the property that for any pair of elements $\{z_\alpha\}$, $\{z'_\alpha\} \varepsilon F$ we have $\{z_\alpha\} \neq \{z'_\alpha\}$ (mod N). Partially order Λ by inclusion and let Γ be any chain in Λ. Suppose $\{z_\alpha\}$, $\{z'_\alpha\} \varepsilon \bigcup_{F \varepsilon \Gamma} F$. Since Γ is a chain there is an $F \varepsilon \Gamma$ such that $\{z_\alpha\}$, $\{z'_\alpha\} \varepsilon F$. This then implies that $\{z_\alpha\} \neq \{z'_\alpha\}$

(mod N). Therefore $\bigcup_{F\varepsilon\Gamma} F$ is an upper bound for Γ in Λ so by Zorn's lemma Λ has a maximal element.

Let F be a maximal element of Λ and suppose F is countable. Enumerate the elements of F by $\{z_\alpha^1\}$, $\{z_\alpha^2\}$, ... where each $\{z_\alpha^i\} = \{x_\alpha^{k(\alpha,i)}\}$. Construct an element $\{z_\alpha\}\varepsilon L_{\mathcal{Q}}$ as follows:

For each $\alpha\not\varepsilon \bigcup_{n=1}^{\infty} X_n$, $z_\alpha = o$. If $\alpha\varepsilon \bigcup_{n=1}^{\infty} X_n$ then since $\bigcap_{n=1}^{\infty} X_n = \emptyset$ there is an n such that $\alpha\varepsilon X_n \setminus X_{n+1}$. Let $z_\alpha = x_\alpha^k$ where x_α^k is chosen so that $x_\alpha^k \neq x_\alpha^{k(\alpha,i)}$ for any $i \leq n$. We may do this since dim $L_\alpha > n$. Then for $\alpha\varepsilon X_n\setminus X_{n+1}$ we have $z_\alpha \neq z_\alpha^i$ for any $i \leq n$. Choose any $m\varepsilon\omega$ and consider $\{z_\alpha - z_\alpha^m\}$. By construction, for any $\alpha\varepsilon \bigcup_{i=m}^{\infty} X_i$ we have $z_\alpha \neq z_\alpha^m$ which as we have already noticed means that $||z_\alpha - z_\alpha^m|| \geq 1$. But $\bigcup_{i=m}^{\infty} X_i \varepsilon\Omega$ so this implies that

$$||\{z_\alpha - z_\alpha^m\}|| \geq 1$$

and hence $\{z_\alpha\} \neq \{z_\alpha^m\}$ (mod N) for any $m\varepsilon\omega$ contradicting the maximality of F. Therefore, F is not countable.

Since F is uncountable there are an uncountable number of elements of the unit sphere of L which are uniformly far apart. This means L is not separable. Q.E.D.

Having examined the structure of ultraproduct spaces it is natural to turn to the question of what the dual of an ultraproduct space is. It appears to be difficult to answer this question completely but some partial results can be given.

Let L be an arbitrary ultraproduct space with index set A and ultrafilter Ω. For each $\alpha \varepsilon A$ let $\hat{L}_\alpha = L^*_\alpha$ and form the ultraproduct space $\hat{L}$ using the $\hat{L}_\alpha$.

<u>Theorem 2.14</u> $\hat{L}$ is a linear subspace of L^*.

Proof: Denote the semi-normed space associated with $\hat{L}$ by $\hat{L}_o$. Define an operation on $L_o \times \hat{L}_o$ by

$$<\{x_\alpha\}, \{x^*_\alpha\}> = \lim_\Omega <x_\alpha, x^*_\alpha>$$

The properties of limits immediately show that this is a bilinear operation on $L_o \times \hat{L}_o$ and hence $\{x^*_\alpha\} \varepsilon \hat{L}_o$ acts as a linear functional on L_o. We need to show this functional is continuous.

Since $|<x_\alpha, x^*_\alpha> - \lim_\Omega <x_\alpha, x^*_\alpha>| < \varepsilon$ implies

$$\big|\, |<x_\alpha, x^*_\alpha>| - |\lim_\Omega <x_\alpha, x^*_\alpha>| \,\big| < \varepsilon \text{ we have}$$

$$\lim_\Omega |<x_\alpha, x^*_\alpha>| = |\lim_\Omega <x_\alpha, x^*_\alpha>|$$

and thus

$$\begin{aligned} |<\{x_\alpha\}, \{x^*_\alpha\}>| &= |\lim_\Omega <x_\alpha, x^*_\alpha>| \\ &= \lim_\Omega |<x_\alpha, x^*_\alpha>| \\ &\leq \lim_\Omega ||x^*_\alpha|| \cdot ||x_\alpha|| \\ &= ||\{x^*_\alpha\}|| \cdot ||\{x_\alpha\}|| \end{aligned}$$

which shows that the linear functional $\{x^*_\alpha\}$ is continuous and further, that its norm in L^*_o is less than or equal to

$||\{x_\alpha^*\}||_{\hat{L}_o}$. On the other hand, given $\varepsilon > o$ we may find in each L_α an element x_α with $||x_\alpha|| = 1$ such that $|\ ||x_\alpha^*|| - \langle x_\alpha, x_\alpha^* \rangle| < \frac{\varepsilon}{3}$. For some set $B_1 \varepsilon \Omega$, $\alpha \varepsilon B_1$ implies

$$|\langle x_\alpha, x_\alpha^* \rangle - \langle \{x_\alpha\}, \{x_\alpha^*\} \rangle| < \frac{\varepsilon}{3}$$

and for some set $B_2 \varepsilon \Omega$, $\alpha \varepsilon B_2$ implies

$$|\ ||x_\alpha^*|| - ||\{x_\alpha^*\}||\ | < \frac{\varepsilon}{3}$$

so for $\alpha \varepsilon B_1 \cap B_2$

$$|\langle \{x_\alpha\}, \{x_\alpha^*\} \rangle - ||\{x_\alpha^*\}||\ | < \varepsilon$$

which means that since

$$||\{x_\alpha^*\}||_{L_o^*} = \sup_{||\{x_\alpha\}||=1} |\langle \{x_\alpha\}, \{x_\alpha^*\} \rangle|$$

the norm of $\{x_\alpha^*\}$ in L_o^* is at least as large as $||\{x_\alpha^*\}||_{\hat{L}_o}$. Hence $||\{x_\alpha^*\}||_{L_o^*} = ||\{x_\alpha^*\}||_{\hat{L}_o}$ so $\hat{L}_o$ is a subspace of L_o^*.

To complete the proof, define an operation on $L \times \hat{L}$ by $\langle \{x_\alpha\} + N, \{x_\alpha^*\} + N \rangle = \langle \{x_\alpha\}, \{x_\alpha^*\} \rangle$. To show this operation is well-defined suppose

$$||\{x_\alpha\} - \{\bar{x}_\alpha\}|| = o \text{ and } ||\{x_\alpha^*\} - \{\bar{x}_\alpha^*\}|| = o.$$

Then $|\langle \{x_\alpha\}, \{x_\alpha^*\} \rangle - \langle \{\bar{x}_\alpha\}, \{\bar{x}_\alpha^*\} \rangle|$

$$= |\langle \{x_\alpha\}, \{x_\alpha^*\} \rangle - \langle \{\bar{x}_\alpha\}, \{x_\alpha^*\} \rangle$$

$$+ \langle \{\bar{x}_\alpha\}, \{x_\alpha^*\} \rangle - \langle \{\bar{x}_\alpha\}, \{\bar{x}_\alpha^*\} \rangle|$$

$$= |<\{x_\alpha\}-\{\bar{x}_\alpha\},\{x^*_\alpha\}>+<\{\bar{x}_\alpha\},\{x^*_\alpha\}-\{\bar{x}^*_\alpha\}>|$$

$$\leq |<\{x_\alpha\}-\{\bar{x}_\alpha\},\{x^*_\alpha\}>|+|<\{\bar{x}_\alpha\},\{x^*_\alpha\}-\{\bar{x}^*_\alpha\}>|$$

$$\leq ||\{x_\alpha\}-\{\bar{x}_\alpha\}||\cdot||\{x^*_\alpha\}||+||\{\bar{x}_\alpha\}||\cdot||\{x^*_\alpha\}-\{\bar{x}^*_\alpha\}||$$

$$= o.$$

The bilinearity and continuity follow immediately from that of $<\cdot,\cdot>$ on $L_o \times \hat{L}_o$.

3. LATTICE PROPERTIES

In order to define a natural lattice structure on an ultraproduct space L we will see that not only do each of the spaces L_α have to be lattices but in fact they must also satisfy property (AB). We will therefore assume for the best of this chapter then each L_α is an (AB) lattice.

<u>Definition 3.1</u> For $\{x_\alpha\}$, $\{y_\alpha\}\varepsilon L_o$ define

$$\{x_\alpha\}v\{y_\alpha\}= \{x_\alpha v y_\alpha\} \text{ in } L_o$$

and

$$(\{ x_\alpha\}+N)v(\{y_\alpha\}+N) = (\{x_\alpha\}v\{y_\alpha\})+N \text{ in } L.$$

This operation is clearly a supremum in L_o which makes L_o a vector lattice since each L_α is a vector lattice. In order to show that 3.1 gives a well-defined operation in L we first observe that (Luxemburg and Zaanen [8], (p. 64)), we have $|(x_\alpha\wedge z_\alpha)-(y_\alpha\wedge z_\alpha)|+|(x_\alpha v z_\alpha)-(y_\alpha v z_\alpha)$

$$= |x_\alpha-y_\alpha|$$

so that $|(x_\alpha v z_\alpha)-(y_\alpha v z_\alpha)|\leq|x_\alpha-y_\alpha||$. Using the (AB) properties in L_α yields

$$||(x_\alpha v z_\alpha)-(y_\alpha v z_\alpha)||\leq||x_\alpha-y_\alpha||.$$

Now $\{x_\alpha\}v\{z_\alpha\}-\{y_\alpha\}v\{z_\alpha\} = \{x_\alpha v z_\alpha-y_\alpha v z_\alpha\}$

so $$||\{x_\alpha\}v\{z_\alpha\}-\{y_\alpha\}v\{z_\alpha\}|| = ||\{x_\alpha v z_\alpha-y_\alpha v z_\alpha\}||$$

$$\leq ||\{x_\alpha\}-\{y_\alpha\}||.$$

Thus, if $\{x_\alpha\} = \{y_\alpha\}$ (mod N), then for any $\{z_\alpha\}\varepsilon L_o$, $\{x_\alpha\}v\{z_\alpha\} = \{y_\alpha\}v\{z_\alpha\}$ (mod N). Now suppose $\{x_\alpha\} = \{\bar{x}_\alpha\}$ (mod N) and $\{y_\alpha\} = \{\bar{y}_\alpha\}$ (mod N) then

$\{\bar{x}_\alpha\}v\{\bar{y}_\alpha\} = \{x_\alpha\}v\{\bar{y}_\alpha\} = \{x_\alpha\}v\{y_\alpha\}$ (mod N) so the operation of 3.1 is well-defined and makes L a vector lattice.

<u>Proposition 3.2</u> L is an AB-lattice.

Proof: To begin with notice that

$$\begin{aligned} |\{x_\alpha\}| &= \{x_\alpha\}v\{o\}+ -\{x_\alpha\}v\{o\} \\ &= \{x_\alpha vo\}+\{-x_\alpha vo\} \\ &= \{x_\alpha vo+(-x_\alpha)vo\} \\ &= \{|x_\alpha|\} \end{aligned}$$

so that

$$\begin{aligned} ||\ |\{x_\alpha\}|\ || &= ||\{|x_\alpha|\}|| = \lim_\Omega||\ |x_\alpha|\ || \\ &= \lim_\Omega||x_\alpha|| \\ &= ||\{x_\alpha\}|| \end{aligned}$$

so L_o has property (A). Furthermore, if $\{x_\alpha\}\leq\{y_\alpha\}$ then

$$\{x_\alpha vy_\alpha\} = \{x_\alpha\}v\{y_\alpha\} = \{y_\alpha\}$$

so $x_\alpha vy_\alpha = y_\alpha$ for all $\alpha\varepsilon A$ which implies that $x_\alpha\leq y_\alpha$ for all $\alpha\varepsilon A$. Thus $\{o\}\leq\{x_\alpha\}\leq\{y_\alpha\}$ implies $o\leq x_\alpha\leq y_\alpha$ for all $\alpha\varepsilon A$ and hence $o\leq||x_\alpha||\leq||y_\alpha||$ for all $\alpha\varepsilon A$ so by proposition 1.13

$$o\leq||\{x_\alpha\}||\leq||\{y_\alpha\}||$$

and thus L_o has property (B) also.

Now since $|\{x_\alpha\}+N| = \{x_\alpha\}+N\vee N + -(\{x_\alpha\}+N)\vee N$

$$= (\{x_\alpha\}\vee o + -\{x_\alpha\}\vee o)+N$$

$$= |\{x_\alpha\}|+N$$

and L_o has property (A) then so does L. Furthermore, $\{x_\alpha\}+N\leq\{y_\alpha\}+N$ means that

$$\{y_\alpha\}+N = \{x_\alpha\}+N\vee\{y_\alpha\}+N = \{x_\alpha\}\vee\{y_\alpha\}+N$$

which implies that there exists a $\{z_\alpha\}\varepsilon N$ such that $\{y_\alpha\}+\{z_\alpha\} = \{x_\alpha\}\vee\{y_\alpha\}$ or $\{y_\alpha+z_\alpha\} = \{x_\alpha\vee y_\alpha\}$ so $y_\alpha+z_\alpha = x_\alpha\vee y_\alpha$ for all $\alpha\varepsilon A$. This implies that

$$x_\alpha\leq x_\alpha\vee y_\alpha = y_\alpha+z_\alpha \text{ for all } \alpha\varepsilon A,$$

which is to say we can find representatives

$$\{x_\alpha\}\varepsilon\{x_\alpha\}+N \text{ and } \{y'_\alpha\}\varepsilon\{y_\alpha\}+N$$

such that $\{x_\alpha\}\leq\{y'_\alpha\}$. Thus if $N\leq\{x_\alpha\}+N\leq\{y_\alpha\}+N$ we may choose representatives $\{x_\alpha\}\varepsilon\{x_\alpha\}+N$ and $\{y_\alpha\}\varepsilon\{y_\alpha\}+N$ so that $\{o\}\leq\{x_\alpha\}\leq\{y_\alpha\}$. Since L_o has property (B), L will have (B) also. Q.E.D.

By the preceding proposition 3.2 together with proposition 1.24 and theorem 2.3 we have the following corollary which implies that $\vee$ is a continuous operation on L x L.

<u>Corollary 3.3</u> L is a Banach lattice.

<u>Proposition 3.4</u> L is Archimedean.

Proof: Suppose $\{x_\alpha\}+N\geq N$, $\{y_\alpha\}+N\geq N$ and $n(\{x_\alpha\}+N)\leq\{y_\alpha\}+N$ for all $n\varepsilon\omega$. Since L is (AB) we then have

$$||n(\{x_\alpha\}+N)|| = n||\{x_\alpha\}+N|| \leq ||\{y_\alpha\}+N||$$

for all $n\varepsilon\omega$ and since R is Archimedean this implies that $||\{x_\alpha\}+N|| = o$. Since L is a normed space, $\{x_\alpha\}+N = N$. Q.E.D.

Proposition 3.5 If each L_α is an (AM)-space, so is L.

Proof: Suppose $\{x_\alpha\}+N \geq N$, $\{y_\alpha\}+N \geq N$. Choose representatives $\{x_\alpha\} \geq o$ and $\{y_\alpha\} \geq o$. Then $x_\alpha \geq o$ and $y_\alpha \geq o$ for all $\alpha\varepsilon A$ so since L_α satisfies property M we have

$$||x_\alpha \vee y_\alpha|| = ||x_\alpha|| \vee ||y_\alpha|| .$$

Then
$$\begin{aligned} ||\{x_\alpha\}+N \vee \{y_\alpha\}+N|| &= ||\{x_\alpha \vee y_\alpha\}|| \\ &= \lim_\Omega ||x_\alpha \vee y_\alpha|| \\ &= \lim_\Omega ||x_\alpha|| \vee ||y_\alpha|| \\ &= \lim_\Omega ||x_\alpha|| \vee \lim_\Omega ||y_\alpha|| \\ &= ||\{x_\alpha\}+N|| \vee ||\{y_\alpha\}+N|| \end{aligned}$$

so L satisfies property (M) also. Q.E.D.

Proposition 3.6 If each L_α is an (AL)-space, so is L.

Proof: Letting $\{x_\alpha\}+N \geq N$, $\{y_\alpha\}+N \geq N$ and choosing representatives as in the proof of the preceding proposition 3.5 we have $x_\alpha \geq o$ and $y_\alpha \geq o$ for all $\alpha\varepsilon A$. Then

$$\begin{aligned} ||\{x_\alpha\}+N+\{y_\alpha\}+N|| &= ||\{x_\alpha+y_\alpha\}|| \\ &= \lim_\Omega ||x_\alpha+y_\alpha|| \\ &= \lim_\Omega ||x_\alpha||+||y_\alpha|| \\ &= \lim_\Omega ||x_\alpha||+\lim_\Omega ||y_\alpha|| \\ &= ||\{x_\alpha\}+N||+||\{y_\alpha\}+N|| \end{aligned}$$

so L has property (L). Q.E.D.

<u>Lemma 3.7</u> Given a sequence $\langle\{x_\alpha^n\}+N\rangle_{n=1}^{\infty}$ of elements of an ultraproduct lattice L with $\{y_\alpha\}+N \leq \{x_\alpha^n\}+N \leq \{x_\alpha^{n-1}\}+N$ for all $n\varepsilon\omega$ and for some $\{y_\alpha\}\varepsilon L_o$, we may choose representatives $\{\bar{x}_\alpha^n\}\varepsilon\{x_\alpha^n\}+N$ so that $\{y_\alpha\} \leq \{\bar{x}_\alpha^n\} \leq \{\bar{x}_\alpha^{n-1}\}$ for all $n\varepsilon\omega$.

Proof: We first notice that we may choose representatives $\{\hat{x}_\alpha^n\}\varepsilon\{x_\alpha^n\}+N$ such that $\{\hat{x}_\alpha^n\} \leq \{\hat{x}_\alpha^{n-1}\}$ for all $n\varepsilon\omega$ as follows. Since $\{x_\alpha^n\}+N \leq \{x_\alpha^{n-1}\}+N$ for every $n\varepsilon\omega$ we have

$$\{x_\alpha^n\}+N = (\{x_\alpha^n\}+N) \wedge (\{x_\alpha^{n-1}\}+N)$$

$$= (\{x_\alpha^n\} \wedge \{x_\alpha^{n-1}\})+N$$

which means that

$$\{x_\alpha^n\} \wedge \{x_\alpha^{n-1}\} \varepsilon\{x_\alpha^n\}+N \quad .$$

Set $\{\hat{x}_\alpha^1\} = \{x_\alpha^1\}$, $\{\hat{x}_\alpha^2\} = \{x_\alpha^2\} \wedge \{\hat{x}_\alpha^1\}$, $\{\hat{x}_\alpha^3\} = \{x_\alpha^3\} \wedge \{\hat{x}_\alpha^2\}$, etc. Continuing in this manner we get the desired representatives. To complete the proof notice that since $\{\hat{x}_\alpha^n\}+N = \{x_\alpha^n\}+N$ we have $\{y_\alpha\}+N \leq \{\hat{x}_\alpha^n\}+N$ for every $n\varepsilon\omega$ so that

$$(\{y_\alpha\}\vee\{\hat{x}_\alpha^n\})+N = \{\hat{x}_\alpha^n\}+N$$

or

$$\{y_\alpha\}\vee\{\hat{x}_\alpha^n\} \;\varepsilon\; \{\hat{x}_\alpha^n\}+N = \{x_\alpha^n\}+N.$$

Also, since $\{x_\alpha^n\} \leq \{x_\alpha^{n-1}\}$ we have

$$\{y_\alpha\}\vee\{\hat{x}_\alpha^n\} \leq \{y_\alpha\}\vee\{\hat{x}_\alpha^{n-1}\}$$

Setting $\{\bar{x}_\alpha^n\} = \{y_\alpha\}\vee\{\hat{x}_\alpha^n\}$ we have the desired representatives since

$$\{\bar{x}_\alpha^n\} = \{y_\alpha\}\vee\{x_\alpha^n\}\varepsilon\{x_\alpha^n\}+N$$

and

$$\{y_\alpha\} \leq \{\bar{x}_\alpha^n\} \leq \{\bar{x}_\alpha^{n-1}\} \text{ for all } n\varepsilon\omega. \quad \text{Q.E.D.}$$

Lemma 3.8 If $X\varepsilon\Omega$ and $x_\alpha \leq y_\alpha$ for all $\alpha\varepsilon X$, then $\{x_\alpha\}+N \leq \{y_\alpha\}+N$.

Proof: For each $\alpha\varepsilon A\setminus X$ define $z_\alpha = x_\alpha - y_\alpha$ and let $z_\alpha = o$ for all $\alpha\varepsilon X$. Then $y_\alpha + z_\alpha \geq x_\alpha$ for all $\alpha\varepsilon A$ and furthermore, $||\{z_\alpha\}|| = o$ since $||z_\alpha|| = o$ for all $\alpha\varepsilon X\varepsilon\Omega$. Thus $\{y_\alpha+z_\alpha\}\varepsilon\{y_\alpha\}+N$ so we have $\{x_\alpha\}+N \leq \{y_\alpha\}+N$. Q.E.D.

If each L_α is σ-complete it is clear that L_o is σ-complete also. Moving up to the space L appears to require more hypothesis.

Proposition 3.9 If each L_α is a σ-complete (AL)-space with semi-continuous norm, then L is σ-complete.

Proof: Let $<\{x_\alpha^n\}+N>_n$ be a decreasing sequence of non-negative elements of L. Denote $x_n = \{x_\alpha^n\}+N$ and $a = \inf_n ||x_n|| = \lim_{n\to\infty} ||x_n||$ since $x_n \leq x_{n-1}$ implies $||x_n|| \leq ||x_{n-1}||$. We will find an element $y\varepsilon L$ with $y \geq o$ and $y \leq x_n$ for all $n\varepsilon\omega$ such that $||y|| = a$.

To define y first choose representatives $\{x_\alpha^n\}\varepsilon x_n$ so that $o \leq \{x_\alpha^n\} \leq \{x_\alpha^{n-1}\}$ for all $n\varepsilon\omega$ by lemma 3.7. Let

$$A_n = \{\alpha\varepsilon A \mid \; |\,||x_\alpha^n|| - ||x_n||\,| \leq \tfrac{1}{n}\}.$$

Then $A_n\varepsilon\Omega$ for each $n\varepsilon\omega$. Furthermore, letting $S_n = \bigcap_{i=1}^{n} A_i$, $S_n\varepsilon\Omega$ also. Define an element of L_o as follows.

$$\begin{aligned} y_\alpha &= o && \text{if } \alpha \notin S_1 \\ y_\alpha &= x_\alpha^n && \text{if } \alpha\varepsilon S_n \setminus S_{n+1} \\ y_\alpha &= \inf_n x_\alpha^n && \text{if } \alpha\varepsilon \bigcap_{n=1}^{\infty} S_n \end{aligned}$$

Then $y_\alpha \le x_\alpha^n$ for all $\alpha \in S_n$ since either $y_\alpha = x_\alpha^m$ for some $m \ge n$ or $y_\alpha = \inf_n x_\alpha^n$. This implies by lemma 3.8 that

$$\{y_\alpha\}+N \le x_n \quad \text{for all } n\varepsilon\omega.$$

Since clearly $o \le \{y_\alpha\}$ all that remains is to show that $||\{y_\alpha\}|| = a$.

Let $\varepsilon > o$ and choose n_o so that

$$|| x_{n_o} || \le a + \frac{\varepsilon}{2} \text{ and } \frac{1}{n_o} < \frac{\varepsilon}{2}.$$

For any $\alpha \varepsilon S_{n_o}$ either $y_\alpha = x_\alpha^m$ for some $m \ge n_o$ or $y_\alpha = \inf_n x_\alpha^n$.

In the first case, $\alpha \varepsilon S_m$ so $\left| ||y_\alpha|| - ||x_m|| \right| \le \frac{1}{m} \le \frac{1}{n_o} < \frac{\varepsilon}{2}$ and

$$\left| ||x_m|| - a \right| \le \frac{\varepsilon}{2} \text{ so } \left| ||y_\alpha|| - a \right| < \varepsilon .$$

In the second case, $y_\alpha = \inf_n x_\alpha^n$ implies $||y_\alpha|| = \inf_n ||x_\alpha^n||$ since each L_α has semi-continuous norm and hence there is a $n_1 \ge n_o$ with $\frac{1}{n_1} < \frac{\varepsilon}{4}$ such that

$$\left| ||x_\alpha^{n_1}|| - ||y_\alpha|| \right| < \frac{\varepsilon}{4}.$$

Since $\alpha \varepsilon S_{n_1}$, $\left| ||x^{n_1}|| - ||x_{n_1}|| \right| < \frac{1}{n_1} < \frac{\varepsilon}{4}$.

But $\left| ||x_{n_1}|| - a \right| \le \frac{\varepsilon}{2}$ so we again have

$$\left| ||y_\alpha|| - a \right| < \varepsilon \quad \text{for all } \alpha \varepsilon S_{n_o} \varepsilon \Omega$$

which implies that

$$||\{y_\alpha\}|| = \lim_\Omega ||y_\alpha|| = a.$$

To complete the proof, suppose $\{z_\alpha\}+N \leq \{x_\alpha^n\}+N$ for all $n\varepsilon\omega$. Then $\{z_\alpha\}+N \vee \{y_\alpha\}+N \leq \{x_\alpha^n\}+N$ for all $n\varepsilon\omega$ so

$$||\{z_\alpha\}+N \vee \{y_\alpha\}+N|| \leq \inf_n ||\{x_\alpha^n\}+N|| = ||\{y_\alpha\}+N||$$

But by property (L) in the space L,

$$||\{z_\alpha\}+N \vee \{y_\alpha\}+N|| = ||\{z_\alpha\}+N \vee \{y_\alpha\}+N-\{y_\alpha\}+N|| + ||\{y_\alpha\}+N||$$

which implies $||\{z_\alpha\}+N \vee \{y\alpha\}+N - \{y_\alpha\}+N|| = o$.
Hence $\{z_\alpha\}+N \vee \{y_\alpha\}+N = \{y_\alpha\}+N$
that is, $\{z_\alpha\}+N \leq \{y_\alpha\}+N$
so $\{y_\alpha\}+N = \inf_n\{x_\alpha^n\} +N$
and thus L is σ-complete. Q.E.D.

It is unknown at this time whether the hypotheses of proposition 3.9 could be weakened. It would be of particular interest if the (AL) condition could be dropped.

<u>Proposition 3.10</u> If each L_α has a positive complete element, then so does L_o.

Proof: Let s_α be a positive complete element in L_α. Since each $s_\alpha>o$, $\{s_\alpha\}>o$ also. Suppose $o = \{s_\alpha\} \wedge |\{x_\alpha\}| = \{s_\alpha\} \wedge \{|x_\alpha|\}$. Then $\{s_\alpha \wedge |x_\alpha|\} = o$ implies $s_\alpha \wedge |x_\alpha| = o$ for all $\alpha\varepsilon A$ which implies $x_\alpha = o$ for all $\alpha\varepsilon A$ so $\{x_\alpha\} = o$. Thus $\{s_\alpha\}$ is a positive complete element in L_o. Q.E.D.

<u>Example 3.11</u> L need not have a positive complete element, even if each L_α does.

Let Ω be a free ultrafilter on ω and for each $i\varepsilon\omega$ let $L_i = \ell_2$ with the usual order on ℓ_2. Since ℓ_2 has a positive complete element, for example $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$ it remains to show that L does not have a positive complete element.

Given $x = \{s_i\}+N\varepsilon L$ with $x\geq o$ we show that x is not a positive complete element by producing a $z\varepsilon L$ such that $z\neq o$ and $z\wedge x = o$. We may suppose that $s_i\geq o$ for all $i\varepsilon\omega$. Let $s_i = (s^i_j)_{j=1,2,\ldots}\varepsilon\ell_2$.

Since $(s^i_j)_j\varepsilon\ell_2$ there is an index j(i) such that $s^i_{j(i)}<\frac{1}{i}$. Let $\omega_i\varepsilon\ell_2$ be $(\omega^i_j)_j$ where

$$\omega^i_{j(i)} = 1 \quad\text{and}\quad \omega^i_j = o \quad\text{for}\quad j\neq j(i)$$

Since $||\omega_i|| = 1$ for all $i\varepsilon\omega$, $\{\omega_i\}\varepsilon L_o$ and $||\{\omega_i\}|| = 1.$ Letting $z = \{\omega_i\}+N$, $||z|| = 1$ so $z\neq o$. Now $z\wedge x = \{\omega_i\wedge s_i\}+N$. But $\omega_i\wedge s_i = (0,\ldots, o, s^i_{j(i)}, o, \ldots)$ so since $o<s^i_{j(i)}<\frac{1}{i}$, $||\omega_i\wedge s_i||<\frac{1}{i}$ which implies that $||\{\omega_i\wedge s_i\}|| = o$ since Ω contains the Fréchet filter and each set in the Fréchet filter contains all integers i from a given point onward. Thus $z\wedge x = o$ so x is not a positive complete element in L. Since x was arbitrary, L has no positive complete element.

<u>Proposition 3.12</u> Let Ω be an ultrafilter with the countable intersection property. The semi-normed lattice L_o does not have semi-continuous norm.

Proof: Let $\{X_i\}_{i=1}^{\infty}$ be a collection of elements of Ω such that each $\alpha\epsilon A$ belongs to at most a finite number of the X_i. Define a sequence $<\{x_\alpha^n\}>_n$ of elements of L_o as follows:

$$\text{Let } x_\alpha^1 = o \qquad \text{for } \alpha \notin \bigcup_{i=1}^{\infty} X_i$$

$$x_\alpha^1 > o,\ ||x_\alpha^1|| = 1 \qquad \text{for } \alpha \epsilon \bigcup_{i=1}^{\infty} X_i ,$$

$$\text{let } x_\alpha^2 = o \qquad \text{for } \alpha \notin \bigcup_{i\geq 2} X_i$$

$$x_\alpha^2 = x_\alpha^1 \qquad \text{for } \alpha \epsilon \bigcup_{i\geq 2} X_i$$

and in general let

$$x_\alpha^k = o \qquad \text{for } \alpha \notin \bigcup_{i\geq k} X_i$$

$$x_\alpha^k = x_\alpha^1 \qquad \text{for } \alpha \epsilon \bigcup_{i\geq k} X_i$$

Let $k\epsilon\omega$. Since $\{\alpha\epsilon A \mid ||x_\alpha^k|| = 1\} = \bigcup_{i\geq k} X_i \epsilon \Omega$ we have $||\{x_\alpha^k\}|| = 1$ and hence

$$\inf_k ||\{x_\alpha^k\}|| = 1.$$

On the other hand our construction shows that

$$o \leq \{x_\alpha^n\} \leq \{x_\alpha^{n-1}\} \quad \text{for all } n\epsilon\omega$$

and for any $\alpha\epsilon A$, α belongs to at most a finite number of the X_i so there exists a k_o such that $x_\alpha^k = o$ for all $k \geq k_o$. Thus $\inf_k x_\alpha^k = o$. But if $\{y_\alpha\} \leq \{x_\alpha^n\}$ for all $n\epsilon\omega$ then fixing α we see that $y_\alpha \leq \inf_k x_\alpha^k = o$ so since $\{o\} \leq \{x_\alpha^n\}$ for all $n\epsilon\omega$ we must in fact have

$$\{o\} = \inf_k \{x_\alpha^k\} .$$

But then $||\inf_n\{x_\alpha^n\}|| = ||\{o\}|| = o \neq \inf_n ||\{x_\alpha^n\}||$

so L_o does not have semicontinuous norm. Q.E.D.

<u>Proposition 3.13</u> If each L_α is σ-complete and has semicontinuous norm, then L has semicontinuous norm also.

Proof: Let $<\{x_\alpha^n\}+N>_n$ be a decreasing sequence of non-negative elements of L with

$$\{x_\alpha\}+N = \inf_n \{x_\alpha^n\}+N.$$

Denote $x = \{x_\alpha\}+N$ and $x_n = \{x_\alpha^n\}+N$ for all $n\varepsilon\omega$. Since $o\leq x\leq x_n$ for all $n\varepsilon\omega$

$$||x|| \leq \inf_n ||x_n||.$$

Furthermore, since $x_n\leq x_{n-1}$ for all $n\varepsilon\omega$,

$$||x_n|| \leq ||x_{n-1}|| \quad \text{for all } n\varepsilon\omega.$$

Let $a = \inf_n ||x_n|| = \lim_{n\to\infty} ||x_n||$. By the proof of proposition 3.9 find a $y\varepsilon L$ with $y\geq o$, $y\leq x_n$ for all $n\varepsilon\omega$ and $||y|| = a$. Then, since $o\leq y\leq x$ we have $||y|| = a \leq ||x||$ and thus $||x|| = a = \inf_n ||x_n||$ so L has semicontinuous norm. Q.E.D.

CONCLUSION

The general study of ultraproduct spaces begun here can be continued in several directions. More work can be done on the lattice properties of ultraproduct spaces. We have already remarked that the hypotheses of proposition 3.9 might be simplified and the same might be true of proposition 3.13. A direction which seems likely to be fruitful is examining the dual of ultraproduct spaces. In addition to considering the dual of specific ultraproduct spaces, the question of when the ultraproduct of dual spaces, which we have called $\hat{L}$, is identical to the dual of the ultraproduct space, that is, L^*, appears particularly interesting. In general, the detailed study of specific ultraproduct spaces should be a useful new source of examples of Banach spaces.

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