9

System Realizations

In Chapter 1, we gave an introduction to the relationship between transfer functions and state equations. In this chapter, we will continue this investigation with added attention to the complications introduced by controllability and observability of systems. Also in Chapter 1, we presented what we called at the time *simulation diagrams*. A simulation diagram, such as Figures 1.7 or 1.8, is sometimes also known as an *integrator realization*, because it shows how a differential equation or transfer function can be *realized* with physical components, in this case, integrators. Note that once a transfer function has been represented by such a diagram, the state equations can be written directly, with state variables taken as the outputs of the integrators. In this chapter, we will explore the details of constructing such realizations of transfer functions. We will show a number of alternatives to the realizations of Figures 1.7 or 1.8, along with the mathematical methods by which they are obtained. In the course of this presentation, we will discover the relationship among controllability, observability, and the relative size of the transfer function and associated state matrices. One such realization method, which uses quantities called Markov parameters, will lead to a method for *system identification*, which allows us to estimate the state matrices from input and output data.

Throughout this chapter, we will concentrate on time-invariant systems. The main reason for this is that time-varying systems are not described by transfer functions, so conversion from transfer function to state space is not necessary. Some results can be obtained by "realizing" the impulse-response matrix for time-varying systems [16], but because this matrix is not often available in closed form, that process is considered an academic exercise.

Unless otherwise indicated by our standard notation, the results shown here apply equally well to SISO and MIMO systems. They also apply to continuous- and discrete-time systems alike, with the necessary change from *s*-domain notation to *z*-domain notation.

9.1 Minimal Realizations

As shown in Chapter 1, we can derive the transfer function representation for the state space system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{9.1}$$

ss2tf(A,B,C,D)

by first assuming zero initial conditions, then taking the LaPlace transform of both sides of the equation (9.1). The result is[M]

$$H(s) = C(sI - A)^{-1}B + D \tag{9.2}$$

For the discrete-time system

$$\begin{aligned} x(k+1) &= A_d x(k) + B_d u(k) \\ y(k) &= C_d x(k) + D_d u(k) \end{aligned} \tag{9.3}$$

the result is much the same,

$$H(z) = C_d(zI - A_d)^{-1}B_d + D_d \tag{9.4}$$

The relationship between a state space system, which we will denote by the quadruple $\{A, B, C, D\}$, and the transfer function $H(s)$ is described by the terminology given in the following definition:

> ***Realization:*** A realization of transfer function $H(s)$ is any state space quadruple $\{A, B, C, D\}$ such that $H(s) = C(sI - A)^{-1}B + D$. If such a set $\{A, B, C, D\}$ exists, then $H(s)$ is said to be *realizable*. (9.5)

Because of the existence of the methods shown in Chapter 1, we can immediately say that a transfer function is realizable if it is *proper* (or, in the multivariable case, if each component transfer function is proper). Also as seen in Chapter 1, it is possible for two different state space descriptions to give the same transfer function. For example, consider the system in (9.1) and an equivalent system given by the similarity transformation provided by nonsingular change of basis matrix *M*:

$$\begin{aligned} \dot{\hat{x}} &= M^{-1}AM\hat{x} + M^{-1}Bu \\ y &= CMx + Du \end{aligned} \tag{9.6}$$

Forming the transfer function of this matrix, we get

$$\begin{aligned}\hat{H}(s) &= CM(sI - M^{-1}AM)^{-1}M^{-1}B + D \\ &= C(sMM^{-1} - MM^{-1}AMM^{-1})B + D \\ &= C(sI - A)^{-1}B + D \\ &= H(s)\end{aligned}$$

Therefore, we have the result that any two systems related by a similarity transformation have the same transfer function.

Now we wish to consider the *order* of these transfer functions relative to the dimension of the state space description, i.e., relative to the value n, which is the size of the matrix A. To motivate this discussion, we will find the transfer function for the differential equation $\ddot{y} + 4\dot{y} + 3y = \dot{u} + 3u$. With zero initial conditions, a LaPlace transform of both sides gives the result

$$H(s) = \frac{Y(s)}{U(s)} = \frac{s+3}{s^2+4s+3} = \frac{s+3}{(s+1)(s+3)} = \frac{1}{s+1} \tag{9.7}$$

This shows the curious fact that the transfer function is of lower order (i.e., the degree of the characteristic polynomial in the denominator) than the order of the differential equation, which would also be the size of the state space description of the system derived from the realization methods of Chapter 1. We could also realize this transfer function with the first-order state space system

$$\begin{aligned}\dot{x} &= -x + u \\ y &= x\end{aligned} \tag{9.8}$$

To discover the reason for this occurrence, consider putting the differential equation into state space form, according to the realization from Figure 1.8:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}-4 & 1 \\ -3 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}1 \\ 3\end{bmatrix}u \\ y &= \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}\end{aligned} \tag{9.9}$$

The first thing we notice about this particular realization is that it is in the observable canonical form. This means that the realization in (9.9) is observable by default. However, if we examine the controllability matrix[M] for (9.9), we get `ctrb(A,B)`

$$P = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$$

Clearly this is a rank-deficient matrix, and we see that this realization is not controllable. If we had chosen the controllable canonical realization from Figure 1.7, we would have found that

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{9.10}$$

which is easily found to be controllable but not observable.

From these examples we make two observations. The first is that for a given transfer function, it is possible to find realizations of different order. This necessitates the following definition:

> ***Minimal Realization:*** A realization $\{A, B, C, D\}$ is called a *minimal realization* (also called an *irreducible realization*) of a transfer function if there is no other realization of smaller size. (9.11)

The second observation we make from the examples above results in the following theorem, which we will prove somewhat less than rigorously:

> THEOREM: A minimal realization is both controllable and observable. (9.12)

`ctrbf(A,B,C)`

To demonstrate this theorem, we note from the Kalman decomposition of Chapter 8 that if a system is not controllable, then it is possible to decompose it into controllable and uncontrollable parts:[M]

$$\begin{aligned} \begin{bmatrix} \dot{x}_c \\ \dot{x}_{\bar{c}} \end{bmatrix} &= \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + \begin{bmatrix} B_c \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} x_c \\ x_{\bar{c}} \end{bmatrix} + Du \end{aligned} \tag{9.13}$$

Deriving the transfer function from this form,

$$
\begin{aligned}
H(s) &= \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} sI - A_c & -A_{12} \\ 0 & sI - A_{\bar{c}} \end{bmatrix}^{-1} \begin{bmatrix} B_c \\ 0 \end{bmatrix} + D \\
&= \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} (sI - A_c)^{-1} & (sI - A_c)^{-1} A_{12} (sI - A_{\bar{c}})^{-1} \\ 0 & (sI - A_{\bar{c}})^{-1} \end{bmatrix} \begin{bmatrix} B_c \\ 0 \end{bmatrix} + D \\
&= \begin{bmatrix} C_c & C_{\bar{c}} \end{bmatrix} \begin{bmatrix} (sI - A_c)^{-1} B_c \\ 0 \end{bmatrix} + D \\
&= C_c (sI - A_c)^{-1} B_c + D
\end{aligned} \tag{9.14}
$$

This demonstrates the interesting fact that the transfer function does not depend on the uncontrollable part of the realization. A similar exercise shows that the transfer function is independent of the unobservable part of the realization as well. Together, the statements can be generalized to say that *the transfer function of the system depends on only the controllable and observable part of the system*. *Conversely, a nonminimal realization is either uncontrollable, unobservable, or both***,** i.e.,

$$H(s) = C_{co}(sI - A_{co})^{-1} B_{co} + D \tag{9.15}$$

The even more general statement can be made:

> THEOREM: (Minimal realization theorem) If $\{A, B, C, D\}$ is a minimal realization and $\{\overline{A}, \overline{B}, \overline{C}, \overline{D}\}$ is another minimal realization, then the two realizations are similar to one another (i.e., they are related by a similarity transformation). (9.16)

The proof of this theorem will be deferred until we investigate Markov parameters in Section 9.3.

In the case of SISO systems, wherein we can denote the transfer function as

$$h(s) = \frac{n(s)}{d(s)} \tag{9.17}$$

we identify the degree of the denominator polynomial $d(s)$ as the highest power of *s*. If a minimal realization is found for (9.17) of size less than *n*, then the two polynomials $n(s)$ and $d(s)$ will have a common (polynomial) factor, which could be canceled in order to arrive at a transfer function of smaller degree, not altering the input-output behavior. Note that this form of pole-zero cancellation is *exact* in the sense that it happens internally and identically within the system.

It is not a violation of the admonishment given in classical control theory that poles and zeros never exactly cancel each other.

If the polynomials $n(s)$ and $d(s)$ have no common factor, they are said to be *coprime*, and no such pole-zero cancellation will occur. If $n(s)$ and $d(s)$ are coprime, and $d(s)$ is of degree *n*, then the minimal realization will have size *n* as well. This is often stated as a theorem whose proof simply follows the explanation of minimal realizations above. The definition of coprimeness and this statement of the size of the minimal realization also extends to matrix-fraction descriptions for MIMO systems, although we do not pursue such descriptions here.

As a final note for this section, we might remark on the implication of modal controllability and observability on BIBO stability. We learned in Chapter 7 that BIBO stability depends on the poles of the transfer function, not on the eigenvalues of the state space representation. In Chapter 8 we discovered that the controllability and observability of the individual modes of the system can be determined by inspection of either the Jordan form or the Kalman decomposition. We have seen in this section that any uncontrollable and or unobservable modes will not appear in the transfer function. Therefore, combining these facts, we can arrive at a method by which we can determine BIBO stability by inspection: *If all of the unstable modes of a system (as determined by their eigenvalues) are found to be uncontrollable, unobservable, or both, by inspection of the canonical form, then the system will be BIBO stable*.

9.2 Specific Realizations

In Chapter 1, we presented two basic “integrator” realizations, which we referred to as simulation diagrams, because they can be used directly to simulate systems with analog integrators. They apply equally well to continuous- and discrete-time systems, with only the substitution of unit delay blocks in the place of the integrators. We see now that these simulations are in fact realizations in the sense they are used here: a technique for converting from transfer function to state space notation. In this section, we will discuss these realizations from a new perspective and add some additional realizations.

We start with the differential equation

$$\begin{aligned} &\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = \\ &\qquad b_n\frac{d^n u(t)}{dt^n} + \cdots + b_0 u(t) \end{aligned} \tag{9.18}$$

We begin with the frequency-domain expression that is obtained by finding the LaPlace transform of both sides of (9.18) while assuming zero initial conditions:

$$h(s)=\frac{y(s)}{u(s)}=\frac{b_n s^n+\cdots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0} \tag{9.19}$$

9.2.1 Controllable Canonical Realization

In Chapter 1, this realization, shown in Figure 9.1, was referred to as the "phase variable" realization. From the perspective of Chapter 6, this term refers to the fact that the state variables defined as the outputs of the integrators are the variables depicted in phase plots, i.e., an output variable and its successive derivatives (or time delays).

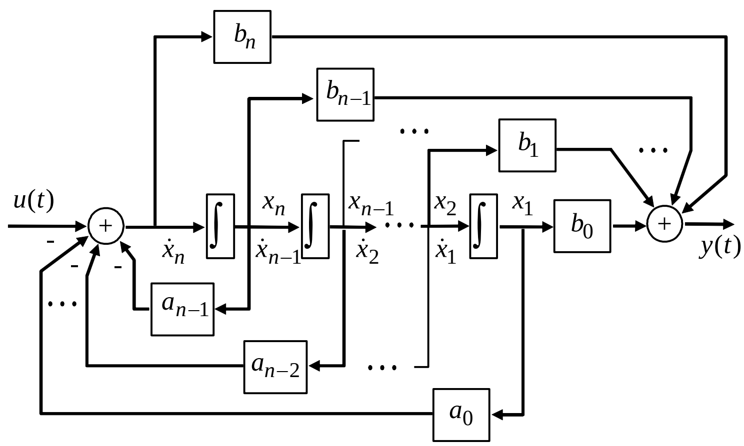

Figure 9.1 Integrator diagram of controllable canonical realization.

From the perspective of Chapter 8, the phase variable realization, written as the set of state equations

$$\begin{bmatrix}\dot{x}_1\\ \vdots\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}0&1&0&0\\ \vdots&\ddots&\ddots&0\\ 0&\cdots&0&1\\ -a_0&-a_1&\cdots&-a_{n-1}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix}+\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}u(t)$$

$$y(t)=\begin{bmatrix}b_0-b_na_0 & b_1-b_na_1 & \cdots & b_{n-1}-b_na_{n-1}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix}+b_n\,u(t) \tag{9.20}$$

is seen to yield the controllable canonical form. Note that this need not yield a *minimal* realization, only a controllable one. If the minimal realization is of size less than n, then (9.20), while being controllable by construction (see Chapter 8), will not be observable because we know that nonminimal realizations must be either uncontrollable, unobservable, or both.

9.2.2 Observable Canonical Realization

Also in Chapter 1 we presented the realization shown in Figure 9.2.

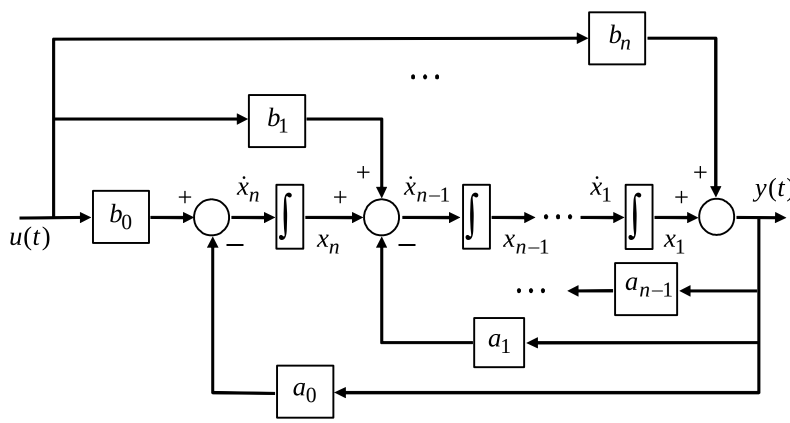

Figure 9.2 Integrator diagram of observable canonical realization.

Here, we need only remark that this realization results in the state equations of the form

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_{n-1} & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ -a_1 & \cdots & 0 & 1 \\ -a_0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_{n-1} - a_{n-1} b_n \\ \vdots \\ b_1 - a_1 b_n \\ b_0 - a_0 b_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_n u(t) \tag{9.21}$$

We refer to this realization as the observable canonical realization, because of its special companion form. (In some texts, the term "observable canonical realization" is given as a companion form with the $-a_i$ coefficients appearing in the *last* column of the *A*-matrix instead of the first and with a somewhat different *b*-matrix [4].)

Again, we note that these realizations (controllable and observable canonical) can be constructed just as easily in discrete-time, with unit delay blocks instead. Also, we should remark that there is a multivariable counterpart to each, which we will discuss in Chapter 10.

9.2.3 Jordan Canonical Realizations

By this time, the reader may have noticed that for every canonical form of the state equations, we have identified a corresponding realization. This is true for the Jordan canonical form as well. To see how the Jordan canonical form becomes the Jordan canonical realization, we will consider the specific example of the strictly proper transfer function:

$$h(s) = \frac{4s^4 + 54s^3 + 247s^2 + 500s + 363}{s^5 + 11s^4 + 63s^3 + 177s^2 + 224s + 100} \tag{9.22}$$

[If we have a non-strictly proper transfer function, it should first be manipulated so that it appears in the form:

$$h(s) = \frac{n(s)}{d(s)} + k \tag{9.23}$$

where the ratio $n(s)/d(s)$ is strictly proper. The following discussion applies to the ratio $n(s)/d(s)$, and the k term is simply a feedthrough constant.] The transfer function (9.22) can be expanded into the partial-fraction expansion^{M} `residue(n,d)`

$$h(s) = \frac{1-j2}{s-(-3+j4)} + \frac{1+j2}{s-(-3-j4)} + \frac{-1}{s+2} + \frac{1}{(s+2)^2} + \frac{3}{s+1} \tag{9.24}$$

which is then realized with the block-diagram realization shown in Figure 9.3.

The first observation we make about the Jordan canonical realization, given Figure 9.3, is that it is not an "integrator" realization since the dynamic blocks contain the form $1/(s-\lambda)$ rather than $1/s$. If it is desired that this realization be constructed entirely from pure integrators, then each $1/(s-\lambda)$ can be individually realized by the subdiagram shown in Figure 9.4. However, this decomposition into pure integrators is usually not necessary, since the operational amplifier circuit of the ideal integrator $1/s$ can be easily converted to the "nonideal integrator" with the addition of a single feedback resistor in parallel with the feedback capacitor.

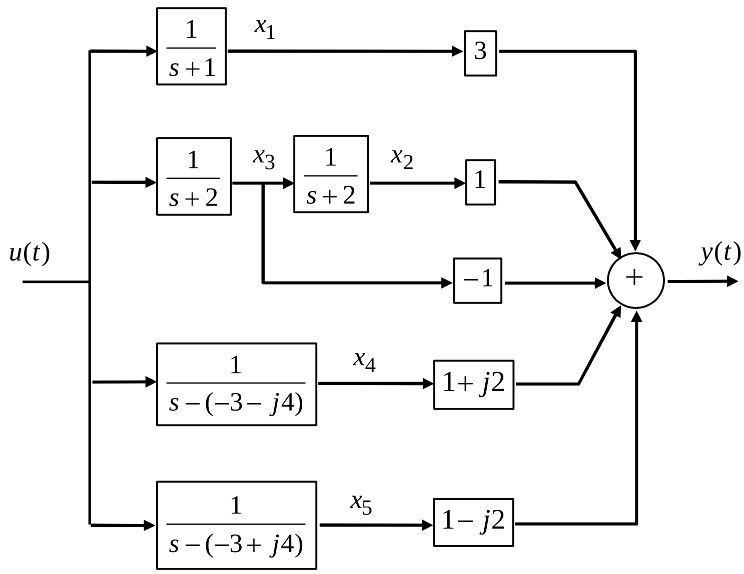

Figure 9.3 Complex-gain version of the Jordan form realization of the transfer function of Equation (9.22).

By defining the state variables as the output of the dynamic blocks, as we have in Figure 9.3, and keeping in mind Figure 9.4 as the internal realization of these blocks, we arrive at the state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{c|cc|c|c} -1 & 0 & 0 & 0 & 0 \\ \hline 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ \hline 0 & 0 & 0 & -3-j4 & 0 \\ \hline 0 & 0 & 0 & 0 & -3+j4 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 3 & 1 & -1 & 1-j2 & 1+j2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \tag{9.25}$$

which is easily recognizable as being in the Jordan form.

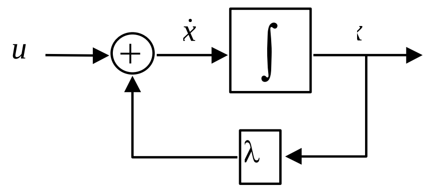

Figure 9.4 Integrator realization of $1/(s-\lambda)$ blocks in Figure 9.3.

The second observation we wish to make about the Jordan form realization of Figure 9.3 and Equation (9.25) is that it contains complex gains. If the need for a realization is physical, i.e., if it is desired that the system be constructed in hardware, then we cannot allow complex numbers. To remedy this, each pair of complex conjugate poles in (9.24) should be left as a quadratic, i.e.,

$$h(s)=\frac{2s+22}{(s+3)^2+4^2}+\frac{-1}{s+2}+\frac{1}{(s+2)^2}+\frac{3}{s+1} \tag{9.26}$$

which can be obtained simply by adding the parallel paths in Figure 9.3. The quadratic term can be realized with any of the other realization methods (e.g., the controllable or observable), and is then considered as a block. The quadratic term then becomes a block in a *block* Jordan form such as[M] `cdf2rdf(V,D)`

$$\dot{x}=\left[\begin{array}{c:cc:c} -1 & & & \\ \hdashline & -2 & 1 & \\ & & -2 & \\ \hdashline & & & \hat{A}_{2\times 2} \end{array}\right]x+\begin{bmatrix}1\\0\\1\\\hat{b}_{2\times 1}\end{bmatrix}u \tag{9.27}$$
$$y=\begin{bmatrix}3 & 1 & -1 & \hat{c}_{1\times 2}\end{bmatrix}x$$

For example, the block corresponding to the quadratic term might be given as

$$\hat{A}_{2\times 2}=\begin{bmatrix}-3 & 4\\ -4 & -3\end{bmatrix} \qquad \hat{b}_{2\times 1}=\begin{bmatrix}2\\4\end{bmatrix} \qquad \hat{c}_{1\times 2}=\begin{bmatrix}1 & 0\end{bmatrix}$$

We should also remark on the controllability and/or observability of such a realization. It is guaranteed to be neither. Note that in Figure 9.3, each eigenvalue is associated with a parallel path from input to output. In the example given, each path has a nonzero gain connecting it to the output and a unity gain connection to the input. We know from Chapter 8 that for this SISO system, which has only a single Jordan block for each distinct eigenvalue, the row (column) of the b-matrix (c-matrix) that corresponds to the last row (first column) of the corresponding Jordan block must be nonzero. In this system, that is the case. A zero in such a row (column) would correspond to one of the parallel paths in Figure 9.3 being "disconnected" from the input, the output, or perhaps both. The special structure of Jordan forms makes them "decoupled," so that the modes do not interact; therefore, any such disconnected path results in a nonminimal realization.

If there were more than one Jordan block for any particular eigenvalue, we would have two separate paths in Figure 9.3 for that eigenvalue. If this were to happen in our system, we already know that the system would be either uncontrollable, unobservable, or both, because there are an insufficient number of inputs and outputs to allow for linearly independent rows (columns) of the b-matrix (c-matrix).

In the case that the system has more than one input and/or output, the Jordan form can still be computed and realized in a form similar to Figure 9.3. Of course, the diagram will include p input channels and q output channels, each with a separate summing junction. The other interconnections will look very similar to those in Figure 9.3. The procedure will also require finding the partial fraction expansion of the $(q \times p)$ *matrix* transfer function $H(s)$, which entails a partial-fraction expansion whose residues (numerator coefficients) are all $(q \times p)$ constant matrices. This, in turn, requires the definition of the characteristic polynomial of the matrix $H(s)$ as the least common denominator of all its minors [4]. Rather than pursue this method in detail, we will proceed to a more common technique for realizing a multivariable system, with *Markov parameters*.

9.3 Markov Parameters

An interesting way to realize a transfer function that has some beneficial side effects is through the power series expansion. Thus, if we have the (SISO) transfer function for a causal LTI system (implying a proper transfer function),

$$g(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \tag{9.28}$$

we seek an expansion in the form:

$$g(s) = h(0) + h(1)s^{-1} + h(2)s^{-2} + \dots \tag{9.29}$$

By simple long division, it is clear that $h(0) = b_n$. The values $h(i)$, with $i = 0,1,\dots$, are known as *Markov parameters*. From (9.29), it is apparent that $g(s)$ could be realized with an infinite chain of integrators, although this would clearly not be a good idea. We suspect instead that the realization should require no more than *n* integrators and hence, the transfer function should be realizable with a system of order *n* or less.

To generate a more suitable expression for Markov parameters, rewrite the transfer function of the system in terms of the matrices in the realization:

$$g(s) = C(sI - A)^{-1}B + D \tag{9.30}$$

[For generality, we allow for the possibility of a MIMO system, although Equation (9.28) cannot be expressed as such in that case, but rather as a matrix of transfer functions.] Manipulating the term that must be inverted by using the *resolvent identity* given in Appendix A, i.e., Equation (A.42),

$$\begin{aligned} g(s) &= D + C\left[\frac{1}{s}\left(I - \frac{A}{s}\right)^{-1}\right]B \\ &= D + C\left[\frac{1}{s}\left(I + \frac{A}{s} + \frac{A^2}{s^2} + \cdots\right)\right]B \\ &= D + \frac{CB}{s} + \frac{CAB}{s^2} + \frac{CA^2B}{s^3} + \cdots \end{aligned} \tag{9.31}$$

(Convergence of the series generated may be ensured by scaling the *A*-matrix if necessary.) This shows that the Markov parameters are $h(0) = D$ and $h(i) = CA^{i-1}B$ for $i = 1, 2, \ldots$. One might also observe that if the inverse LaPlace transform were performed on the last line of (9.31), then the impulse response would result:

$$\begin{aligned} g(t) &= D\delta(t) + CB + CABt + \frac{CA^2Bt^2}{2!} + \cdots \\ &= D\delta(t) + C\left[1 + At + \frac{A^2t^2}{2!} + \cdots\right]B \\ &= D\delta(t) + Ce^{At}B \end{aligned} \tag{9.32}$$

just as we would expect. Equivalently, we can see that for $i = 1, 2, \ldots$,

$$h(i) = CA^{i-1}B = \left.\frac{d^{i-1}g(t)}{dt^{i-1}}\right|_{t=0} \tag{9.33}$$

Again, this expression takes the form of a *matrix* in the case of a MIMO system.

An alternative method to derive the Markov parameters (for SISO systems) is to realize that from (9.28) and (9.29),

$$\begin{aligned} &b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0 \\ &\qquad = \left(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right)\left(h(0) + h(1)s^{-1} + h(2)s^{-2} + \cdots\right) \end{aligned}$$

Equating coefficients of powers of s^{-1}, we have

$$\begin{aligned} b_n &= h(0) \\ b_{n-1} &= h(1) + h(0)a_{n-1} \\ b_{n-2} &= h(2) + h(1)a_{n-1} + h(0)a_{n-2} \\ &\vdots \\ b_i &= h(n-i) + h(n-i-1)a_{n-1} + \cdots + h(0)a_i \end{aligned}$$

so the recursive relationship can be given as:

$$\begin{aligned} h(0) &= b_n \\ h(1) &= b_{n-1} - h(0)a_{n-1} \\ h(2) &= b_{n-2} - h(1)a_{n-1} - h(0)a_{n-2} \\ &\vdots \\ h(n) &= b_0 - h(n-1)a_{n-1} - \cdots - h(0)a_0 \end{aligned} \tag{9.34}$$

An important observation suggested by this result is that the Markov parameters $h(i),\ i = 0,1,\ldots$, ought to be invariant to the particular realization $\{A,B,C,D\}$, i.e., they are unique to the transfer function. We will use this fact in the proof of the minimal realization theorem (on page 371) that follows.

9.3.1 Proof of the Minimal Realization Theorem

At this point we will reconsider the statement made in the minimal realization theorem. In that theorem it was claimed that any two minimal realizations of the same transfer function were similar to one another. We can now demonstrate this using the newly defined Markov parameters. We will first demonstrate some interesting properties of Markov parameters.

First, note that if $\{A,B,C,D\}$ and $\{\bar{A},\bar{B},\bar{C},\bar{D}\}$ are two different realizations of the same transfer function $g(s)$, then (even if they are of different size), $D = \bar{D}$ [or $h(0) = \bar{h}(0)$], and, by equating powers of s^{-1} in the respective series expansions (9.31),

$$\bar{h}(i) = \bar{C}\bar{A}^{i-1}\bar{B} = h(i) = CA^{i-1}B \tag{9.35}$$

Thus, Markov parameters themselves are unique to a transfer function, even though the matrices from which they may be computed might differ. This same result is even more apparent from direct application of the similarity transformation:

$$\bar{C}\bar{A}^{i-1}\bar{B} = (CT)\left(T^{-1}A^{i-1}T\right)\left(T^{-1}B\right) = CA^{i-1}B$$

Next, consider two minimal realizations $\{A, B, C, D\}$ and $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$ so that we are sure each is of the same size. Also neglect $h(0)$ and consider only $h(i)$, $i = 1, 2, \ldots$. If we form the product of the observability and controllability matrices

$$\begin{aligned} QP &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \\ &= \begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \\ CAB & CA^2B & & \\ \vdots & & \ddots & \vdots \\ CA^{n-1}B & & \cdots & CA^{2(n-1)}B \end{bmatrix} \end{aligned} \tag{9.36}$$

so of course for $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}\}$,

$$\bar{Q}\bar{P} = \begin{bmatrix} \bar{C}\bar{B} & \bar{C}\bar{A}\bar{B} & \cdots & \bar{C}\bar{A}^{n-1}\bar{B} \\ \bar{C}\bar{A}\bar{B} & \bar{C}\bar{A}^2\bar{B} & & \\ \vdots & & \ddots & \vdots \\ \bar{C}\bar{A}^{n-1}\bar{B} & & \cdots & \bar{C}\bar{A}^{2(n-1)}\bar{B} \end{bmatrix} \tag{9.37}$$

Now because of (9.35), we have

$$QP = \bar{Q}\bar{P} \tag{9.38}$$

Because both realizations are minimal, they are known to be controllable and observable, so that the $(n \times qn)$ matrix $\bar{Q}$ is full rank n. Then multiplying both side of (9.38) by $\bar{Q}^{\mathrm{T}}$ gives

$$\overline{Q}^{\mathrm{T}} QP = \overline{Q}^{\mathrm{T}} \overline{Q}\,\overline{P} \tag{9.39}$$

Being observable, the $(n \times n)$ matrix $\overline{Q}^{\mathrm{T}}\overline{Q}$ is invertible, so (9.39) can be solved to get

$$\begin{aligned} \overline{P} &= (\overline{Q}^{\mathrm{T}}\overline{Q})^{-1}\overline{Q}^{\mathrm{T}} QP \\ &\stackrel{\Delta}{=} TP \end{aligned} \tag{9.40}$$

Thus we have a relationship between the controllability matrices P and $\overline{P}$.

Performing a similar procedure to solve (9.38) for Q,

$$QPP^{\mathrm{T}} = \overline{Q}\,\overline{P}P^{\mathrm{T}} \tag{9.41}$$

$$Q = \overline{Q}\,\overline{P}P^{\mathrm{T}}\left(PP^{\mathrm{T}}\right)^{-1} \tag{9.42}$$

Now substituting (9.40) into (9.42),

$$\begin{aligned} Q &= \overline{Q}TPP^{\mathrm{T}}\left(PP^{\mathrm{T}}\right)^{-1} \\ &= \overline{Q}T \end{aligned} \tag{9.43}$$

Together, (9.40) and (9.43) demonstrate that

$$\begin{aligned} P = T^{-1}\overline{P} &= T^{-1}\left[\overline{B} \quad \overline{A}\,\overline{B} \quad \cdots \quad \overline{A}^{n-1}\overline{B}\right] \\ &= \left[T^{-1}\overline{B} \quad T^{-1}\overline{A}\,\overline{B} \quad \cdots \quad T^{-1}\overline{A}^{n-1}\overline{B}\right] \\ &= \left[T^{-1}\overline{B} \quad \left(T^{-1}\overline{A}T\right)T^{-1}\overline{B} \quad \cdots \quad \left(T^{-1}\overline{A}^{n-1}T\right)T^{-1}\overline{B}\right] \\ &= \left[B \quad AB \quad \cdots \quad A^{n-1}B\right] \end{aligned} \tag{9.44}$$

and

$$\begin{aligned} Q &= \overline{Q}T \\ &= \begin{bmatrix} \overline{C} \\ \overline{C}\,\overline{A} \\ \vdots \\ \overline{C}\,\overline{A}^{n-1} \end{bmatrix} T = \begin{bmatrix} \overline{C}T \\ \overline{C}\,\overline{A}T \\ \vdots \\ \overline{C}\,\overline{A}^{n-1}T \end{bmatrix} = \begin{bmatrix} \overline{C}T \\ \overline{C}T\left(T^{-1}\overline{A}T\right) \\ \vdots \\ \overline{C}T\left(T^{-1}\overline{A}^{n-1}T\right) \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{9.45}$$

These relationships imply that $A = T^{-1}\overline{A}T$, $B = T^{-1}\overline{B}$, and $C = \overline{C}T$, meaning that $\{A, B, C, D\}$ and $\{\overline{A}, \overline{B}, \overline{C}, \overline{D}\}$ are indeed related by the similarity transformation T. This proves the minimal realization theorem.

9.3.2 Hankel Matrices and System Order

We have shown that for transfer function $g(s)$, a minimal realization is one that is of minimal order, and we know that this realization will be both controllable and observable. However, we have not yet shown that this order will be, as we have been implicitly assuming since Chapter 1, the same as the degree of the denominator in the SISO case, i.e., n. This can be shown with Markov parameters. We will do this explicitly only for the SISO situation.

We form an $(n_r \times n_c)$ matrix from the Markov parameters, called a *Hankel* matrix,[M] as follows: `hankel(C)`

$$H(n_r, n_c) = \begin{bmatrix} h(1) & h(2) & \cdots & h(n_c) \\ h(2) & h(3) & \cdots & h(n_c+1) \\ \vdots & \vdots & \ddots & \vdots \\ h(n_r) & h(n_r+1) & \cdots & h(n_r+n_c-1) \end{bmatrix}$$

$$= \begin{bmatrix} cb & cAb & \cdots & cA^{n_c-1}b \\ cAb & cA^2b & \cdots & cA^{n_c}b \\ \vdots & \vdots & \ddots & \vdots \\ cA^{n_r-1}b & cA^{n_r}b & \cdots & cA^{n_r+n_c-2}b \end{bmatrix} \tag{9.46}$$

The construction of this matrix is easier to justify for discrete-time systems than for continuous-time systems and will be presented in Section 9.5. For now, we assume that we can perform a power series expansion on the transfer function to obtain the Markov parameters analytically. The main result of this section can now be stated in terms of this Hankel matrix as the following theorem:

> THEOREM: Given a sequence of Markov parameters in terms of the system matrices in (9.33), the order n of the

transfer function that is realized by these matrices (and hence, the size of the minimal realization) is

$$n = \max_{n_r, n_c} r\left(H(n_r, n_c)\right) \tag{9.47}$$

i.e., the maximal rank of the Hankel matrix as more rows and columns are considered.

This theorem implies that if we begin constructing the so-called "infinite Hankel matrix,"

$$H(n_r, n_c) = \begin{bmatrix} h(1) & h(2) & h(3) & \cdots \\ h(2) & h(3) & & \\ h(3) & & \ddots & \\ \vdots & & & \end{bmatrix} \tag{9.48}$$

its rank will grow up to a maximum value of n, which will be the size of the minimal realization. (Why we would happen to have such a sequence of Markov parameters for a system as opposed to some other information will become clear in Section 9.5.) The theorem is easily proven by observing that if indeed we have a minimal realization (of order n), then by (9.36),

$$H(n, n) = QP \tag{9.49}$$

where Q and P are, respectively, the rank n observability and controllability matrices constructed from A, B, and C. If an additional row or column is added to (9.49), then by the Cayley-Hamilton theorem, that row or column must be linearly dependent on a previous row or column, so the rank of (9.48) cannot exceed n. Thus, $r\left(H(n_r, n_c)\right) = n$ for all n_r and $n_c > n$.

Realizations from Hankel Matrices

In addition to indicating the size of the minimal realization of a system, the Hankel matrix may be used to give a realization similar to those seen in Section 9.2. We will present this method as an algorithm applicable to MIMO as well as SISO systems, because it is considerably easier to treat single-variable systems as a special case than it would be to generalize to multivariable systems had we given a presentation that applied only to single-variable systems.

We therefore begin by assuming that we have available a Hankel matrix $H(n_r, n_c)$, with $n_r > n+1$, $n_c > n$, and n being the size of the minimal realization. If n is not known, we can successively test the rank of the Hankel matrix as we add rows and columns to it, until it appears to stop growing in rank. This might result in a realization that is *smaller* than the actual minimal realization, but it would probably qualify as a reasonable approximation, since it would lack only in high-order Markov parameters, i.e., the ones most likely to be negligible in the power series expansion in (9.31). It is quite reasonable to seek an m^{th}-order realization that approximates an n^{th}-order realization, $m<n$, especially if the transfer functions are similar. This is known as the *partial realization* problem and will be addressed again in Section 9.4. The algorithm is as follows:

1. Search the rows of the $(n_r p \times n_c q)$ (p inputs, q outputs) matrix $H(n_r, n_c)$, from top to bottom, for the first n linearly independent rows. (Again, if we do not know n, we can choose an integer likely to approximate n.) Denote these rows as rows $r_1, r_2, \cdots, r_n$. Gather these rows into a single $(n \times n_c q)$ matrix called H_r.
2. Gather into another $(n \times n_c q)$ matrix the rows $r_1+q, r_2+q, \cdots, r_n+q$ of $H(n_r, n_c)$ and call this matrix H_{r+q}. We asked that $n_r > n+1$ in order to have sufficient rows available to perform this step.
3. From matrix H_r, search from left to right for the first n linearly independent columns and refer to these as columns $c_1, c_2, \cdots, c_n$. Gather these columns together and call the resulting invertible $(n \times n)$ matrix M.
4. Create the $(n \times n)$ matrix M_A as columns $c_1, c_2, \cdots, c_n$ from the matrix H_{r+q}.
5. Create the $(n \times p)$ matrix M_B as the first p columns of matrix H_r.
6. Create the $(q \times n)$ matrix M_C as columns $c_1, c_2, \cdots, c_n$ from the matrix $H(1, n_c)$, i.e., from the first *block* row of the Hankel matrix.

From the matrices thus defined, the realization can be given as:

$$A \stackrel{\Delta}{=} M_A M^{-1} \qquad B \stackrel{\Delta}{=} M_B \qquad C \stackrel{\Delta}{=} M_C M^{-1} \tag{9.50}$$

Example 9.1: Realization from a Hankel Matrix

The SISO transfer function

$$g(s) = \frac{s+2}{s^2+4s+3} \tag{9.51}$$

can be expanded into a power series that yields the following sequence of Markov parameters:

$$\begin{aligned} h(0) &= 0 \\ h(1) &= 2 \\ h(2) &= -7 \\ h(3) &= 22 \\ h(4) &= -67 \\ h(5) &= 202 \\ &\vdots \end{aligned} \tag{9.52}$$

Find a realization from this set of Markov parameters.

Solution:

The fact that the Markov parameter $h(0)=0$ indicates that the system will have no feedthrough term, i.e., $d=0$. If we construct the Hankel matrices

$$H(1,1)=\begin{bmatrix}2\end{bmatrix} \qquad H(2,2)=\begin{bmatrix}2 & -7\\ -7 & 22\end{bmatrix} \qquad H(3,3)=\begin{bmatrix}2 & -7 & 22\\ -7 & 22 & -67\\ 22 & -67 & 202\end{bmatrix} \;\ldots \tag{9.53}$$

we find that $r\big(H(3,3)\big)=r\big(H(2,2)\big)=2$. We assume that a realization of order 2 is therefore desired. (In fact, it is easy to show from rank arguments on the expression in (9.46) that, for noiseless data, when the rank first stops growing, it is sufficient to stop; no further rank increase can be expected.) We will therefore seek a second-order realization. Using a Hankel matrix with an extra row, we will start with

$$H(2,3)=\begin{bmatrix}2 & -7\\ -7 & 22\\ 22 & -67\end{bmatrix} \tag{9.54}$$

Searching for the $n=2$ linearly independent rows of this matrix gives the result that $r_1=1$, $r_2=2$, and

$$H_r=\begin{bmatrix}2 & -7\\ -7 & 22\end{bmatrix}$$

Correspondingly, because $q = 1$,

$$H_{r+q} = \begin{bmatrix} -7 & 22 \\ 22 & -67 \end{bmatrix}$$

The first two linearly dependent columns of H_r are columns $c_1 = 1$ and $c_2 = 2$. Steps 3, 4, and 6 in the algorithm give, respectively,

$$M = \begin{bmatrix} 2 & -7 \\ -7 & 22 \end{bmatrix} \qquad M_A = \begin{bmatrix} -7 & 22 \\ 22 & -67 \end{bmatrix} \qquad M_C = \begin{bmatrix} 2 & -7 \end{bmatrix}$$

and the $p = 1^{\text{st}}$ column of H_r is

$$M_B = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

From all these values, we have the system matrices

$$A = M_A M^{-1} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \qquad b = M_B = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

$$c = M_C M^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad d = h(0) = 0\,, \tag{9.55}$$

These matrices give a second-order realization for the transfer function in (9.51).

9.4 Balanced Realizations

We now know that uncontrollable or unobservable modes do not contribute to the transfer function of a system, and that realizations of minimal order therefore result in completely controllable and observable models. We could, in principle, reduce the order of our nonminimal state space model by performing the Kalman decomposition and retaining only the fully controllable and observable subsystems, thus achieving *model reduction* without altering the input/output behavior of the system (i.e., the transfer function). However, it has been found that the controllability and observability of a system (and in particular, a mode) can depend on arbitrarily small variations in the parameters of the system matrices. That is, arbitrarily small perturbations of a parameter might change a controllable system to an uncontrollable system, thereby changing a minimal realization to a nonminimal one, without significantly changing the transfer function. However, up to this point, we have no way of determining which mode is *more* or *less* controllable and/or observable.

balreal(sys)

We now investigate further this notion of model reduction with a special realization known as a *balanced realization*[M] [13]. The balanced realization identifies the modes that are *almost* unobservable and uncontrollable, making them candidates for deletion. However, as revealed by the next section, we must be careful to delete only the modes that are almost *both* uncontrollable and unobservable.

9.4.1 Grammians and Signal Energy

gram(sys,'c')

gram(sys,'o')

Restricting our discussion to continuous-time LTI systems, we will use the controllability and observability grammians[M] discussed in Chapter 8. The continuous-time grammians, Equations (8.57) and (8.69), are defined as

$$G_c(t_0, t_1) = \int_{t_0}^{t_1} e^{A(t_0 - t)} B B^{\mathrm{T}} e^{A^{\mathrm{T}}(t_0 - t)}\, dt \tag{9.56}$$

and

$$G_o(t_0, t_1) = \int_{t_0}^{t_1} e^{A^{\mathrm{T}}(t - t_0)} C^{\mathrm{T}} C e^{A(t - t_0)}\, dt \tag{9.57}$$

respectively. From Chapter 8, the system is controllable and observable in the interval $[t_0, t_1]$ if and only if these two symmetric matrices are invertible.

Suppose we are given an arbitrary initial state $x(t_0) = x_0$ and wish to find a control that drives the system from this state to any other state $x(t_1) = x_1$. It is easily verified by substitution into the state equation solution, i.e., Equation (6.6), that a control signal that accomplishes this can be written as:

$$u(t) = B^{\mathrm{T}} e^{A^{\mathrm{T}}(t_0 - t)} G_c^{-1}(t_0, t_1) \left[e^{A(t_0 - t_1)} x_1 - x_0 \right] \tag{9.58}$$

[a similar control was used in Equation (8.58)]. If we define the *energy* content of this control signal as

$$E_u = \int_{t_0}^{t_1} \|u(t)\|^2\, dt = \int_{t_0}^{t_1} u^{\mathrm{T}}(t) u(t)\, dt \tag{9.59}$$

then it can be shown that the energy of the particular control signal in (9.58) is smaller than or equal to the energy of any other control signal that accomplishes the same transfer from x_0 to x_1 in the same time [4]. This energy becomes

$$\begin{aligned} E_u &= \int_{t_0}^{t_1} u^{\mathrm{T}}(t)u(t)\,dt \\ &= \int_{t_0}^{t_1} \left[[e^{A(t_0-t_1)}x_1 - x_0]^{\mathrm{T}} G_c^{-1}(t_0,t_1) e^{A(t_0-t)} B \right] \\ &\qquad \cdot \left[B^{\mathrm{T}} e^{A^{\mathrm{T}}(t_0-t)} G_c^{-1}(t_0,t_1)[e^{A(t_0-t_1)}x_1 - x_0] \right] dt \\ &= [e^{A(t_0-t_1)}x_1 - x_0]^{\mathrm{T}} G_c^{-1}(t_0,t_1) \left[\int_{t_0}^{t_1} e^{A(t_0-t)} BB^{\mathrm{T}} e^{A^{\mathrm{T}}(t_0-t)}\, dt \right] \\ &\qquad \cdot G_c^{-1}(t_0,t_1)[e^{A(t_0-t_1)}x_1 - x_0] \\ &= [e^{A(t_0-t_1)}x_1 - x_0]^{\mathrm{T}} G_c^{-1}(t_0,t_1)[e^{A(t_0-t_1)}x_1 - x_0] \end{aligned}$$

Taking $t_0 = 0$ and $x_1 = 0$, then the energy to reach the origin in time t_1 becomes simply

$$E_u = x_0^{\mathrm{T}} G_c^{-1}(0,t_1)x_0 \tag{9.60}$$

In a similar fashion, suppose we wish to compute the energy in the output as the system decays from the initial condition x_0 to zero in the absence of any input. This energy would be:

$$\begin{aligned} E_y &= \int_0^{t_1} \|y(t)\|^2\, dt \\ &= \int_0^{t_1} x_0^{\mathrm{T}} e^{A^{\mathrm{T}}t} C^{\mathrm{T}} C e^{At} x_0\, dt \\ &= x_0^{\mathrm{T}} G_o(0,t_1)x_0 \end{aligned} \tag{9.61}$$

Thus, the control energy and the "observation" energy are quadratic forms on the initial condition. Because the grammians are both symmetric positive-definite, we can interpret the expressions in (9.60) and (9.61) according the geometric interpretation of quadratic forms as discussed in Chapters 4 and 5. Recall that a symmetric positive-definite quadratic form defines the surface of an ellipsoid whose major axes lie along the eigenvectors, with lengths along those axes equal to the corresponding eigenvalues. Equation (9.60) implies that certain initial conditions require more energy to control than others, namely, those at the long axes of the ellipsoid defined by the *inverse* controllability grammian. Initial conditions at such locations are "harder" to control than others. Equation (9.61) implies that certain initial conditions provide more observation energy or are "easier" to observe. These conditions are at the long end of the ellipsoid defined by the observability grammian.

In general, though, the controllability and observability grammians are independent of each other, so a mode that is easy to control might be difficult to observe (i.e., it might take little control energy, but also contributes little to the output) or vice versa. This is why we cannot simply use grammians to deduce that modes that are almost uncontrollable *or* almost unobservable do not contribute to the transfer function; on the contrary, their corresponding observability or controllability properties may give just the opposite effect.

9.4.2 Internal Balancing

The definitions for the grammians, Equations (9.56) and (9.57), are often given with limits of integration of 0 and ∞, which results in no loss of generality for time-invariant systems. If this is the case and if the system is asymptotically stable, then the grammians can also be found as the positive-definite solutions to the two Lyapunov equations[M]

`lyap(A,Q)`

$$AG_c + G_c A^{\mathrm{T}} = -BB^{\mathrm{T}} \tag{9.62}$$

and

$$A^{\mathrm{T}} G_o + G_o A = -C^{\mathrm{T}} C \tag{9.63}$$

Using these equations, we will investigate the effect of a (not necessarily orthonormal) similarity transformation, $\tilde{x} = Tx$. We already know that the state matrices transform as

$$\tilde{A} = T^{-1}AT \qquad \tilde{B} = T^{-1}B \qquad \tilde{C} = CT \tag{9.64}$$

In the new basis, the Lyapunov equation that gives the controllability grammian becomes

$$\begin{aligned} &\tilde{A}\tilde{G}_c + \tilde{G}_c\tilde{A}^{\mathrm{T}} = -\tilde{B}\tilde{B}^{\mathrm{T}} \\ &T^{-1}AT\tilde{G}_c + \tilde{G}_c T^{\mathrm{T}} A^{\mathrm{T}} \left(T^{-1}\right)^{\mathrm{T}} = -T^{-1}BB^{\mathrm{T}}\left(T^{-1}\right)^{\mathrm{T}} \end{aligned} \tag{9.65}$$

Multiplying this equation from the left by T and from the right by T^{T}, we get

$$A\left(T\tilde{G}_cT^{\mathrm{T}}\right)+\left(T\tilde{G}_cT^{\mathrm{T}}\right)A^{\mathrm{T}}=-BB^{\mathrm{T}}$$

implying that

$$\begin{aligned}G_c&=T\tilde{G}_cT^{\mathrm{T}}\\ \tilde{G}_c&=T^{-1}G_c\left(T^{-1}\right)^{\mathrm{T}}\end{aligned}\tag{9.66}$$

Performing analogous steps on the observability grammian and on Equation (9.63) results in

$$\begin{aligned}G_o&=\left(T^{-1}\right)^{\mathrm{T}}\tilde{G}_oT^{-1}\\ \tilde{G}_o&=T^{\mathrm{T}}G_oT\end{aligned}\tag{9.67}$$

From (9.66) and (9.67), it is apparent that the controllability and observability grammians can be scaled by a similarity transformation, so their eigenvalues can be changed by appropriate selection of transformation T. Note however that their product

$$\begin{aligned}\tilde{G}_c\tilde{G}_o&=T^{-1}G_c\left(T^{-1}\right)^{\mathrm{T}}T^{\mathrm{T}}G_oT\\ &=T^{-1}G_cG_oT\end{aligned}\tag{9.68}$$

which implies that the eigenvalues of the product G_cG_o are the same as the eigenvalues of the product $\tilde{G}_c\tilde{G}_o$. There therefore exists a trade-off between the controllability ellipsoid and the observability ellipsoid.

In order to exploit this trade-off, we seek a transformation T that results in

$$\tilde{G}_c=\tilde{G}_o=\Sigma$$

where Σ is a diagonal matrix containing the square roots of the eigenvalues of the product G_cG_o. This will accomplish two things. First, because the new grammians will be diagonal, their major axes will be aligned with the coordinate axes in the new basis. Second, because the grammians will be equal, each mode (i.e., along each coordinate axis) will be just *as controllable as it is observable*. Modes with large control energies will have small observation energies, and vice versa. Modes with large control energies, being *almost uncontrollable*, and with small observation energies, being *almost unobservable*, will thus be identified. They can then be assumed to have a relatively small effect on the transfer function and might be deleted from a realization. Such a realization is said to be *internally balanced*. It is important to note that we are restricted to stable systems, so that we are safe from discarding unstable modes.

In [13], an algorithm to accomplish this balancing is given as follows:

1. Determine the singular value decomposition of the positive-definite matrix G_c. Because of its symmetry, the result will be of the form $G_c = M_c \Sigma_c M_c^T$, where Σ_c is the diagonal matrix of eigenvalues of G_c.

2. Define a preliminary transformation as $T_1 = M_c \Sigma_c^{1/2}$ and apply this transformation[M] to the original realization $\{A, B, C\}$ to get $\{A_1, B_1, C_1\} = \{T_1^{-1} A T_1, T_1^{-1} B, C T_1\}$.

ss2ss(sys,T)

3. Compute the new observability grammian $\hat{G}_o$ from the transformed system and determine its singular value decomposition $\hat{G}_o = M_o \hat{\Sigma}_o M_o^{\mathrm{T}}$, where $\hat{\Sigma}_o$ is a diagonal matrix of the eigenvalues of $\hat{G}_o$.

4. Define the transformation $T_2 = M_o \Sigma_o^{-1/4}$. Apply this new transformation to the previously transformed system to result in a composite transformation $T = T_1 T_2$. The balanced realization is then

$$
\begin{aligned}
\tilde{A} &= T^{-1} A T = T_2^{-1} T_1^{-1} A T_1 T_2 \\
\tilde{B} &= T^{-1} B = T_2^{-1} T_1^{-1} B \\
\tilde{C} &= C T = C T_1 T_2
\end{aligned}
\tag{9.69}
$$

Suppose that when a system is thus balanced, the transformation matrices are arranged such that $\tilde{G}_c = \tilde{G}_o = \Sigma = diag(\sigma_1, \ldots, \sigma_j, \sigma_{j+1}, \ldots, \sigma_n)$, with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Suppose now that $\sigma_j \gg \sigma_{j+1}$. Then it can be concluded that modes corresponding to state variables $\tilde{x}_{j+1}$ through $\tilde{x}_n$ are much less controllable *and* observable than are modes corresponding to variables $\tilde{x}_1$ through $\tilde{x}_j$. Then in the partitioned state equations

$$
\begin{bmatrix} \dot{\tilde{x}}_1 \\ \vdots \\ \dot{\tilde{x}}_j \\ \hdashline \dot{\tilde{x}}_{j+1} \\ \vdots \\ \dot{\tilde{x}}_n \end{bmatrix}
=
\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{12} & \tilde{A}_{22} \end{bmatrix}
\begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_j \\ \hdashline \tilde{x}_{j+1} \\ \vdots \\ \tilde{x}_n \end{bmatrix}
+
\begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u
$$

$$y = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_j \\ \hdashline \tilde{x}_{j+1} \\ \vdots \\ \tilde{x}_n \end{bmatrix} \tag{9.70}$$

the true, exact transfer function $H(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}$ can be approximated by the transfer function of the simplified, reduced system:

$$\tilde{H}(s) = \tilde{C}_1(sI - \tilde{A}_{11})^{-1}\tilde{B}_1 \tag{9.71}$$

This system can also be shown to be asymptotically stable. (Note that the same concepts apply to discrete-time systems, but the computations are different because of the different Lyapunov equations. See [14].)

9.5 Discrete-Time System Identification

We close this chapter with a discussion of one of the most useful applications of Markov parameters: discrete-time system identification. System identification is the name given to the process of determining a system description (e.g., the matrices *A*, *B*, and *C*) from input/output data (the feedthrough matrix *D* is trivially determined by direct measurement). There are many approaches to this problem, including both frequency-domain (e.g., curve-fitting frequency responses) and time-domain (i.e., finding a realization). System identification has become an important topic for controls engineers, because often in practical problems the description of the plant is not analytically available. Without a mathematical model for the plant, most control strategies are useless.

The methods we use here follow directly from the definitions and uses of Markov parameters for realizations as discussed above. In fact, the algorithm given here can be used interchangeably with the algorithm given in Section 9.3.2; they will however result in different realizations. In Section 9.3.2, we assumed that the Markov parameters were available as a result of a series expansion on the transfer function. Here, we assume that we do not know the transfer function but that we have access to input and output data. We consider discrete-time systems because in most situations, empirical input and output measurements are obtained at discrete-time instants. It is therefore natural to seek a discrete-time model, even if the plant itself is continuous.

9.5.1 Eigensystem Realization Algorithm

For the first method we will use a sequence of noiseless measurements of input/output pairs of a discrete-time system [7]. As in Example 3.11, we can assume a zero-state response and generate, in response to the known input sequence $u(0), u(1), \cdots$, the following output sequence:

$$\begin{aligned} y(0) &= Du(0) \\ y(1) &= CBu(0) + Du(1) \\ y(2) &= CABu(0) + CBu(1) + Du(0) \\ &\vdots \end{aligned} \tag{9.72}$$

or

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(j-1) \end{bmatrix} = \begin{bmatrix} D & 0 & 0 & 0 & 0 \\ CB & D & 0 & 0 & 0 \\ CAB & CB & D & 0 & 0 \\ \vdots & \ddots & \ddots & D & 0 \\ CA^{j-1}B & \cdots & CAB & CB & D \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ u(2) \\ \vdots \\ u(j-1) \end{bmatrix} \tag{9.73}$$

Clearly, in this case, the natural form of the input/output data in discrete-time is the Markov parameter. Given such pairs of input/output data, we can rearrange (9.73) to give the formula:

$$\begin{aligned} &\begin{bmatrix} y(0) & y(1) & \cdots & y(j-1) \end{bmatrix} \\ &= \begin{bmatrix} h(0) & h(1) & \cdots & h(j-1) \end{bmatrix} \begin{bmatrix} u(0) & u(1) & \cdots & u(j-1) \\ 0 & u(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & u(1) \\ 0 & \cdots & 0 & u(0) \end{bmatrix} \end{aligned}$$

Defining this equation symbolically as $Y \stackrel{\Delta}{=} H_1 U$, then if the matrix U is invertible, we can solve for the Markov parameters:

$$H_1 = \begin{bmatrix} h(0) & h(1) & \cdots & h(j-1) \end{bmatrix} = YU^{-1} \tag{9.74}$$

provided, of course, that matrix U is indeed invertible. The invertibility of U is related to the question of whether the input is *sufficiently exciting* for the equation to be solved. If our simple system is single input, then we clearly would require the property $u(0) \neq 0$.

Even if the initial condition on the state is nonzero, the input/output sequence is still sufficient for identifying the system. This is because we will be computing a minimal realization, which is therefore controllable and observable. For a controllable and observable system, we will always be able to *make* the initial state zero by choosing appropriate input signals in negative time. Then because the starting time for a time-invariant system is irrelevant for identification purposes, we can include the negative-time input/output data into (9.73) above while estimating the system matrices *A*, *B*, and *C* [4].

Thus, assuming that the Markov parameters for a MIMO system are known, we can begin constructing, after *k* samples have been acquired, the $(kq \times kp)$ block Hankel matrix as in Equation (9.46):

$$\begin{aligned} H_k = H(k,k) &= \begin{bmatrix} CB & CAB & \cdots & CA^{k-1}B \\ CAB & CA^2B & & \\ \vdots & & \ddots & \vdots \\ CA^{k-1}B & & \cdots & CA^{2(k-1)}B \end{bmatrix} \\ &= \begin{bmatrix} H(1) & H(2) & \cdots & H(k) \\ H(2) & H(3) & & \\ \vdots & & \ddots & \vdots \\ H(k) & & \cdots & H(2k-1) \end{bmatrix} \end{aligned} \tag{9.75}$$

As mentioned before, we will increase the size of this matrix by acquiring new data until the rank of (9.75) stops growing. When this occurs, the order of the minimal realization will be equal to the order of the block Hankel matrix when it stops growing, i.e., $n = k$ such that $n = r(H_k) = r(H_{k+1}) = r(H_{k+2}) = \cdots$. According to the eigensystem realization algorithm (ERA) [7], the next step is to perform a singular value decomposition on H_k:

$$H_k = U\Sigma V^{\mathrm{T}} \tag{9.76}$$

To carry out the rest of the algorithm, we compute the following matrices:

- U_1 $(kq \times n)$ is defined as the first *n* columns from the $(kq \times kq)$ matrix *U*.
- V_1 $(kp \times n)$ is defined as the first *n* columns from the $(kp \times kp)$ matrix *V*.
- U_{11} $(q \times n)$ is defined as the first *q* rows of the matrix U_1 (where *q* is the number of outputs).
- V_{11} $(p \times n)$ is defined as the first *p* rows of the matrix V_1 (where *p* is the number of inputs).

- S is defined as the $(n \times n)$ invertible, diagonal matrix as in Equation (4.34):

$$\Sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$$

where $S = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$.

- $\tilde{H}_k$ is defined as the modified Hankel matrix

$$\tilde{H}_k = \begin{bmatrix} H(2) & H(3) & \cdots & H(k+1) \\ H(3) & H(4) & & \\ \vdots & & \ddots & \vdots \\ H(k+1) & & \cdots & H(2k) \end{bmatrix}$$

Given these computations, the system matrices may be derived as

$$\begin{aligned} A &= S^{-1/2} U_1^{\mathrm{T}} \tilde{H}_k V_1 S^{-1/2} \\ B &= S^{1/2} V_{11}^{\mathrm{T}} \\ C &= U_{11} S^{1/2} \end{aligned} \tag{9.77}$$

Again, if there is a feedthrough term, it is measured directly, just like the Markov parameters.

9.5.2 Simplified ERA Identification

The basic procedure spelled out in the algorithm above can be altered to give different (minimal) realizations for the same set of Markov parameters [5]. The only modification to the ERA algorithm above is that instead of first performing the SVD on the block Hankel matrix, only selected rows and columns of the Hankel matrix are used; this step is similar to the procedure used for the realization based on the Hankel matrix in Section 9.3.2. The algorithm then offers some flexibility in the form of the resulting realization. The steps in this modified algorithm are:

1. Choose n linearly independent columns of H_k. Arrange them into a matrix defined as H_1. Denote the chosen columns as $\nu_1, \nu_2, \ldots, \nu_n$.

2. Select from H_k the columns $\nu_1 + p, \nu_2 + p, \ldots, \nu_n + p$ and arrange them into a matrix called H_2.

3. The system matrices in a special canonical form (the "pseudo-controllable canonical form") are computed as follows:

$$A = \left(H_1^{\mathrm{T}} H_1\right)^{-1} H_1^{\mathrm{T}} H_2$$
$$B = \begin{bmatrix} I_{p\times p} \\ 0_{(n-p)\times p} \end{bmatrix} \tag{9.78}$$

where A is the least-squares solution to the equation $H_1 A = H_2$ and C is the first q rows of H_1.

The flexibility offered by this approach is that the choice of the particular columns of H_k can be guided by special integers known as (*pseudo*)*controllability indices*. These numbers pertain to the number of modes that can be controlled by each individual input, i.e., each column of the *B*-matrix. There may be several sets of controllability indices so the designer will have certain freedoms in selecting the form of the resulting system matrices. In fact, an analogous procedure can be written in terms of the (*pseudo*)*observability indices* and the *rows* of the Hankel matrix. However, because we have not yet encountered controllability and observability indices, we will not delve further into these options. Although this algorithm does indeed offer such flexibilities, the computational complexity is comparable to the original ERA, because the best way to accurately compute (9.78) is through the SVD.

9.6 Summary

In this chapter we have tied together the two most common representations for linear systems: the transfer function and the state space description. In essence, the material presented in this chapter represents the more rigorous theory behind the material we presented in Chapter 1. In doing so, we have answered some questions posed or suggested in that chapter. For example, we have discovered why transfer functions derived from state matrices do not necessarily have degree equal to the order of the *A*-matrix. We also learned that there are as many different realizations as there are similarity transformations (infinite).

Perhaps more importantly, this chapter introduces the practical techniques necessary for the actual physical construction of linear systems (and, as we shall see, their controllers). By constructing the realizations herein, time-domain systems can be modeled from frequency-domain or empirical data. (Usually, these are implemented on a computer, although analog state variable controllers are becoming more common.) This led us naturally into the subject of system identification, which is itself the topic of many books and papers, e.g., [10] and [11]. The small subset of system identification procedures mentioned here is the construction of a minimal realization from the Markov parameters that naturally result from the measurement of input and output data in a discrete-time system.

Summarizing the important results of this chapter:

- It was found that uncontrollable and/or unobservable modes do not contribute to the transfer function of a system, so that if it is only the input/output behavior that is of interest, such modes (provided they are stable) can be discarded from the state space model. Thus, we have a *minimal* realization when the model is both controllable and observable.

- Several specific realizations were shown, including the controllable canonical, observable canonical, and Jordan canonical forms. We see from these realizations that there is a direct correspondence between similarity transformations and integrator realizations (i.e., simulation diagrams). The further significance of the controllable and observable canonical realizations will be revealed in the next chapter.

- Markov parameters were introduced as the coefficients of a power series expansion of the (causal) transfer function. It was seen that Markov parameters are invariant to similarity transformations. Arranged into the Hankel matrix, Markov parameters provide a means of generating a realization either from this series expansion or from empirical input/output data from discrete-time systems.

- The Hankel matrix itself is also shown to be the product of the observability and controllability matrices for a realization. Its rank, as more rows and columns are added to it, grows to a maximal size that is equal to the order of the minimal realization. An algorithm is given to generate state matrices based on the Hankel matrix.

- Balanced realizations were introduced to tie together the ideas of controllability, observability, and system order. If a partial realization, i.e., one with smaller order than the minimal realization, is desired, a balanced realization can be computed that illustrates the extent to which individual modes are almost uncontrollable and unobservable. Balanced realizations are particularly important to the study of large-scale systems, where model reduction is sometimes the first order of business.

- As we have mentioned, system identification is sometimes the first step in the design of a controller for a practical physical system. Many times, systems are either too complex or too poorly modeled to provide an analytical description of a plant. In these situations, it is first necessary to perform system identification so that a working model is available as the basis for controller design. By acquiring input/output data in discrete-time, a sequence of Markov parameters can be used to generate a model for "blackbox" data. Although this technique nicely illustrates some of the contents of this chapter, it is neither the most efficient nor common method for performing realistic system identification.

In the next chapter, we will at last be able to use the tools we have developed so far to design controllers and observers. The concepts of control via state feedback and observation are fundamental to time-domain linear system studies, and we will explore them in Chapter 10.

9.7 Problems

9.1 Find one minimal and one nonminimal realization for the transfer function

$$g(s)=\frac{s^3+6s^2+3s-10}{s^3+4s^2+s-6}$$

9.2 Find a minimal realization for the transfer function

$$g(z)=\frac{z+3}{z^2+7z+10}$$

9.3 Show using the Kalman observability decomposition that the unobservable part of a realization does not contribute to the transfer function for that system. [Similar to the construction of Equation (9.14).]

9.4 Given below are two different realizations.

a) Show that they are each minimal realizations of the same system.

b) Find the similarity transformation matrix T that relates the two.

$$A_1=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -12 & -19 & -8\end{bmatrix} \qquad b_1=\begin{bmatrix}0 & 1\\ 1 & 1\\ 0 & -1\end{bmatrix} \qquad c_1=\begin{bmatrix}1 & 0 & 0\\ -1 & 0 & 0\end{bmatrix}$$

$$A_2=\begin{bmatrix}-90 & 29 & -55\\ -3 & 1 & -2\\ 133 & -42 & 81\end{bmatrix} \qquad b_2=\begin{bmatrix}-1 & -2\\ 0 & 1\\ 1 & 3\end{bmatrix} \qquad c_2=\begin{bmatrix}0 & 1 & 0\\ 0 & -1 & 0\end{bmatrix}$$

9.5 Compute a minimal realization for the sequence of Markov parameters $h(i) = 0, 1, -4, 16, -64, 256, \ldots$, for $i = 0, 1, 2, \ldots$.

9.6 A Fibonacci sequence is a sequence of numbers, each of which is the sum of the previous two, for example: $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$. Consider these numbers as a sequence of Markov parameters and determine a discrete-time state variable system that produces such a sequence. Express the result as a difference equation.

9.7 The *Padé approximation* used to approximate a time-delay with a transfer function [5] can be viewed as a partial realization problem that uses the Hankel matrix methods outlined in this chapter. Use these methods to determine a second-order Padé approximation to the time delay operator e^{-Ts}. (Hint: expand e^{-Ts} into a McClaurin series in terms of the parameter $\sigma = 1/s$.)

9.8 Write a MATLAB routine (or code in another language) to perform the realization based on a Hankel matrix as in Section 9.3.2.

9.9 Write a MATLAB routine (or code in another language) to perform the ERA as in Section 9.5.1.

9.10 Write MATLAB code (or code in another language) that can be used to determine the Markov parameters from an arbitrary proper transfer function. (Hint: Use the DECONV[M] command.)

`deconv(p,q)`

9.11 For a SISO time-invariant system, show that a realization based on Markov parameters can be written as

$$A = \begin{bmatrix} h(2) & h(3) & \cdots & h(n+1) \\ h(3) & h(4) & & \\ \vdots & & \ddots & \vdots \\ h(n+1) & & \cdots & h(2n-1) \end{bmatrix} \begin{bmatrix} h(1) & h(2) & \cdots & h(n) \\ h(2) & h(3) & & \\ \vdots & & \ddots & \vdots \\ h(n) & & \cdots & h(2n) \end{bmatrix}^{-1}$$

$$b = \begin{bmatrix} h(1) \\ h(2) \\ \vdots \\ h(n) \end{bmatrix} \qquad c = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \qquad d = h(0)$$

9.12 Show that if the LTI system $\dot{x} = Ax + Bu$ is controllable and asymptotically stable, then there exists a symmetric, positive-definite solution G to the Lyapunov equation

$$AG + GA^{\mathrm{T}} = -BB^{\mathrm{T}}$$

9.13 Show that if the LTI system $x(k+1) = A_d x(k) + B_d u(k)$ is reachable and asymptotically stable, then there exists a symmetric, positive-definite solution G to the Lyapunov equation

$$AGA^{\mathrm{T}} - G = -BB^{\mathrm{T}}$$

9.14 Use Hankel matrix methods to derive realizations for the following transfer functions and draw the integrator realizations for each.

$$G_1(s) = \begin{bmatrix} \dfrac{s}{(s+1)^2} \\ \dfrac{(s+3)}{(s+1)(s+2)} \end{bmatrix} \qquad G_2(s) = \begin{bmatrix} \dfrac{s}{(s+1)^2} & \dfrac{1}{(s+1)^2(s+2)} \\ \dfrac{(s+3)}{(s+1)(s+2)} & \dfrac{(s-2)}{(s+2)} \end{bmatrix}$$

9.15 Show that step 2 in the model-balancing algorithm results in a transformed controllability grammian

$$\hat{G}_c = T_1^{-1} G_c \left(T_1^{-1}\right)^{\mathrm{T}} = I$$

9.16 Perform a model balancing on the following system:

$$\dot{x} = \begin{bmatrix} -3 & 5 & 4 & 0 & 0 \\ 0 & -4 & 2 & 0 & 1 \\ 2 & 0 & -5 & 1 & 0 \\ -2 & 3 & 0 & -8 & 1 \\ 2 & 0 & 3 & 0 & -9 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 1 & 0 & 1 & 0 \end{bmatrix}$$

9.17 The following set of input/output data is collected from a single-input, two-output discrete-time system. Determine a realization and a transfer function that identify the system.

k	0	1	2	3	4	5	6	7	8	9
$u(k)$	1	0.5	0	0.5	1	0.5	0	.5	1	0.5
$y_1(k)$	1	1.5	0.5	1	1.25	1.5	0.375	0.875	1.3125	1.5
$y_2(k)$	0	1	1.5	0.5	0.25	0.5	1.125	0.375	0.3125	0.375

9.8 References and Further Reading

The realization theory we present here can be found in its original form in Kalman [9]. Our discussion of the minimal realization and the significance of controllability and observability are based on that work. It is also discussed in similar terms in the texts [4], [8], and [16].

For a good introduction to Markov parameters and Hankel matrix methods, see [4], [8], [16], and [17]. In addition to obtaining the realizations themselves, we have applied these methods to system identification problems (using the ERA algorithm). However, system identification is a much broader science, and the interested reader should consult [10] and [11]. The ERA algorithm itself can be pursued in further detail, in its original description [7] and in some of the variations and enhancements of it in [1], [2], and [5]. Model balancing and balanced realizations were introduced by Moore in [13].

As was briefly mentioned in Chapter 1, we do not discuss most of the details of frequency-domain representations in this book. However, considerable literature is available, including introductory but comprehensive treatments in [3], [4], [8], [12], and [15].

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