10

State Feedback and Observers

We are now ready to consider the *control* of state space systems. In an overly general but realistic way, we can define the term "control" as the alteration of a dynamic system such that it behaves as if it were a different dynamic system. Often, the motivation for control results from more specific criteria, such as stability, the tuning of a transient response, the reduction of error in the output, or improvement in the system's tolerance to disturbance and unmodeled dynamics. Nevertheless, control is achieved through the introduction of a controller, or compensator, which changes the equations to achieve a desired behavior.

In this chapter, we introduce the most fundamental form of control for state space systems: *state feedback*. Using state feedback, we will be able to change the *A*-matrix of a system, under certain conditions. This will require access to the state variables, which sometimes are not available in physical systems. However, we can reconstruct the state variables using a construct known as an *observer*. As might well be guessed from these introductory remarks, the conditions of controllability and observability will be necessary for the design of controllers and observers.

We will consider in this chapter only time-invariant systems, both continuous-time and discrete-time. At the end of the chapter, we will have discussed all the necessary tools for the design of state-feedback controllers for SISO and MIMO systems. Throughout, we should keep in mind that the concept of observers and controllers is not necessarily an exercise in electronics or mechanical design, but is applicable to mathematical models that are quite broad. For example, state space models (as well as observers and controllers) have been developed for physiological systems, economic models, and social behaviors. It is interesting to consider what the consequences of *control* and *observations* are in those domains [2].

10.1 State Feedback for SISO Systems

The concepts and procedures for state feedback are the same for both continuous-time and discrete-time systems, although there are some distinctions in the results that might be achieved with each. We will therefore present state feedback in continuous-time, as the notation is more compact.

To begin, we resort to an example that will be familiar to readers with any classical controls systems experience. Consider the control configuration in Figure 10.1.

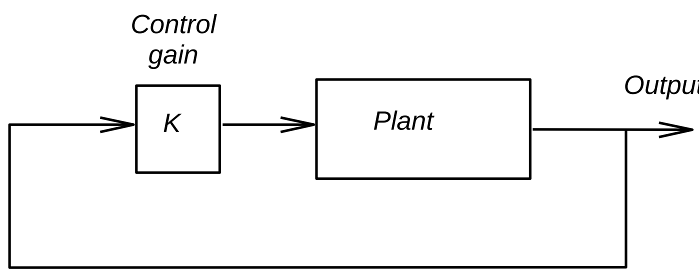

Figure 10.1 Basic regulator configuration for scalar systems.

This is the so-called “regulator system,” wherein there is no input to track; there is only a plant output that we wish to maintain at zero. In this type of control, the output is fed back to the input and multiplied by a gain factor. By changing the value of the feedback gain K, the poles of the plant can be moved to different locations in the s- (or z-)plane. Techniques such as root locus may be used to determine the sets on which the poles may move.

If the system in Figure 10.1 is scalar, i.e., one-dimensional, then the output can be made equal to the single state variable, and the location of the single pole can be moved arbitrarily along the real line by choice of gain K. Continuing this idea to state space notation, consider the SISO first-order system

$$\begin{aligned} \dot{x} &= ax + bu \\ y &= cx + du \end{aligned} \tag{10.1}$$

This system can be depicted in the simple block diagram shown in Figure 10.2. We cannot directly access the summing junction at the input, but we can connect the input signal u to a feedback path. That is, we can feed the output y back to the input u, or, under certain circumstances, we may have access to the state variable x and the ability to feed it back to the input. Feeding y back to u is known as *output feedback*, and feeding x back to u is known as *state feedback*. This is shown in the dotted connection in Figure 10.2.

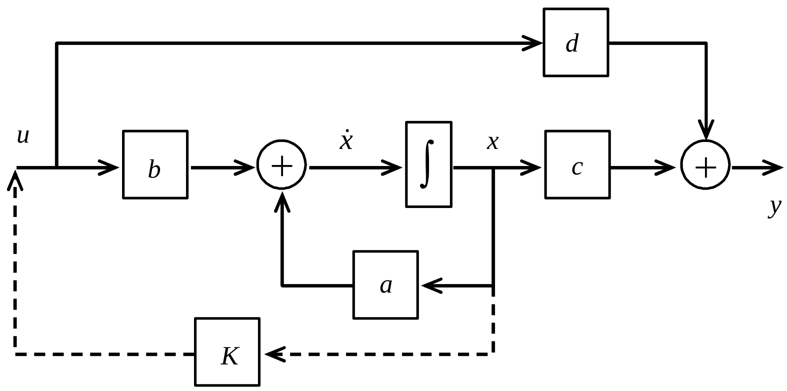

Figure 10.2 Block diagram for a scalar state space system.

State feedback is expressed by the assignment $u = Kx$, and by rewriting the state equations in the presence of such feedback, we get the "closed-loop" system

$$\begin{aligned} \dot{x} &= ax + bKx = (a + bK)x \\ y &= cx + dKx = (c + dK)x \end{aligned} \tag{10.2}$$

It is clear from (10.2) that the behavior of the system will be altered by the introduction of state feedback, and that the single eigenvalue (pole) is dependent on the feedback gain *K*. It is apparent that by appropriately choosing $K = (a^{\text{des}} - a)/b$, where a^{des} is the *desired* eigenvalue, feedback can be used to give the system any pole at all, provided that $b \neq 0$, which is equivalent to asking that the system be controllable.

Extending this idea further, we can consider nonscalar systems. In general time-invariant state space systems, we propose to use feedback of the form

$$u(t) = Kx(t) \tag{10.3}$$

where, as usual, $x \in \Re^n$, so *K* will be a $1 \times n$ matrix of *feedback gains*. More generally, because we may want to preserve the possibility of an exogenous input, which, like $u(t)$ in the original system, might be multiplied by a system matrix, we model the state feedback input as

$$u(t) = Kx(t) + Ev(t) \tag{10.4}$$

where $v(t)$ is the exogenous (i.e., externally applied) input and *E* is its input matrix, just as *B* is the input matrix to signal $u(t)$.

Substituting (10.4) into the general state equations

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx + du \end{aligned} \tag{10.5}$$

we arrive at

$$\begin{aligned} \dot{x} &= Ax + b(Kx + Ev) = (A + bK)x + bEv \\ y &= cx + d(Kx + Ev) = (c + dK)x + dEv \end{aligned} \tag{10.6}$$

whose transfer function is consequently

$$H(s) = (c + dK)\left(sI - A - bK\right)^{-1} bE + dE \tag{10.7}$$

As is usually the case, K is assumed here to be constant (i.e., "static" state feedback). The block diagram of the new system is pictured in Figure 10.3.

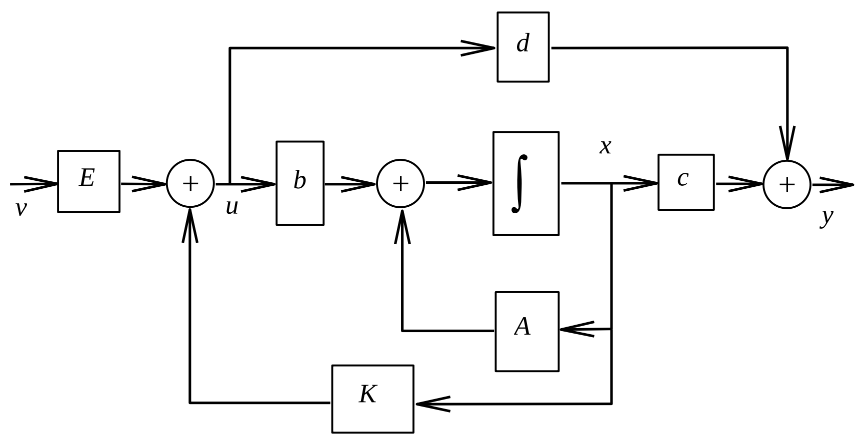

Figure 10.3 Block diagram of state feedback in a state space system.

We know from our stability studies of Chapter 7 that asymptotic stability depends only on the eigenvalues of a time-invariant system, and we also know from the behavior of differential equations that a system's transient response is also dependent on the eigenvalues (or poles, if the realization is minimal). Therefore, we need to study the procedure by which gain matrix K can be found such that the matrix $A + bK$ has a desired set of eigenvalues. Because this matrix does not appear in the output equation for y, we can concentrate on the state equation alone: $\dot{x} = (A + bK)x + bEv$.

10.1.1 Choosing Gain Matrices

The task of selecting an appropriate gain matrix K is facilitated by the controllable canonical form. Recall that the state matrices in this canonical form are:

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + du(t) \tag{10.8}$$

Using state feedback expanded explicitly as

$$u = Kx = \begin{bmatrix} k_0 & k_1 & \cdots & k_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{10.9}$$

(and neglecting temporarily the exogenous input term and the output equation), we can substitute (10.9) into (10.8) to achieve

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_0 & k_1 & \cdots & k_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \\ k_0 & k_1 & \cdots & k_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ -a_0 + k_0 & -a_1 + k_1 & \cdots & -a_{n-1} + k_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \end{aligned} \tag{10.10}$$

Knowing that the bottom row of constant coefficients in this matrix is the reverse-ordered set of negative coefficients in the characteristic polynomial for the system, it is clear from (10.10) that the gain matrix K allows us to completely specify the terms in the characteristic polynomial

$$\phi(s) = s^n + (a_{n-1} - k_{n-1})s^{n-1} + \cdots + (a_1 - k_1)s + (a_0 - k_0) \qquad (10.11)$$

place(A,B,P)

and hence, allows us to arbitrarily select the poles (eigenvalues) of the system. State feedback is thus a method by which we can achieve *pole placement*.[M]

If the system (10.5) is not originally in the controllable canonical form, then the transformation technique presented in Chapter 8 can be used to transform it to controllable form. Specifically, transformation matrix U – see Equation (8.38) – will be used to transform the general system (10.5) into the controllable canonical form

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u \qquad (10.12)$$

where, as usual, $x = U\bar{x}$, $\bar{A} = U^{-1}AU$, etc. The feedback applied to this system in order to place the poles will then be denoted by $\bar{K}$, i.e., $u = \bar{K}\bar{x}$, resulting in a system matrix $\bar{A} + \bar{b}\bar{K}$ with the desired eigenvalues. However, it is important to remember that, as in most cases, the change of basis just performed is a mathematical convenience, and the state variables $\bar{x}$ might not be accessible. Rather, only the state variables x will be accessible. In this case, we must finish the feedback design by undoing the similarity transformation:

$$\begin{aligned} u &= \bar{K}\bar{x} = \bar{K}U^{-1}x \\ &\triangleq Kx \end{aligned} \qquad (10.13)$$

Thus, feedback $K = \bar{K}U^{-1}$ is used instead of the feedback $\bar{K}$ directly.

The convenience of the controllable form will be illustrated in the following example, which shows how difficult it might be to compute feedback, even for a simple system, if the matrices are not first converted to controllable canonical form.

Example 10.1: SISO State Feedback

Find a state feedback gain matrix that will place the eigenvalues of the following system at the new locations $\lambda_1 = -5$ and $\lambda_2 = -6$.

$$\dot{x} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \tag{10.14}$$

Solution:

As an exercise, we will first attempt to place the poles of this system at the desired location *without* first transforming to controllable form. Defining $K = \begin{bmatrix} k_0 & k_1 \end{bmatrix}$, we have by direct substitution into (10.14),

$$\begin{aligned} A + bK &= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_0 & k_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} k_0 & k_1 \\ k_0 & k_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + k_0 & 3 + k_1 \\ 4 + k_0 & 2 + k_1 \end{bmatrix} \end{aligned} \tag{10.15}$$

Computing the eigenvalues of (10.15) above in the usual way:

$$\begin{aligned} \left| \lambda I - (A + bK) \right| &= \det\left(\begin{bmatrix} \lambda - 1 - k_0 & -3 - k_1 \\ -4 - k_0 & \lambda - 2 - k_1 \end{bmatrix} \right) \\ &= \lambda^2 + \lambda(-k_0 - k_1 - 3) - (10 + k_0 + 3k_1) \end{aligned} \tag{10.16}$$

Knowing that the desired eigenvalues are to be at -5 and -6, we must set (10.16) equal to the *desired* characteristic polynomial,

$$\begin{aligned} \phi^{des}(\lambda) &= (\lambda + 5)(\lambda + 6) \\ &= \lambda^2 + 11\lambda + 30 \\ &= \lambda^2 + \lambda(-k_0 - k_1 - 3) - (10 + k_0 + 3k_1) \end{aligned}$$

By matching coefficients in this equation, we arrive at the necessary conditions for the two gains:

$$\begin{aligned} 11 &= -k_0 - k_1 - 3 \\ 30 &= -10 - k_0 - 3k_1 \end{aligned} \tag{10.17}$$

We will not solve these equations because as they appear in (10.17), they illustrate the difficulty of not using the controllable form. The result in (10.17) is a coupled set of algebraic equations for the feedback gains, which we do know how to solve.

However, the solution is not apparent by inspection, and it gets computationally more intensive as the system size increases.

Instead, we will compute the transformation matrix U that converts (10.14) to controllable canonical form. Using the results of Section 8.2.4, we find that

$$U = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$$

resulting in

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{10.18}$$

Applying feedback $\bar{K} = [\bar{k}_0 \quad \bar{k}_1]$ to this system,

$$\begin{aligned} \bar{A} + \bar{b}\bar{K} &= \begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \bar{k}_0 & \bar{k}_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 10 + \bar{k}_0 & 3 + \bar{k}_1 \end{bmatrix} \end{aligned} \tag{10.19}$$

The characteristic equation of this matrix is, by inspection, $\phi(\lambda) = \lambda^2 + \lambda(-3 - \bar{k}_1) + (-10 - \bar{k}_0)$. It is now easy to set this $\phi(\lambda)$ equal to $\phi^{des}(\lambda) = \lambda^2 + 11\lambda + 30$ to give the much simpler system of linear equations:

$$\begin{aligned} 11 &= -3 - \bar{k}_1 \\ 30 &= -10 - \bar{k}_0 \end{aligned} \tag{10.20}$$

Equations (10.20) may be solved by inspection, giving $\bar{K} = [\bar{k}_0 \quad \bar{k}_1] = \begin{bmatrix} -40 & -14 \end{bmatrix}$. Now we must remember to undo the similarity transform via (10.13):

$$\begin{aligned} K = \begin{bmatrix} k_0 & k_1 \end{bmatrix} &= \bar{K}U^{-1} \\ &= \begin{bmatrix} -1 & -13 \end{bmatrix} \end{aligned} \tag{10.21}$$

This value for K can be seen to satisfy the original design equations in (10.17). It is also easy to verify that the eigenvalues of $A + bK$ are indeed –5 and –6 as desired.

Formulas for State Feedback Gains

In addition to the above procedure, which requires the designer to actually transform the system into the controllable form, there are two other formulas for finding the gain K for a SISO system that do not require the complete transformation process. They are similar in that they both require knowledge of the original and the desired characteristic polynomials, but if the desired eigenvalues are known, then these will be available. In addition, they each require knowledge of the original controllability matrix[M] and, as we might expect, assurance of its invertibility.

`ctrb(A,B)`

Ackermann's Formula

To derive the first formula, known as *Ackermann's formula*,[M] we denote the *desired* characteristic polynomial as

`acker(A,B,P)`

$$\phi^{\text{des}}(\lambda) = \lambda^n + a_{n-1}^{\text{des}}\lambda^{n-1} + \cdots + a_1^{\text{des}}\lambda + a_0^{\text{des}} \tag{10.22}$$

while the *original* characteristic polynomial is denoted, as usual, as

$$\phi(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \tag{10.23}$$

We know from the Cayley-Hamilton theorem that the controllable-form A-matrix, i.e., $\bar{A}$, must satisfy its own characteristic polynomial, so it must satisfy (10.23):

$$\phi(\bar{A}) = \bar{A}^n + a_{n-1}\bar{A}^{n-1} + \cdots + a_1\bar{A} + a_0 I = 0$$

or

$$\bar{A}^n = -a_{n-1}\bar{A}^{n-1} - \cdots - a_1\bar{A} - a_0 I \tag{10.24}$$

Although $\bar{A}$ does not satisfy the *desired* characteristic polynomial in (10.22), we can nevertheless evaluate (10.22) at $\bar{A}$:

$$\begin{aligned}\phi^{\text{des}}(\bar{A}) &= \bar{A}^n + a_{n-1}^{\text{des}}\bar{A}^{n-1} + \cdots + a_1^{\text{des}}\bar{A} + a_0^{\text{des}} I \\ &= \left(-a_{n-1}\bar{A}^{n-1} - \cdots - a_1\bar{A} - a_0 I\right) + a_{n-1}^{\text{des}}\bar{A}^{n-1} + \\ &\qquad \cdots + a_1^{\text{des}}\bar{A} + a_0^{\text{des}} I \\ &= \left(a_{n-1}^{\text{des}} - a_{n-1}\right)\bar{A}^{n-1} + \cdots + \left(a_1^{\text{des}} - a_1\right)\bar{A} + \left(a_0^{\text{des}} - a_0\right)I\end{aligned} \tag{10.25}$$

Now we note that *if* we were to use the controllable form for our computations, as above, we would exploit the form of (10.11) to write the equalities

$$\begin{aligned} -a_{n-1}^{\text{des}} &= -a_{n-1} + \bar{k}_{n-1} \\ &\vdots \\ -a_1^{\text{des}} &= -a_1 + \bar{k}_1 \\ -a_0^{\text{des}} &= -a_0 + \bar{k}_0 \end{aligned}$$

which in turn imply

$$\begin{aligned} a_{n-1}^{\text{des}} - a_{n-1} &= -\bar{k}_{n-1} \\ &\vdots \\ a_1^{\text{des}} - a_1 &= -\bar{k}_1 \\ a_0^{\text{des}} - a_0 &= -\bar{k}_0 \end{aligned} \tag{10.26}$$

This allows us to express (10.25) as

$$\phi^{\text{des}}(\bar{A}) = -\bar{k}_{n-1}\bar{A}^{n-1} - \cdots - \bar{k}_1\bar{A} - \bar{k}_0 I \tag{10.27}$$

A clever use of the special structure of controllable form A-matrices allows us to recognize that if we define the *selection* vectors as

$$\begin{aligned} e_1 &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^{\text{T}} \\ e_2 &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^{\text{T}} \\ &\vdots \\ e_n &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^{\text{T}} \end{aligned}$$

then

$$e_1^{\text{T}}\bar{A} = e_2^{\text{T}} \qquad e_2^{\text{T}}\bar{A} = e_3^{\text{T}} \qquad \ldots \qquad e_{n-1}^{\text{T}}\bar{A} = e_n^{\text{T}} \tag{10.28}$$

By repeatedly multiplying the equations in (10.28) from the right by $\bar{A}$, we generate the identities

$$e_1^{\text{T}}\bar{A}^{n-1} = e_2^{\text{T}}\bar{A}^{n-2} = e_3^{\text{T}}\bar{A}^{n-3} = \cdots = e_n^{\text{T}} \tag{10.29}$$

Multiplying (10.27) from the left by e_1^{T} therefore results in

$$
\begin{aligned}
e_1^{\mathrm{T}}\phi^{\mathrm{des}}(\overline{A}) &= -\overline{k}_{n-1}e_1^{\mathrm{T}}\overline{A}^{n-1} - \overline{k}_{n-2}e_1^{\mathrm{T}}\overline{A}^{n-2} - \cdots - \overline{k}_1 e_1^{\mathrm{T}}\overline{A} - \overline{k}_0 e_1^{\mathrm{T}} \\
&= -\overline{k}_{n-1}e_n^{\mathrm{T}} - \overline{k}_{n-2}e_{n-1}^{\mathrm{T}} - \cdots - \overline{k}_0 e_1^{\mathrm{T}} \\
&= -\begin{bmatrix} \overline{k}_0 & \overline{k}_1 & \cdots & \overline{k}_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \\
&= -\begin{bmatrix} \overline{k}_0 & \overline{k}_1 & \cdots & \overline{k}_{n-1} \end{bmatrix} \\
&= -\overline{K}
\end{aligned}
\qquad (10.30)
$$

While (10.30) may at first seem to be a suitable formula for feedback gain, we must realize that we have again found $\overline{K}$ and not K, so that the reverse similarity transformation in (10.13) must be applied. However, the entire derivation was motivated by the desire to not have to compute $\overline{A}$ or the similarity transformation matrix U. Fortunately, a few more identities will simplify Equation (10.30) even more.

Knowing from (10.13) that $K = \overline{K}U^{-1}$, and that $\overline{A} = U^{-1}AU$, we can rewrite (10.30) as

$$
e_1^{\mathrm{T}}\phi^{\mathrm{des}}(U^{-1}AU) = -KU
$$

or

$$
\begin{aligned}
e_1^{\mathrm{T}}U^{-1}\phi^{\mathrm{des}}(A)U &= -KU \\
e_1^{\mathrm{T}}U^{-1}\phi^{\mathrm{des}}(A)UU^{-1} &= -K \\
e_1^{\mathrm{T}}U^{-1}\phi^{\mathrm{des}}(A) &= -K
\end{aligned}
\qquad (10.31)
$$

In Equation (8.45), we found that $U^{-1} = \overline{P}P^{-1}$, where $\overline{P}$ is the controllability matrix of the controllable canonical form and P is the original controllability matrix. Therefore, (10.31) gives

$$
K = -e_1^{\mathrm{T}}U^{-1}\phi^{\mathrm{des}}(A) = -e_1^{\mathrm{T}}\overline{P}P^{-1}\phi^{\mathrm{des}}(A)
$$

Within this expression, the term $e_1^{\mathrm{T}}\overline{P}$ will be equal to the first row of the controllable form controllability matrix $\overline{P}$, which will be

$$
e_1^{\mathrm{T}}\overline{P} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^{\mathrm{T}} = e_n^{\mathrm{T}}
$$

So finally, we have the final version of Ackermann's formula:

$$K = -e_n^{\mathrm{T}} P^{-1} \phi^{\mathrm{des}}(A) \tag{10.32}$$

where, as derived, $\phi^{\mathrm{des}}(A)$ is the *desired* characteristic polynomial (i.e., with the *desired* poles), evaluated at the *original* A-matrix. This matrix polynomial evaluation is relatively easy to perform with the aid of a computer and an analysis tool such as MATLAB.[M]

`polyvalm(V,X)`

We should remark on the use of (10.32) for feedback computation. For "manual" calculations, (10.32) is convenient and accurate provided that the controllability matrix P is well-conditioned, so that its inverse may be accurately computed. In cases where numerical accuracy is questionable because of the inversion operation, more numerically stable algorithms are available,[M] such as treating (10.32) as a simultaneous equation problem and solving it through the use of SVDs.

`place(A,B,P)`

Bass-Gura Formula

A second but similar formula for the gain matrix results from equally clever uses of matrix identities, but without the need to even consider the transformation U that converts the system to controllable form. This formula is known as the *Bass-Gura formula* [7] and again starts with the expression for the *desired* characteristic polynomial:

$$\phi^{\mathrm{des}}(\lambda) = \det\left[\lambda I - (A + bK)\right] \tag{10.33}$$

Appealing to the identity that $\det(AB) = \det(A)\det(B)$, we can factor (10.33) to get

$$\begin{aligned} \phi^{\mathrm{des}}(\lambda) &= \det\left[(\lambda I - A)\left(I - (\lambda I - A)^{-1} bK\right)\right] \\ &= \det\left[\lambda I - A\right] \det\left[I - (\lambda I - A)^{-1} bK\right] \\ &= \phi(\lambda) \det\left[I - (\lambda I - A)^{-1} bK\right] \end{aligned} \tag{10.34}$$

where $\phi(\lambda)$ is the original characteristic polynomial. Another identity from Appendix A, Equation (A.18), and the fact that a scalar quantity is also its own determinant allow us to express (10.34) as

$$\begin{aligned}\phi^{\text{des}}(\lambda) &= \phi(\lambda)\det\left[1 - K(\lambda I - A)^{-1}b\right] \\ &= \phi(\lambda)\left(1 - K(\lambda I - A)^{-1}b\right) \\ &= \phi(\lambda) - \phi(\lambda)K(\lambda I - A)^{-1}b\end{aligned}$$

or

$$\phi^{\text{des}}(\lambda) - \phi(\lambda) = -\phi(\lambda)K(\lambda I - A)^{-1}b \tag{10.35}$$

Another identity, known as a *resolvent* identity, Equation (A.42), gives the expansion of (10.35) as

$$\begin{aligned}\phi^{\text{des}}(\lambda) - \phi(\lambda) &= -\phi(\lambda)K\frac{1}{\phi(\lambda)}\Big[\lambda^{n-1}I + (A + a_{n-1}I)\lambda^{n-2} + \\ &\qquad \cdots + (A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I)\Big]b \\ &= -K\Big[\lambda^{n-1}I + (A + a_{n-1}I)\lambda^{n-2} + \\ &\qquad \cdots + (A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I)\Big]b\end{aligned} \tag{10.36}$$

Recalling from (10.22) and (10.23) that

$$\begin{aligned}\phi^{\text{des}}(\lambda) - \phi(\lambda) &= \lambda^n + \left(a_{n-1}^{\text{des}} - a_{n-1}\right)\lambda^{n-1} + \\ &\qquad \cdots + \left(a_1^{\text{des}} - a_1\right)\lambda + \left(a_0^{\text{des}} - a_0\right)\end{aligned} \tag{10.37}$$

we can match coefficient of like powers of λ in (10.36) and (10.37) to get

$$\begin{aligned}a_{n-1}^{\text{des}} - a_{n-1} &= -Kb \\ a_{n-2}^{\text{des}} - a_{n-2} &= -K(A + a_{n-1}I)b = -KAb - Ka_{n-1}b \\ a_{n-3}^{\text{des}} - a_{n-3} &= -K(A^2 + a_{n-1}A + a_{n-2}I)b \\ &= -KA^2b - a_{n-1}KAb - a_{n-2}Kb \\ &\;\;\vdots\end{aligned} \tag{10.38}$$

Rearranging (10.38) into a matrix equation,

$$\begin{bmatrix} a_{n-1}^{\text{des}} - a_{n-1} \\ \vdots \\ a_1^{\text{des}} - a_1 \\ a_0^{\text{des}} - a_0 \end{bmatrix}^{\mathrm{T}} = -K\begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}\begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ 0 & 1 & a_{n-1} & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \quad (10.39)$$

`toeplitz(C,Z)`

The matrix on the right of Equation (10.39) is known as an upper triangular Toeplitz[M] matrix and is obviously invertible. The controllability matrix appearing in (10.39) is clearly invertible as well, so that Equation (10.39) can be rewritten more compactly as

$$K = -\begin{bmatrix} a_{n-1}^{\text{des}} - a_{n-1} \\ \vdots \\ a_1^{\text{des}} - a_1 \\ a_0^{\text{des}} - a_0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ 0 & 1 & a_{n-1} & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & 0 & 0 & 1 \end{bmatrix}^{-1} P^{-1} \quad (10.40)$$

which is the Bass-Gura formula.

Although we have taken some pains to present the gain computation matrices for SISO systems, it should be clear that from (10.32) and (10.40) (in particular, the fact that the inverse controllability matrix appears in each), these formulas do not extend to MIMO systems. For MIMO systems, there are a number of numerical techniques, as well as the methods we present in Section 10.2, which essentially reduce the p-input state feedback problem into p single-input feedback problems. First, though, we will make some important observations about the properties of state feedback in general. These remarks *do* apply to single-input and multi-input systems alike.

10.1.2 Properties of State Feedback

Obviously, once we have introduced state feedback into a system, we have fundamentally altered it. Primarily, its stability and transient response properties are altered because the eigenvalues have been changed. This fact raises a number of questions that we can easily address.

Equivalence of Controllability and Pole Placement

From the form of the system in (10.10) that includes state feedback gains, we have already concluded that when state feedback is applied to a controllable system, each individual coefficient of the characteristic polynomial can be specified. If the locations of the new poles are chosen such that complex poles appear in conjugate pairs, then it is clear that *controllability implies the ability to arbitrarily place poles*.

Through a simple example, we can illustrate what typically happens when state feedback is applied to a system that is *not* controllable.

Example 10.2: State Feedback for an Uncontrollable System

Consider the system

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \tag{10.41}$$

Apply state feedback and determine the location of the resulting eigenvalues.

Solution:

The system is in Jordan form, with two Jordan blocks corresponding to the same eigenvalue. However, because there is only a single input, we know from Chapter 8 that the system must be uncontrollable. Taking a state feedback gain matrix of the usual structure, $K = \begin{bmatrix} k_0 & k_1 \end{bmatrix}$, and substituting $u = Kx$ into (10.41) give

$$\dot{x} = (A + bK)x = \begin{bmatrix} -2 + k_0 & k_1 \\ k_0 & -2 + k_1 \end{bmatrix} x \tag{10.42}$$

Finding the eigenvalues:

$$\begin{aligned} \det\left(\lambda I - (A + bK)\right) &= \begin{vmatrix} \lambda + 2 - k_0 & -k_1 \\ -k_0 & \lambda + 2 - k_1 \end{vmatrix} \\ &= \left(\lambda + 2 - k_0\right)\left(\lambda + 2 - k_1\right) - k_0 k_1 \\ &= \lambda^2 + \lambda\left(4 - k_0 - k_1\right) + \left(4 - 2k_0 - 2k_1\right) \\ &= \left(\lambda + 2\right)\left(\lambda + \left(2 - k_0 - k_1\right)\right) \end{aligned} \tag{10.43}$$

By setting (10.43) equal to zero, it is apparent that one eigenvalue may be arbitrarily located via state feedback, but the other one is fixed at $\lambda = -2$. Thus, when state feedback is applied to an uncontrollable system, one or more of the eigenvalues will be unaffected by the state feedback gains. (In fact, it has been proposed that an easy way to test for controllability is to apply a random feedback gain matrix and check to see if all the eigenvalues have changed. If one or more have not changed, it is very likely that the system is uncontrollable. If those that do *not* move are in the left half-plane already, then the system is probably stabilizable.)

Controllability and Observability After Feedback

Unless there is a compelling physical reason to do otherwise, the exogenous input applied to a system after state feedback is often applied with $E = I$. Considering Equation (10.10), then, it is clear that if the system is assumed to be in controllable canonical form before feedback, then it will be in controllable canonical form after feedback. If indeed $E = I$, as we will assume hereafter, then the input matrix (i.e., the matrix through which input v is applied) is again the same as the original input matrix:

$$\bar{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

We conclude, therefore, that *a system that is controllable before state feedback will be controllable after state feedback*. This will be true as well if $E \neq I$ provided that E is full rank.

A similar question can be asked of the observability of the system, i.e., is an observable system still observable after the introduction of state feedback? This question can be answered through the related question: Does state feedback alter the *zeros* of a system? To answer this question, we recall that the relationship between the transfer function and the state space matrices is given Equations (9.20) and (9.21). Therefore, *without* feedback, the transfer function of a system (10.8) is

$$\begin{aligned} g(s) &= \frac{c_n s^{n-1} + c_{n-1}s^{n-2} + \cdots + c_2 s + c_1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + d \\ &= \frac{d\left(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0\right) + c_n s^{n-1} + c_{n-1}s^{n-2} + \cdots + c_2 s + c_1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \\ &= \frac{ds^n + \left(da_{n-1} + c_n\right)s^{n-1} + \cdots + \left(da_0 + c_1\right)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \end{aligned} \tag{10.44}$$

After the introduction of state feedback, the new system equations are (assuming again that $E = I$),

$$\begin{aligned}\dot{x} &= (A+bK)x+bv \\ y &= (c+dK)x+dv\end{aligned} \tag{10.45}$$

Since the "new" matrix $A+bK$ has the same form as (10.10), the system in (10.45) will give a transfer function analogous to (10.44):

$$\begin{aligned}g(s) &= \frac{(c_n+dk_{n-1})s^{n-1}+(c_{n-1}+dk_{n-2})s^{n-2}+\cdots+(c_2+dk_1)s+(c_1+dk_0)}{s^n+(a_{n-1}-k_{n-1})s^{n-1}+\cdots+(a_1-k_1)s+(a_0-k_0)}+d \\ &= \frac{ds^n+(da_{n-1}-dk_{n-1}+c_n+dk_{n-1})s^{n-1}+\cdots+(da_0-dk_0+c_1+dk_0)}{s^n+(a_{n-1}-k_{n-1})s^{n-1}+\cdots+(a_1-k_1)s+(a_0-k_0)} \\ &= \frac{ds^n+(da_{n-1}+c_n)s^{n-1}+\cdots+(da_0+c_1)}{s^n+(a_{n-1}-k_{n-1})s^{n-1}+\cdots+(a_1-k_1)s+(a_0-k_0)}\end{aligned} \tag{10.46}$$

The numerator of this transfer function is clearly equal to that of (10.44). We conclude therefore that *the zeros of a SISO transfer function are not affected by state feedback*. This is true for MIMO system as well, although we have not shown this here.

With this fact, we can immediately conclude that *observability is not preserved by state feedback*. Why is this obvious? If we start with a controllable and observable system (i.e., a minimal realization), the feedback gain can be used to place the poles but the zeros remain unaffected. Suppose therefore that the poles move to locations that coincide with one or more zeros. Then the system is no longer minimal. Because we have already determined that controllability cannot have changed because of the introduction of state feedback, then observability must have been lost.

The final remark we make regarding state feedback is that it follows exactly the same procedure in continuous-time as in discrete-time. Of course, the design criteria will change because the stability region in the *z*-plane is inside the unit circle as opposed to being in the left-half of the *s*-plane. Other than that, the process for converting to controllable form, computing state feedback matrices, and for converting back again is exactly the same. It is only the *desired* characteristic polynomial that will likely be different and that is at the choice of the designer.

10.2 Multivariable Canonical Forms and Feedback

In both of the feedback formulas above (Ackermann and Bass-Gura), it is necessary to invert the controllability matrix P in order to compute appropriate gain matrices. This implies both that P is full rank (which it must be if the system is controllable) *and* that it is square. If the system has $p > 1$ inputs, then matrix P will be $n \times pn$, so it will of course not be invertible. Furthermore, the matrix structures found in the controllable canonical form are particular to single-input systems, i.e., those with a single-column b-matrix. The concept of state feedback is equally applicable to multiple-input systems, and Figure 10.3 still represents an accurate diagram of the feedback structure. However, the implementation and design are somewhat more complex.

10.2.1 Controllability Indices and Canonical Forms

To accommodate such difficulties with multi-input systems and enable us to compute state-feedback gains, we use *multivariable canonical forms*. A multivariable canonical form takes into account the differences in the different inputs, and the extent to which each input contributes to the control of the entire system. We can think of each input as having a certain capacity to control the system as a whole such that the controllable subspace is the sum of the spaces controllable by the individual inputs. This will enable us to design control signals one input at a time. The mechanism by which we perform this analysis is the *controllability index*. In this section we will define controllability indices and learn to use them to transform a multi-input system to multivariable canonical form.

Consider the $n \times pn$ controllability matrix for a multi-input system:

$$P = \left[B \;\vdots\; AB \;\vdots\; \cdots \;\vdots\; A^{n-1}B \right] \tag{10.47}$$

Assume that the system is controllable. Then this matrix must have rank n or, equivalently, n linearly independent columns. If we wish to derive a similarity transformation from this matrix, we will need n such linearly independent columns, but of course there will be many ways that we might select this set from (10.47). Denote the B-matrix column-wise as

$$B = \left[b_1 \;\vdots\; b_2 \;\vdots\; \cdots \;\vdots\; b_p \right] \tag{10.48}$$

Then (10.47) becomes

$$P = \left[b_1 \quad \cdots \quad b_p \;\vdots\; Ab_1 \quad \cdots \quad Ab_p \;\vdots\; \cdots \;\vdots\; A^{n-1}b_1 \quad \cdots \quad A^{n-1}b_p \right] \tag{10.49}$$

Because we can select n linearly independent columns from this matrix in many ways, the canonical form we produce might have an equal number of variations in structures. We will present one common method for seeking out our n columns and discuss possible alternatives later.

We proceed by examining the columns of this matrix from left to right, noting how many columns in each partition of the matrix are linearly dependent on columns appearing to their left. For example, in the partition of (10.49) containing the columns $Ab_1, \ldots, Ab_p$, we will probably find that at least one column is linearly dependent on those to its left, which includes the columns in the partition $b_1 \cdots b_p$. We will denote the number of these dependent columns we find in the i^{th} partition as q_i, $i = 0, \ldots, n-1$. (The q is reminiscent of the notation for nullity, and the subscript indicates that this is the i^{th} partition of P.) Thus, if $r(B) = p$, then $q_0 = 0$, but q_i for $i > 0$ depends not only on the rank of $A^i B$, but also on the partitions to the left as well.

Continuing this procedure, we will generate a set of integers $\{q_0, q_1, \ldots, q_{n-1}\}$. Note that if a single column in particular, say Ab_2, is found to be linearly dependent on the columns to its left, then of course $A^2 b_2$ will also be dependent on the columns to its left. Therefore it must be true that

$$0 \le q_0 \le q_1 \le \cdots \le q_{n-1} \le p \tag{10.50}$$

Because there can be at most n linearly independent columns among the np total columns, there must be some partition at which we stop encountering any more *in*dependent columns in P. We denote by μ the number of partitions that have at least one independent column. Then,

$$0 \le q_0 \le q_1 \le \cdots \le q_{\mu-1} < q_\mu = q_{\mu+1} = \cdots = q_{n-1} = p \tag{10.51}$$

Because we have searched the P matrix from left to right, the linearly dependent partitions are going to be the rightmost, so it would be possible to eliminate partitions $\mu, \mu+1, \cdots, n$, and consider only the reduced controllability matrix

$$P = \left[B \;\vdots\; AB \;\vdots\; \cdots \;\vdots\; A^{\mu-1}B \right]$$

when continuing to search for columns to use in the similarity transformation. This integer μ is known as the overall *controllability index*.

Next, having identified all the linearly independent columns, from left to right, we will gather them into a separate matrix and rearrange them according to the input (i.e., the column of B) with which they are associated. Of course, there will not be n columns for each input. The number of columns associated with input i will be denoted μ_i. Therefore, the collection of *in*dependent columns is expressed as the matrix

$$M = \left[b_1 \quad Ab_1 \quad \cdots \quad A^{\mu_1 - 1} b_1 \;\vdots\; b_2 \quad Ab_2 \quad \cdots \quad A^{\mu_2 - 1} b_2 \;\vdots\; \cdots \right. \\ \left. \cdots \;\vdots\; b_p \quad Ab_p \quad \cdots \quad A^{\mu_p - 1} b_p \right] \tag{10.52}$$

The integers $\{\mu_1, \cdots, \mu_p\}$ are known as the *individual controllability indices*. With a little thought, it can be observed that

$$\mu = \max\{\mu_1, \cdots, \mu_p\}$$

and

$$\mu_1 + \mu_2 + \cdots + \mu_p \le n$$

where

$$\mu_1 + \mu_2 + \cdots + \mu_p = n$$

only for a controllable system. These controllability indices serve as our measure of "how controllable" the system is from each input acting alone. For example, if $\mu_1 = n$, then the controllability matrix would contain n linearly independent columns resulting from the *first* input alone, i.e., the entire system would be controllable even if the first input were the only one present. The remaining inputs would not contribute to the controllability of the system. The higher the controllability index of an input, the larger is the controllability subspace corresponding to that input.

Following the procedure set forth in [3] and [9], the next step in the derivation of the transformation to multi-input canonical form is to compute the inverse of the matrix M [Equation (10.52)]. Symbolically, this matrix can be written row-by-row with the notation

$$M^{-1} = \begin{bmatrix} m_{11} \\ m_{12} \\ \vdots \\ m_{1\mu_1} \\ \hdashline \vdots \\ \hdashline m_{p1} \\ \vdots \\ m_{p\mu_p} \end{bmatrix} \tag{10.53}$$

Notice that this matrix is arranged into partitions, where the i^{th} partition has μ_i

rows. By extracting the last row from each partition of this matrix (i.e., the rows denoted $m_{i\mu_i}$, for $i = 1, 2, \ldots, p$), we construct the matrix

$$T = \begin{bmatrix} m_{1\mu_1} \\ m_{1\mu_1} A \\ \vdots \\ m_{1\mu_1} A^{\mu_1 - 1} \\ \hline \vdots \\ \hline m_{p\mu_p} \\ \vdots \\ m_{p\mu_p} A^{\mu_p - 1} \end{bmatrix} \tag{10.54}$$

Using this transformation matrix to change the basis[M] in the sense that $\bar{x} = Tx$ will result in transformed system matrices $\bar{A} = TAT^{-1}$, $\bar{B} = TB$, and $\bar{C} = CT^{-1}$ such that $\bar{A}$ and $\bar{B}$ have the following multi-input canonical form:

`ss2ss(sys,T)`

$$\bar{A} = \left[\begin{array}{c:c:c:c} \bar{A}_{\mu_1 \times \mu_1} & \begin{matrix} \varnothing \\ \# \cdots \# \end{matrix} & \cdots & \begin{matrix} \varnothing \\ \# \cdots \# \end{matrix} \\ \hdashline \begin{matrix} \varnothing \\ \# \cdots \# \end{matrix} & \bar{A}_{\mu_2 \times \mu_2} & \ddots & \vdots \\ \hdashline \vdots & \ddots & \ddots & \begin{matrix} \varnothing \\ \# \cdots \# \end{matrix} \\ \hdashline \begin{matrix} \varnothing \\ \# \cdots \# \end{matrix} & \cdots & \begin{matrix} \varnothing \\ \# \cdots \# \end{matrix} & \bar{A}_{\mu_p \times \mu_p} \end{array}\right]$$

$$\bar{B} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & \# & \cdots & \# \\ \hdashline 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 1 & \# & \cdots \\ \hdashline & & \vdots & \\ \hdashline 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \tag{10.55}$$

In this notation, $\bar{A}_{\mu_i \times \mu_i}$ is a $(\mu_i \times \mu_i)$ companion form block matrix, i.e., in the exact same structure with which we are familiar for the single-input system, Equation (10.8). Its bottom row, as usual, is filled with essentially arbitrary numbers. Along each such row of matrix $\bar{A}$ is an entire row of arbitrary numbers, indicated by the nonspecific indicator # (this does not imply that these numbers are all the same). The remaining rows of all the off-diagonal blocks are equal to zero, as indicated by the symbol $\varnothing$. The $(i, j)^{\text{th}}$ block has dimension $(\mu_i \times \mu_j)$
.

In the $\bar{B}$-matrix, all the elements are zero except for the rows corresponding to the rows with #-elements in $\bar{A}$, i.e., rows $\mu_1, \mu_1 + \mu_2, \ldots$. The nonzero row that appears in the i^{th} block of $\bar{B}$ will be zeros up to its i^{th} column, which will be a one. The remaining columns of each such row can again be any number (again denoted by #). (In fact, the j^{th} column in the i^{th} block will be zero if $\mu_j > \mu_i$ [3].) The reader may want to relate this procedure to the analogous procedure for a single-input system as given in Section 8.2.4.

The benefit of this canonical form can be seen by considering the rows and columns of $\bar{B}$ one at a time. The first row to have a nonzero element will be the $\mu_1{}^{\text{st}}$ row. Multiplying a $p \times n$ matrix $\bar{K}$ by the $n \times p$ matrix $\bar{B}$ will therefore give a product $\bar{B}\bar{K}$ that has all zero rows until the $\mu_1{}^{\text{st}}$ row, which will be filled with gains. These gains will be added to the $\mu_1{}^{\text{st}}$ row of $\bar{A}$. This will be true thereafter, as well: the next nonzero row of $\bar{B}\bar{K}$ will be the $(\mu_1+\mu_2)^{st}$ row, which can again be used to alter the # values in the $(\mu_1+\mu_2)^{st}$ row of $\bar{A}$, etc. By judicious choice of the elements in $\bar{K}$, these special rows of $\bar{A}$ can be arbitrarily specified. Usually, this specification is such that overall, $\bar{A}$ becomes block-diagonal and subsets of eigenvalues are independently determined within the distinct blocks.

At this stage, a numerical example is the best illustration of how the entries of $\bar{K}$ are chosen, given a multivariable canonical form.

Example 10.3: Multi-input Pole Placement

Find at least two different gain matrices that will place the eigenvalues of the following controllable system at the set $\{-5,-5,-3\pm j3\}$.

$$\dot{x}=\begin{bmatrix}2&-11&12&31\\-3&13&-11&-33\\4&-25&14&51\\-3&16&-11&-36\end{bmatrix}x+\begin{bmatrix}-1&-4\\1&6\\-1&-11\\1&7\end{bmatrix}u \tag{10.56}$$

Solution:

The first computation performed in any state-space controller design is likely to be the controllability matrix:[M] `ctrb(A,B)`

$$P=\left[\begin{array}{cc:cc:cc:cc}-1&-4&6&11&-12&-27&16&73\\1&6&-6&-20&14&59&-24&-175\\-1&-11&8&37&-20&-111&36&331\\1&7&-6&-23&14&68&-24&-202\end{array}\right] \tag{10.57}$$

It is easily verified by computer[M] that $r(P)=n=4$. Using the independent column counting procedure, we find (by checking the rank of progressively more columns in a collection beginning at the left) that the first four columns of this P matrix are linearly independent. Therefore, $\mu_1=\mu_2=2$ and $\mu=2$. The matrix M is constructed according to (10.52) as the first, third, second, and fourth columns of P, in that order. Then, `rank(P)`

$$M^{-1} = \begin{bmatrix} 4/3 & -1 & 3 & 19/3 \\ 1/3 & -1 & 1/2 & 11/6 \\ \hline -1 & -4 & 0 & 3 \\ -1/3 & -1 & 0 & 2/3 \end{bmatrix} \tag{10.58}$$

Taking the second (i.e., μ_1) and fourth (i.e., $\mu_1 + \mu_2$) rows of this matrix as m_{12} and m_{22}, respectively, the following transformation matrix is found:

$$T = \begin{bmatrix} m_{12} \\ m_{12}A \\ m_{22} \\ m_{22}A \end{bmatrix} = \left[\begin{array}{cc:cc} 1/3 & -1 & 1/2 & 11/6 \\ 1/6 & 1/6 & 11/6 & 17/6 \\ \hdashline -1/3 & -1 & 0 & 2/3 \\ 1/3 & 4/3 & -1/3 & -4/3 \end{array}\right] \tag{10.59}$$

which produces the transformed system:

$$\bar{A} = TAT^{-1} = \left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ -4/3 & -7/3 & 8/3 & 7/6 \\ \hdashline 0 & 0 & 0 & 1 \\ -2 & -2/3 & -5 & -14/3 \end{array}\right] \qquad \bar{B} = TB = \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hdashline 0 & 0 \\ 0 & 1 \end{array}\right]$$

At this point we are ready to design the control. With an $n = 4$ and $p = 2$ system, the state-feedback gain matrix is written (remembering to retain the "bar" basis notation):

$$\bar{K} = \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} & \bar{k}_{14} \\ \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} & \bar{k}_{24} \end{bmatrix} \tag{10.60}$$

Directly applying this feedback,

$$\bar{A}+\bar{B}\bar{K}=\left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ -4/3 & -7/3 & 8/3 & 7/6 \\ \hdashline 0 & 0 & 0 & 1 \\ -2 & -2/3 & -5 & -14/3 \end{array}\right]+\left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \hdashline 0 & 0 \\ 0 & 1 \end{array}\right]\begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} & \bar{k}_{14} \\ \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} & \bar{k}_{24} \end{bmatrix}$$

$$=\left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ -4/3 & -7/3 & 8/3 & 7/6 \\ \hdashline 0 & 0 & 0 & 1 \\ -2 & -2/3 & -5 & -14/3 \end{array}\right]+\begin{bmatrix} 0 & 0 & 0 & 0 \\ \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} & \bar{k}_{14} \\ 0 & 0 & 0 & 0 \\ \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} & \bar{k}_{24} \end{bmatrix} \tag{10.61}$$

$$=\left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ -4/3+\bar{k}_{11} & -7/3+\bar{k}_{12} & 8/3+\bar{k}_{13} & 7/6+\bar{k}_{14} \\ \hdashline 0 & 0 & 0 & 1 \\ -2+\bar{k}_{21} & -2/3+\bar{k}_{22} & -5+\bar{k}_{23} & -14/3+\bar{k}_{24} \end{array}\right]$$

The matrix in (10.61) should be examined closely before deciding on any feedback gains. The locations of the gain terms at our disposal determine the flexibility we have with the structure of the "closed-loop" system. In particular, it should be noted that we can easily transform this system into two different structures:

$$\left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ -\alpha_0 & -\alpha_1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 \\ 0 & 0 & -\beta_0 & -\beta_1 \end{array}\right] \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix}$$

That is, we can specify that the system consist of two (2×2) block-diagonal parts, each in companion form, or as a single (4×4) block, also in companion form. Each option, of course, dictates different sets of $\bar{k}_{ij}$'s. To determine the desired values for the bottom row of each companion block, we should consider the desired eigenvalues two at a time, or all four at a time, i.e.,

$$\left\{\begin{array}{c} (s+5)(s+5)=s^2+10s+25\overset{\Delta}{=}s^2+\alpha_1 s+\alpha_0 \\ \text{and} \\ (s+3+j3)(s+3-j3)=s^2+6s+18\overset{\Delta}{=}s^2+\beta_1 s+\beta_0 \end{array}\right\} \tag{10.62}$$

or

$$\begin{aligned}(s+5)(s+5)(s+3+j3)(s+3-j3) &= s^4+16s^3+103s^2+330s+450 \\ &= s^4+\alpha_3 s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0\end{aligned} \tag{10.63}$$

In order to achieve the two-block structure using the two separate characteristic polynomials in (10.62), the gain matrix $\bar{K}$ should be chosen such that

$$\bar{A}+\bar{B}\bar{K} = \left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ -25 & -10 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 \\ 0 & 0 & -18 & -6 \end{array}\right] \tag{10.64}$$

By inspection and by simple term-by-term matching, such a gain matrix is seen to be:

$$\bar{K}_1 = \begin{bmatrix} -23\tfrac{2}{3} & -7\tfrac{2}{3} & -\tfrac{8}{3} & -\tfrac{7}{6} \\ 2 & \tfrac{2}{3} & -13 & -\tfrac{4}{3} \end{bmatrix} \tag{10.65}$$

On the other hand, creating a single block requires that (10.63) be matched term-by-term with

$$\bar{A}+\bar{B}\bar{K}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -450 & -330 & -103 & -16 \end{bmatrix} \tag{10.66}$$

This matching results in a different gain matrix that gives the closed-loop system the same eigenvalues:

$$\bar{K}_2 = \begin{bmatrix} \tfrac{4}{3} & \tfrac{7}{3} & -\tfrac{8}{3} & -\tfrac{1}{6} \\ -448 & -329\tfrac{1}{3} & -98 & -\tfrac{34}{3} \end{bmatrix} \tag{10.67}$$

The reader can verify that in each case, the eigenvalues of $\bar{A}+\bar{B}\bar{K}$ are in the desired locations.

A few remarks are in order. First, we must remember that this system was transformed before the state feedback gains were found. Therefore the problem is not finished with the calculation of the gain; we must undo the transformation with $K=\bar{K}T$. Second, we observe from (10.65) and (10.67) that of the two

`cond(K)`

feedback matrices we generated, $\bar{K}_1$ will be much better conditioned[M] (i.e., have a smaller condition number) than $\bar{K}_2$. This can be guessed because the numbers in $\bar{K}_1$ are much more similar to one another than are the numbers in $\bar{K}_2$. This may have practical implications in physical circuits, wherein it is desirable to have amplifier gains all of the same order of magnitude. Also, the transient response of a system with smaller gains is likely to be better than the transient response of a system with larger gains. Further considerations of the factors that lead one to prefer one feedback gain over another can be addressed with the concept of *optimal control*, which we introduce only briefly in the next chapter. Finally, we point out that although our method for computing these gains resulted in only two "obvious" options, *as long as* $\mu > 1$, *there are an infinite number of ways in which different gain matrices might be found*. (If $\mu = 1$, the system has essentially a single input.) This is simply an easy "manual" method for finding a few possibilities. The opposite can be said about single-input systems: *In a single-input system, the feedback that places the eigenvalues at a specific set of locations is unique.*

10.3 Observers

A significant assumption taken for granted in all of the above development is the availability of the state vector *x*. The implementation of state feedback very obviously relies on the physical presence of the complete state. However, we must not forget the *output* equation. In particular, the output equation models the way in which the system presents its internal workings to the outside world. More often than not, it is the output alone that is physically measurable in the physical system. Then how can we perform state feedback without the presence of the state vector? The answer is in a separate system called an *observer*. Sometimes called an *estimator*,* an observer is itself a linear system whose task is to accept as inputs the original system's (i.e., the plant's) input and output signals, and produce as its output an estimate of the plant's state vector. This state vector estimate should asymptotically track the exact state vector. In this way, the output of the observer, rather than the true state, can be used to compute state feedback. The observer and state feedback combination is often thought of by controls engineers as the most fundamental control system available in state space.

10.3.1 Continuous-Time Observers

We begin by considering a continuous-time LTI system. Discrete-time system observers, like state feedback, follow analogous equations but also admit

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* We will reserve the term *estimator* for observers that operate in systems that also include noise models. Observation of such systems is more literally an *estimation* process.

variations on the design of the observer. They will therefore be treated separately later. For the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{10.68}$$

we seek an estimate of $x(t)$ from knowledge of $u(t)$ and $y(t)$. A naive approach, the *open-loop observer*, is to assume knowledge of the system's initial state, $x(t_0)$, and use the known system matrices to create an exact copy of the plant in (10.68). If we call the state estimate $\hat{x}(t)$, then such an open-loop observer might be written as

$$\dot{\hat{x}} = A\hat{x} + Bu \qquad \hat{x}(t_0) = x(t_0) \tag{10.69}$$

The difficulty with an estimator such as this (i.e., one that does not take into account the performance *y* or the output of the plant assuming the estimated state, $\hat{y} = C\hat{x} + Du$) is that any errors it produces are never measured nor compensated. For example, the initial condition guess $\hat{x}(t_0)$ will invariably be inaccurate and the plant might not be asymptotically stable. This would create a signal $\hat{x}(t)$ that diverges from $x(t)$ such that they very soon have no resemblance at all. Creating a feedback $u = K\hat{x}$ would have unpredictable, and very likely disastrous, results.

Instead we will design a *closed-loop* observer.[M] A closed-loop observer continually compares its estimate (as measured via the output equation) with the true plant in order to create an error signal. The error signal is used to drive the observer toward asymptotic convergence with the plant. Only then will we be able to use the estimate for state feedback $u = K\hat{x}$.

`estim(sys,L)`

We have already stated that the observer is a linear system whose inputs are the plant input and the plant output. But because the observer's output should track the plant and the error in the outputs should drive the observer, it is also possible to think of the observer as having the same basic structure as the plant, but with the additional input $y - \hat{y}$:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ &= A\hat{x} + Bu + L(y - C\hat{x} - Du) \\ &= (A - LC)\hat{x} + (B - LD)u + Ly \end{aligned} \tag{10.70}$$

where the matrix *L* is a matrix of gains that remain to be determined.

How do we enforce convergence of this system's state $\hat{x}$ to the plant state *x*? Suppose the state *error* could be measured: $\tilde{x} \triangleq x - \hat{x}$. Then, of course, $\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}}$ as well. Substituting the state matrices into this relationship gives

$$\begin{aligned}\dot{\tilde{x}} &= \dot{x} - \dot{\hat{x}} \\ &= Ax + Bu - (A - LC)\hat{x} - (B - LD)u - L(Cx + Du) \\ &= A(x - \hat{x}) - LC(x - \hat{x}) \\ &= (A - LC)(x - \hat{x}) \\ &= (A - LC)\tilde{x}\end{aligned} \tag{10.71}$$

If the autonomous system in (10.71), with system matrix $A - LC$, is asymptotically stable, then $\tilde{x} = x - \hat{x} \to 0$, or equivalently, $\hat{x} \to x$ as $t \to \infty$. In order to guarantee the asymptotic stability of (10.71), the only choice we have is the selection of the gain matrix *L*. Matrix *L* must be chosen such that the eigenvalues of $A - LC$ lie in the open left half of the complex plane. To facilitate this, notice the resemblance between the familiar quantity $A + BK$ and the quantity $A - LC$, or more usefully, $(A - LC)^{\mathrm{T}} = A^{\mathrm{T}} - C^{\mathrm{T}}L^{\mathrm{T}}$. (The transpose is immaterial because the eigenvalues of any matrix are same as those of its transpose.) Making the correspondences $C^{\mathrm{T}} \sim B$ and $-L^{\mathrm{T}} \sim K$, the problem of placing the eigenvalues of $A^{\mathrm{T}} - C^{\mathrm{T}}L^{\mathrm{T}}$ by choosing L^{T} is exactly the same (save for the minus sign) as the problem of placing the eigenvalues of $A + BK$ by choosing *K*. The matrix dimensions are analogous, and *A* (A^{T}) is available in each case. For the placement of the observer poles, it may be apparent by analogy that certain canonical forms for the A^{T}-matrix are more convenient than others. As might be guessed, it is the *observable canonical form* – see Equation (9.21) – that facilitates observer design, exactly as the controllable canonical form was appropriate for state-feedback design.

Naturally, it must first be established that the eigenvalues of the observer system *can* be arbitrarily placed, which, by analogy, matches the question of controllability of the pair $(A^{\mathrm{T}}, C^{\mathrm{T}})$. However, construction of a *controllability* matrix for $(A^{\mathrm{T}}, C^{\mathrm{T}})$ is identical to construction of the *observability* matrix for (A, C). Therefore, it can be said that *the eigenvalues of the observer error system in* (10.70) *can be arbitrarily placed in the left half plane if the system in* (10.68) *is observable*. Notice that the observer system in (10.70) must be given an initial condition. If indeed the error system is made asymptotically stable, then unlike the open-loop observer, even an inaccurate initial condition will cause only a transient error, because this error can be proven to asymptotically approach zero.

Full-Order Augmented Observer and Controller

The construction of the observer is motivated by the need for an estimated state vector that we can feed back to the input. However, feeding back the true state vector is undeniably not the same thing as feeding back the estimated state vector, because the two vectors are not identical. This raises the question of whether the observer and feedback together accomplish the goal of placing the plant eigenvalues and whether the two systems interact to produce unexpected results. While the estimates are still converging, we must ensure that the composite system (i.e., observer and controller together) behaves according to the K and L gains we have chosen. To address this concern, the plant, with state feedback of the form $u = K\hat{x} + v$, and the observer can be combined together into an *augmented* system. First, considering the plant itself, the closed-loop equation is:

$$\dot{x} = Ax + Bu = Ax + B(K\hat{x} + v) = Ax + BK\hat{x} + Bv \tag{10.72}$$

Then the observer equation begins with (10.70) and becomes:

$$\begin{aligned}
\dot{\hat{x}} &= (A - LC)\hat{x} + (B - LD)u + Ly \\
&= (A - LC)\hat{x} + (B - LD)(K\hat{x} + v) + L(Cx + Du) \\
&= \left[A - LC + (B - LD)K\right]\hat{x} + (B - LD)v + LCx + LD(K\hat{x} + v) \\
&= \left[A - LC + (B - LD)K + LDK\right]\hat{x} + LCx + \left[B - LD + LD\right]v \\
&= \left[A - LC + BK\right]\hat{x} + LCx + Bv
\end{aligned} \tag{10.73}$$

The two systems, (10.72) and (10.73), together are written in augmented state space form as:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BK \\ LC & A - LC + BK \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} v \tag{10.74}$$

Such a system can be depicted in the block diagram of Figure 10.4.

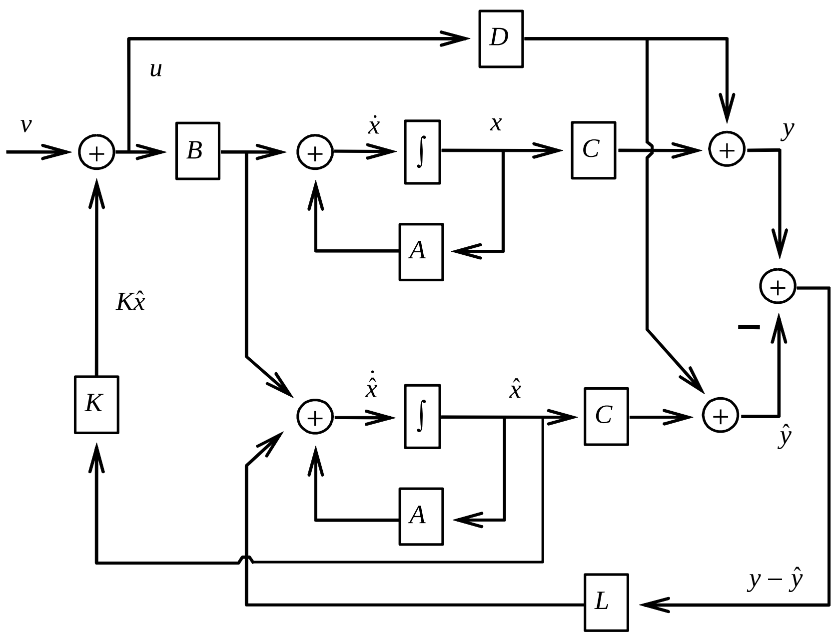

Figure 10.4 Block diagram of observer and state feedback combination.

To check whether the eigenvalues of this composite system are truly at the desired locations, we must determine the zeros of the determinant:

$$\det\begin{bmatrix} \lambda I_{n\times n}-A & -BK \\ -LC & \lambda I_{n\times n}-A+LC-BK \end{bmatrix} \tag{10.75}$$

To facilitate the computation of this determinant, we will perform elementary row and column operations, known to not alter the value of the determinant. First, we subtract the second row from the first, without altering the second row itself:

$$\det\begin{bmatrix} \lambda I_{n\times n}-A+LC & -\lambda I_{n\times n}+A-LC \\ -LC & \lambda I_{n\times n}-A+LC-BK \end{bmatrix}$$

Next, add the first column to the second, without altering the first column:

$$\begin{aligned} &\det\begin{bmatrix} \lambda I_{n\times n}-A+LC & 0 \\ -LC & \lambda I_{n\times n}-A-BK \end{bmatrix} \\ &\qquad = \det\left[\lambda I_{n\times n}-A+LC\right]\cdot\det\left[\lambda I_{n\times n}-A-BK\right] \end{aligned} \tag{10.76}$$

where the last factorization of the determinant is made possible by the block of zeros in the upper right corner. Equation (10.76) indicates that the eigenvalues of the composite system are the union of the eigenvalues of a system with matrix $A-LC$ and those of a system with matrix $A+BK$. Therefore, even though the feedback control $u=K\hat{x}+v$ does not contain the *true* plant state, but rather the *observed* plant state, the closed-loop system still has the correctly placed eigenvalues. This result is known as the *separation principle*. A consequence of the separation principle is that the observer and the controller need not be designed simultaneously; the controller gain K can be computed independently of the observer gain L.

One must be careful, however, when thinking of the controller and observer gains as independent. It would make little sense for the closed-loop plant eigenvalues to be "faster" than the observer eigenvalues. If this were the case, the plant would "outrun" the observer. Rather, the observer eigenvalues should be faster than the plant eigenvalues, in order that the observer state converges to the plant state faster than the plant state converges to zero (or approaches infinity, if it is unstable). A workable rule of thumb is to place the observer eigenvalues two to five times farther left on the complex plane than the closed-loop plant eigenvalues. Eigenvalues with more disparity than that might result in poorly conditioned system matrices and difficulties with physical implementations. Fast eigenvalues also imply a large bandwidth, making the system susceptible to noise and poor transient performance.

Another consideration in the design of the closed-loop observer and controller system is the steady-state behavior. We can examine steady-state behavior by considering the transfer function of the system in (10.74), which presumes zero initial conditions. Computing the transfer function from the system equations in (10.74) would be extremely tedious. However, as we have already discovered, transfer functions are invariant to similarity transformations, which are simply changes of coordinates. Because $\tilde{x}=x-\hat{x}$ is a linear combination of x and $\hat{x}$, we can transform the system to use state variable x and $\tilde{x}$ instead of x and $\hat{x}$. Equation (10.72) will become

$$\begin{aligned}\dot{x} &= Ax+BK\hat{x}+Bv\\ &= Ax+BK\hat{x}-BKx+BKx+Bv\\ &= (A+BK)x-BK(x-\hat{x})+Bv\\ &= (A+BK)x-BK\tilde{x}+Bv\end{aligned} \tag{10.77}$$

whereas (10.71) can be used directly. Together, their combination is the state space system

$$\begin{bmatrix}\dot{x}\\ \dot{\tilde{x}}\end{bmatrix}=\begin{bmatrix}A+BK & -BK\\ 0 & A-LC\end{bmatrix}\begin{bmatrix}x\\ \tilde{x}\end{bmatrix}+\begin{bmatrix}B\\ 0\end{bmatrix}v \tag{10.78}$$

with output equation

$$
\begin{aligned}
y &= Cx + Du \\
&= Cx + D(K\hat{x} + v) \\
&= Cx + DK\hat{x} - DKx + DKx + Dv \\
&= (C + DK)x - DK\tilde{x} + Dv \\
&= \begin{bmatrix}(C + DK) & -DK\end{bmatrix}\begin{bmatrix} x \\ \tilde{x}\end{bmatrix} + Dv
\end{aligned}
\tag{10.79}
$$

`ctrbf(A,B,C)`

Equations (10.78) and (10.79) can be recognized as being in the form of a Kalman controllability decomposition[M*] (see Chapter 8). The realization theory of Chapter 9 then indicates that the transfer function of the system is independent of the uncontrollable part. The transfer function resulting from (10.78) and (10.79) is therefore

$$H(s) = (C + DK)\left(sI - A - BK\right)^{-1} B + D \tag{10.80}$$

which is the multivariable version of the transfer function derived for the SISO case, Equation (10.7), assuming that $E = I$.

It might at first seem surprising that the transfer function for the closed-loop observer and controller system is completely independent of the observer. The reason that this is true is that transfer functions always assume zero initial conditions. If the initial conditions for (10.71) are zero, then $\tilde{x}(t_0) = x(t_0) - \hat{x}(t_0) = 0$, or $\hat{x}(t_0) = x(t_0)$. The asymptotically stable "error system" in (10.71), with this zero initial condition, will therefore remain in a zero state forever and will not affect the system output. This however does not prevent the observer from affecting the transient response, as we will illustrate in an example.

Example 10.4: An Observer and Controller With Simulation

Compute state feedback to place the eigenvalues of the system below at $-5 \pm j5$ and -10. Use an observer to estimate the state of the system and simulate the observer and controller composite system, comparing the true state to the estimated state.

---

* It might also be observed that this form would have made the computation of the eigenvalues of (10.74) much easier.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 24 & 14 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} x \tag{10.81}$$

Solution:

The system is already in controllable canonical form so it provides a shortcut in the computations. (It is not often that one will compute feedback gains "by hand" anyway, so there is little sense in going through the exercise of performing the similarity transformation.) MATLAB has two commands for placing the poles of a system, `ACKER` and `PLACE`. The `ACKER` command uses Ackermann's formula (although warning of its numerical difficulties). Using this command or some other program to compute feedback gains, we find that

$$K = \begin{bmatrix} -524 & -164 & -19 \end{bmatrix} \tag{10.82}$$

Using the same command but with A^{T} and C^{T} as arguments, the observer gain can be computed as well:

$$L = \begin{bmatrix} -47.9 & 186.6 & 341.4 \end{bmatrix}^{\mathrm{T}} \tag{10.83}$$

For the observer, we selected all eigenvalues to be located at -20, at least two times farther left on the complex plane than the desired plant eigenvalues.

Together, the augmented system may be written[M] as in (10.74) and numerically simulated. One can verify (again, by computer) that the eigenvalues of the system matrix in (10.74) are indeed the union of the sets of desired plant and desired observer eigenvalues: $\{-10, -5 \pm j5, -20, -20, -20\}$. In the simulation shown in Figure 10.5, a zero input is assumed and the initial conditions are given as $x(0) = \begin{bmatrix} -3 & -3 & -3 \end{bmatrix}^{\mathrm{T}}$ and $\hat{x}(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$.

`reg(sys,K,L)`

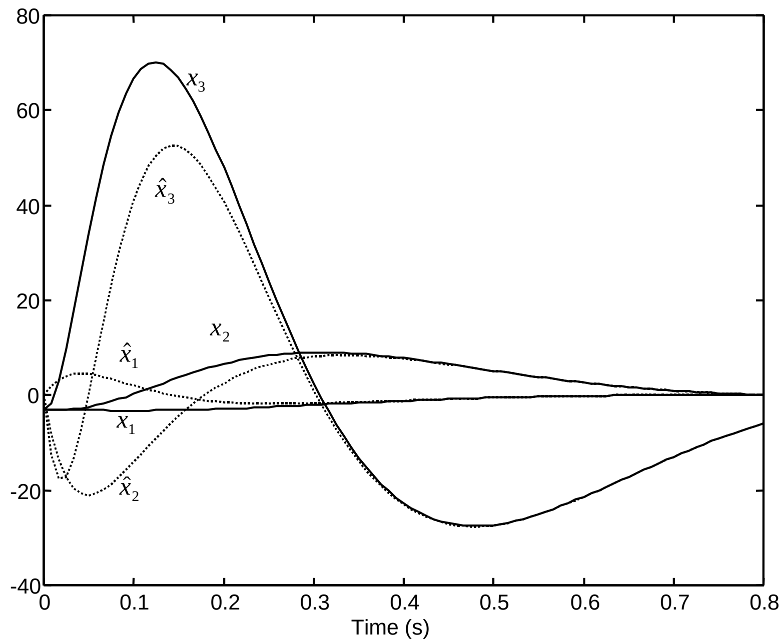

Figure 10.5 Simulated trajectories for the three (true) state variables of the plant, along with their three estimates, as provided by the observer.

The observers developed in this section are referred to as *full-order observers* because if $x \in \Re^n$, then $\hat{x} \in \Re^n$. That is, the observer estimates the full state of the system, not just a portion of it. In the event that only some of the state variables need to be estimated, a *reduced-order observer* can be designed, which will estimate only the subspace of the state space that is not already measured through the output equations.

Reduced-Order Observers

The output equation of a state space description can be viewed as a q-dimensional projection of the n-dimensional state vector. If the coordinate axes (i.e., basis vectors) of this reduced space are judiciously chosen, then the output y can be used to directly measure any q state variables. In this case, the state equations would have the form

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix}u \\ y &= \begin{bmatrix}I_{q\times q} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + Du = x_1 + Du\end{aligned} \tag{10.84}$$

so that the output directly provides the q state variables $x_1(t)$. This portion of the state vector is readily available for feedback, but the rest of the vector, $x_2(t)$, must still be observed. The system that observes such a partial state is called a *reduced-order observer*, and we will describe it below.

First, however, we should note that not all systems will have the form seen in (10.84). If not, a simple transformation matrix may be obtained to transform any system into one resembling (10.84). To construct such a transformation, we will create a new basis from the linearly independent rows of the C-matrix. Usually, there will be q such rows (otherwise, some outputs would be redundant). To create a new basis, then, we will collect these q rows and augment them with $n-q$ additional rows, linearly independent of the others, in order to create a nonsingular transformation. That is, we create the transformation W as:

$$W = \begin{bmatrix}C\\ R\end{bmatrix} \begin{matrix}\text{original output matrix}\\ \text{any } n-q \text{ additional linearly indpendent rows}\end{matrix} \tag{10.85}$$

Then with the similarity transformation $\bar{A} = WAW^{-1}$, $\bar{B} = WB$, and $\bar{C} = CW^{-1}$, the new system will be of the form in (10.84). Hereafter, we will assume that the system starts in the form in (10.84).

The strategy for observing $x_2(t)$ will be to find a state space equation for it alone and to build an observer similar to the full-order observer. To find the equations for the $x_2(t)$ subsystem, we will take advantage of the fact that $x_1(t)$ can be measured directly from y in (10.84), so that it can be treated as a known signal. If this is the case, then the two subsystem equations in (10.84) can be rewritten as

$$\begin{aligned} A_{12}x_2 &= \dot{x}_1 - A_{11}x_1 - B_1u \\ \dot{x}_2 &= A_{22}x_2 + \left[B_2u + A_{21}x_1\right]\end{aligned} \tag{10.86}$$

Simplifying the notation, if we define the (known) signals

$$\bar{u} \stackrel{\Delta}{=} A_{21}x_1 + B_2 u$$
$$\bar{y} \stackrel{\Delta}{=} \dot{x}_1 - A_{11}x_1 - B_1 u$$

then (10.86) may be written as

$$\begin{aligned} \dot{x}_2 &= A_{22}x_2 + \bar{u} \\ \bar{y} &= A_{12}x_2 \end{aligned} \tag{10.87}$$

This set of equations should now be viewed as a state space system in and of itself. As such, the design of an observer for it begins with the specification of the observer equation as in (10.70). Writing this equation for the system in (10.87) and performing the necessary substitutions,

$$\begin{aligned} \dot{\hat{x}}_2 &= A_{22}\hat{x}_2 + \bar{u} + L(\bar{y} - \hat{\bar{y}}) \\ &= A_{22}\hat{x}_2 + \bar{u} + L(\bar{y} - A_{12}\hat{x}_2) \\ &= (A_{22} - LA_{12})\hat{x}_2 + \bar{u} + L\bar{y} \end{aligned} \tag{10.88}$$

The definition of $\bar{y}$ as a "known" signal, while true, is impractical. In practice, the value of $x_1(t)$ is indeed known, but because $\bar{y}$ depends on $\dot{x}_1(t)$, and pure derivatives are difficult to realize, we wish to avoid the use of $\bar{y}$ in (10.88). Toward this end, we will introduce a change of variables $z = \hat{x}_2 - Lx_1$. Now if we use the state vector $\dot{z}$ instead of $\dot{\hat{x}}_2$, the state equation for the observer in (10.88) will simplify:

$$\begin{aligned} \dot{z} &= \dot{\hat{x}}_2 - L\dot{x}_1 \\ &= (A_{22} - LA_{12})\hat{x}_2 + \bar{u} + L\bar{y} - L\dot{x}_1 \\ &= (A_{22} - LA_{12})\hat{x}_2 + \bar{u} - L(A_{11}x_1 + B_1 u) \\ &= (A_{22} - LA_{12})(\hat{x}_2 - Lx_1) + (A_{22} - LA_{12})Lx_1 + \bar{u} - L(A_{11}x_1 + B_1 u) \\ &= (A_{22} - LA_{12})z + \left[(A_{22} - LA_{12})L - LA_{11}\right]x_1 + \bar{u} - LB_1 u \\ &= (A_{22} - LA_{12})z + \left[(A_{22} - LA_{12})L + A_{21} - LA_{11}\right]x_1 + (B_2 - LB_1)u \end{aligned} \tag{10.89}$$

(The transition from the third line to the fourth is a result of adding and subtracting the term $(A_{22} - LA_{12})Lx_1$ from the equation, and the transition from the fifth line to the sixth is a result of substitution for $\bar{u}$.) This is the equation that can be simulated or realized in order to generate the signal $z(t)$, after which the estimated state vector is computed from $\hat{x}_2 = z + Lx_1$. (Recall that the vector signal $x_1(t)$ is directly measured from (10.84) as $x_1(t) = y(t) - Du(t)$.)

The reduced-order observer (10.89) has an observer gain L in it, as did the full-order observer. In order to find a suitable gain, the "error" system must again be constructed, which will ensure that the estimated vector approaches the true vector. In this case, the error system is $e = x_2 - \hat{x}_2$, so the "error dynamics" are [with substitutions from (10.87) and (10.88)]:

$$\begin{aligned} \dot{e} &= \dot{x}_2 - \dot{\hat{x}}_2 \\ &= \left[A_{22}x_2 + \bar{u}\right] - \left[A_{22}\hat{x}_2 + \bar{u} + L(A_{12}x_2 - A_{12}\hat{x}_2)\right] \\ &= A_{22}(x_2 - \hat{x}_2) - LA_{12}(x_2 - \hat{x}_2) \\ &= (A_{22} - LA_{12})(x_2 - \hat{x}_2) \\ &= (A_{22} - LA_{12})e \end{aligned} \tag{10.90}$$

Comparing (10.90) to (10.71), it is apparent that the gain L is again computed in a manner analogous to the computation of the state feedback gains; i.e., by using eigenvalue placement methods[M] on the transposed system matrix $A_{22}^{\mathrm{T}} - A_{12}^{\mathrm{T}}L^{\mathrm{T}}$, exactly as done for the full-order observer. This will ensure that the system (10.90) is made asymptotically stable, so that $\hat{x}_2 \to x_2$ as $t \to \infty$. The question that should arise with the observant reader is that of observability of the pair of matrices (A_{22}, A_{12}). Provided that the original system (A, C) is observable, this will always be the case, as is shown in Problem 10.7 at the end of this chapter.

```
place(A,B,P)
```

Summarizing, assuming that the state equations are in the form (10.84), the "observed" state vector is

$$\begin{bmatrix} x_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} y - Du \\ z + Lx_1 \end{bmatrix} \tag{10.91}$$

where the signal $y(t)$ is the output of the original system, and the signal $z(t)$ is the state of the system given by (10.89). One can then proceed to compute state feedback in the form

$$u = K\begin{bmatrix} x_1 \\ \hat{x}_2 \end{bmatrix} + v \tag{10.92}$$

Although the reduced-order observer has fewer state variables than the full-order observer, one might question whether the equations in (10.89) are actually more efficient than using the larger but simpler full-order observer. In fact, they might not be, but the comparison depends on a number of other factors, such as the sparsity of the system matrices and the size of the system relative to the number of output variables q. The choice of full- versus reduced-order observer should also be taken in consideration of engineering factors as well. Because the signal x_1 appears directly in the estimate for the state variables x_2 (or z) in (10.91), and x_1 is derived from the system output y, there is "direct feedthrough" of the output to the input in the reduced-order observer. Theoretically, this presents no difficulty at all. From a practical viewpoint, however, this can be a source of excess noise in the observed value $\hat{x}_2$. The output of many physical systems is measured with physical sensors, which almost always include some noise. In the reduced-order observer, this noise passes through directly to the estimate. In the full-order observer, the estimated state also relies on the output signal, but only as an input to the observer state equations. Thus, the observer (assuming it is stable) always has a smoothing effect on y. The noise in y does not pass directly to the estimate. Finally, although we do not provide an example to illustrate the point here, it is often found that the reduced-order observer has a better transient response. This is because some of the state variables (i.e., x_1) are directly measured, and have no transient responses at all: $\hat{x}_1 \equiv x_1$. Thus, there are fewer contributions to the error dynamics early in time, while the transients in $x_2 - \hat{x}_2$ are occurring. Because of all these factors, the selection of the reduced- versus full-order observer is not automatic.

10.3.2 Discrete-Time Observers

As is usually true, the discrete-time observer is different from the continuous-time observer in notation only. To demonstrate this, we will derive the full-order observer using the same sequence of steps and the same reasoning as above, but we will leave the reduced-order observer as an exercise.

Full- and Reduced-Order Observers

Given the discrete-time plant equations (assumed to be observable):

$$\begin{aligned} x(k+1) &= A_d x(k) + B_d u(k) \\ y(k) &= C_d x(k) + D_d u(k) \end{aligned} \tag{10.93}$$

a full-order observer can be modeled after (10.70):

$$\begin{aligned} \hat{x}(k+1) &= A_d \hat{x}(k) + B_d u(k) + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C_d \hat{x}(k) + D_d u(k) \end{aligned} \tag{10.94}$$

Substituting for $\hat{y}(k)$, this simplifies to the same form as (10.70):

$$\begin{aligned}\hat{x}(k+1) &= A_d\hat{x}(k)+B_d u(k)+L\left[y(k)-C_d\hat{x}(k)-D_d u(k)\right]\\ &= (A_d-LC_d)\hat{x}(k)+(B_d-LD_d)u(k)+Ly(k)\end{aligned} \tag{10.95}$$

To determine the proper choice of the observer gain *L*, we again form the error signal $e(k)=x(k)-\hat{x}(k)$ and the error-dynamics equations:

$$\begin{aligned}e(k+1) &= x(k+1)-\hat{x}(k+1)\\ &= A_d x(k)+B_d u(k)-A_d\hat{x}(k)-B_d u(k)-L\left[y(k)-\hat{y}(k)\right]\\ &= A_d\left[x(k)-\hat{x}(k)\right]-L\left[C_d x(k)-C_d\hat{x}(k)\right]\\ &= (A_d-LC_d)(x(k)-\hat{x}(k))\\ &= (A_d-LC_d)e(k)\end{aligned} \tag{10.96}$$

Exactly as for the continuous-time case, the procedure for making this time-invariant system asymptotically stable is to place the eigenvalues of the matrix (A_d-LC_d) using the gain matrix *L*. Of course, this implies that the eigenvalues will be inside the unit circle, not in the left half-plane. Also, the eigenvalues should be "faster" than the closed-loop eigenvalues of the plant, as specified with the feedback gain matrix *K*; faster implies a smaller *magnitude* for these eigenvalues, i.e., closer to the origin on the complex plane. Naturally, all the same gain matrix calculation tools will apply. Other properties of observers remain the same as well, such as the fact that the transfer function of the combined controller and observer contains only the closed loop plant poles, and not the observer poles, and the fact that *K* and *L* can be computed independently, i.e., the separation principle.

Current Estimators

The development of this discrete-time full-order observer might serve to convince the reader that the reduced-order discrete-time observer is analogous to the reduced-order continuous-time observer. It is. However, there is a unique peculiarity of discrete-time systems that we can use to our advantage when designing observers. This inherent difference is in the moment at which the feedback is computed. In particular, there is a single one-period time lag between the time that the state estimate $\hat{x}(k)$ is computed, and the time that it is applied to the system through the feedback equation $u(k)=K\hat{x}(k)+r(k)$. Thus, at the instant in time $t=kT$, the value $\hat{x}(k)$ is found, based on the known values of $\hat{x}(k-1)$, $u(k-1)$, and $y(k-1)$. This value is computed in a relatively short time, probably considerably less than the sample interval *T*. Once $\hat{x}(k)$ is determined, the feedback $K\hat{x}(k)$ *could* be immediately determined. It is not used, however, until the *next* computation time, which occurs at the instant $t=(k+1)T$, at which time $\hat{x}(k+1)$ is computed. The result is that each computation of $\hat{x}$ is always based on data that is at least one sample period old.

An alternative formulation has been developed that incorporates an estimate of the state data at the "current" time: a *current estimator* [6]. In the current estimator, the process of observation is separated into two components: *prediction* of the *estimated* state at time $k+1$ and the correction of that estimate with the system's true and estimated outputs at time $k+1$. Explicitly, we create the predicted state estimate as:

$$\bar{x}(k+1) = A_d\hat{x}(k) + B_d u(k) \tag{10.97}$$

which can be computed at time *k*, not $k+1$. This can be interpreted as the "time update" portion of the observer equation. Then at the moment sampling instant $k+1$ occurs, we perform the "measurement update" of this estimate with the expression:

$$\begin{aligned}\hat{x}(k+1) &= \bar{x}(k+1) + L(y(k+1) - \bar{y}(k+1)) \\ &= \bar{x}(k+1) + L\left[y(k+1) - C_d\bar{x}(k+1) - D_d u(k+1)\right]\end{aligned} \tag{10.98}$$

Substituting (10.97) into (10.98) gives the observer equation:

$$\begin{aligned}\hat{x}(k+1) = {} & A_d\hat{x}(k) + B_d u(k) \\ & + L\left\{y(k+1) - C_d\left[A_d\hat{x}(k) + B_d u(k)\right] - D_d u(k+1)\right\}\end{aligned} \tag{10.99}$$

Because this is significantly different from the previous observer equations we have been using, we will again write the error equations to determine the appropriate conditions on gain matrix *L*:

$$\begin{aligned}e(k+1) &= x(k+1) - \hat{x}(k+1) \\ &= A_d x(k) + B_d u(k) - A_d\hat{x}(k) - B_d u(k) \\ &\qquad - L\left\{C_d x(k+1) + D_d u(k+1)\right. \\ &\qquad \left. - C_d\left[A_d\hat{x}(k) + B_d u(k)\right] - D_d u(k+1)\right\} \\ &= A_d x(k) - A_d\hat{x}(k) \\ &\qquad - L\left\{C_d\left[A_d x(k) + B_d u(k)\right] + D_d u(k+1)\right. \\ &\qquad \left. - C_d\left[A_d\hat{x}(k) + B_d u(k)\right] - D_d u(k+1)\right\}\end{aligned}$$

$$\begin{aligned} &= \left(A_d - LC_d A_d\right)\left[x(k) - \hat{x}(k)\right] \\ &= \left(A_d - LC_d A_d\right)e(k) \end{aligned} \tag{10.100}$$

Note that this error equation is somewhat different from (10.71). In this case, it is the eigenvalues of the matrix $A_d - LC_d A_d$ that must be placed using appropriate selection of gain *L*. This is not guaranteed by the observability of the system, as it was with the conventional observer in Section 10.3.1. However, if system matrix A_d is of full rank, then it will not change the space spanned by the rows of C_d. If that is the case, we can think of the product $C_d A_d$ as a "new" output matrix

$$\overline{C}_d \triangleq A_d C_d$$

Then the eigenvalues of $A_d - L\overline{C}_d$ can be placed as usual. If A_d is *not* of full rank, then arbitrary placement of all the eigenvalues of $A_d - LC_d A_d$ will not be possible.

10.3.3 Output Feedback and Functional Observers

Before closing this chapter, we will briefly consider a different type of feedback and observer. We have seen that full state feedback is able to arbitrarily place the eigenvalues of a controllable system, and that if an observer is constructed, the full state will either be available or estimated. However, what can be said about the capabilities of feedback if an observer is *not* used, i.e., if only the output is fed back, in the form $u = K_o y$?

If $u = K_o y$, then the state equation in (10.68) would become

$$\begin{aligned} \dot{x} &= Ax + Bu \\ &= Ax + B(K_o y) \\ &= Ax + B\left[K_o(Cx + Du)\right] \\ &= (A + BK_o C)x + BK_o Du \end{aligned} \tag{10.101}$$

Of course, from (10.101) it is apparent that the eigenvalues of the matrix $A + BK_o C$ must be selectable through proper choice of the gain K_o. Matrix K_o in this case will be $p \times q$ rather than $p \times n$ as was the state-feedback matrix *K*. In the situation that $q = 1$, for example, only a single feedback gain will be available, so of course we could not possibly independently specify the locations of *n* eigenvalues. This is the situation in classical control that is illustrated by the root locus technique: A change in gain will alter all the system poles, but not independently of one another. In general, output feedback will be able to assign *some* eigenvalues, but not all, unless the number of inputs (p) and outputs (q) are both large. If C happens to be square and invertible, then of course the output feedback problem is equivalent to the state feedback problem, because the output is simply a transformed version of the state. However, pole assignment using output feedback is not a completely solved problem (see [5] and [13]).

Another variation on the concepts of controllers and observers is the *functional observer* [11]. The design of the full- and reduced-order observers above represents a time investment to extract the n state variables of a plant, whereas the real goal of the designer is actually to find the control signal $K\hat{x}$ that has only p components. The functional observer combines these functions by directly observing not the state vector, but a linear combination of the state variables, i.e., $K\hat{x}$. While it is only a single computational step to multiply the estimated state $\hat{x}$ by the gain K, the functional estimator can do this with a system of order $\mu_o - 1$, where μ_o is the observability index of the system (defined analogously to the controllability index discussed above) [10]. Thus, some economy can be realized via the use of the functional observer. In fact, if a system with a large n has a small observability index, then the functional observer will guarantee the existence of a low-order observer and controller that can still achieve arbitrary pole placement.

10.4 Summary

In this chapter, we have presented the most fundamental of all state space control system tools: feedback and state observation. State feedback is the vector generalization of "position" feedback in classical systems, although it is understood that the state vector might contain variables quite different from the concept of position. The state vector might contain positions, velocities, accelerations, or any other abstract type of state variable. Because we have defined the state vector of a system as the vector of information sufficient to describe the complete behavior of the system, feedback of the state is a general type of control.

However, the feedback we have considered in this chapter is sometimes referred to as *static* state feedback. The term *static* refers to the fact that the state vector is not filtered by a dynamic system before reintroducing it as a component of the input. This is in contrast to *dynamic* state feedback, which is the type of control usually studied in frequency-domain. In dynamic state feedback, the state vector or state estimate is processed by a dynamic system (a compensator), which has a transfer function (and hence, a state space) of its own. More often, dynamic feedback is applied to the *output* of a system, which is the classical control problem. One can view the observer and controller combined system as dynamic output feedback, because the output is "filtered" by the observer dynamics before resulting in the control input.

In the same sense, it might be argued that classical dynamic compensators, represented by a controller transfer function in the forward path of the feedback loop, really represent a form of observer and gain combination. The dynamics of the compensator "prefilters" the error signal in order to generate more information about the underlying system, thereby providing more data on which the gains can act.

The important topics presented in this chapter include:

- Computational models and formulas were given for placing the eigenvalues of a system using full state feedback. Ackermann's formula and the Bass-Gura formula are the two most common "pencil-and-paper" techniques, but these are not often the most numerically desirable. Computer-aided engineering programs such as MATLAB include one-word commands that can efficiently compute feedback gains for single- and multiple-input systems.

- It was found that the use of state feedback does not change the zeros of a system. Thus, because the poles are free to move and, potentially, cancel the zeros, it is possible that observability is lost by the introduction of state feedback. Controllability, however, is preserved by state feedback.

- The placement of eigenvalues in multi-input systems is difficult if performed "by hand." In order to facilitate this process, the concept of controllability indices was introduced. Using a transformation based on controllability indices, a system can be transformed into a multivariable controllable canonical form in which the feedback gains can be determined by considering their effects one input at a time. Again, this is a manual process that may be dissimilar from the numerical algorithms written into modern computer-aided control system design (CACSD) programs. However, the definitions and properties of controllability indices are useful in broader applications in multivariable control.

- The observer is a dynamic system that produces an estimate of the state variable of the plant system when that state is not directly measured. The full-order observer produces estimates of all the state variables, while the reduced-order observer estimates only those variables that are not measured via output y. The reduced-order observer is therefore of lower dimension than the full-order observer, but this does not necessarily guarantee that it requires fewer computations because of the transformations involved in its construction. There are other factors to consider in the choice of full- versus reduced-order observer as well, such as transient response and noise levels.

- In discrete time, the state feedback and observer calculations are exactly the same as in continuous time; it is only the desired eigenvalue locations that change. However, discrete-time systems offer the additional possibility of the *current* observer. The current observer separates the so-called "time update" from the "measurement update." This distinction will appear again in the Kalman filter, which will be discussed again in the next and final chapter.

The next chapter will explore two concepts that were mentioned only in passing in this chapter. The first is the problem of choosing a state feedback gain K for multi-input systems. Although we showed in this chapter a technique to select a few possible gains, we noted that there are many choices. In order to select a unique gain, a secondary criterion can be introduced: the cost function. By minimizing this cost function rather than placing the system eigenvalues, a unique gain can be determined. This is the basis for the *linear-quadratic regulator* (discussed in Chapter 11).

The second concept is embedded in the idea of the current estimator in discrete time. For systems with noise, provided that plant noise and sensor (output) noise have known statistical properties, a special observer is available that finds (statistically) the best possible observer given the noise model. This type of observer will be referred to as an *estimator*, and in discrete time, it will have the property of separating the computations into "time-update" and "measurement update" components. This estimator is known as a *Kalman filter* and is the standard for designing observers in noisy environments.

Together, the linear-quadratic regulator and the Kalman filter form a controller and observer pair, similar in concept to those described in this chapter, except that the feedback gain and the observer gain are both designed according to more sophisticated criteria. The combination is known as the linear-quadratic-gaussian controller (discussed in Chapter 11). Whereas the state feedback and full-order observer were described here as the most fundamental of the state space control techniques, the linear-quadratic-gaussian controller is often regarded as the most useful.

10.5 Problems

10.1 An uncontrollable state-space equation is given below. If possible, determine a gain matrix that transforms the system such that its eigenvalues are $\{-3,-3,-3,-3\}$. Repeat for eigenvalues $\{-3,-3,-2,-2\}$ and for $\{-2,-2,-2,-2\}$. If the solution is impossible, explain why.

$$\dot{x} = \begin{bmatrix} 3 & 3 & 0 & 2 \\ 0 & 87 & 0 & 60 \\ 6 & 3 & -3 & 2 \\ 0 & -126 & 0 & -87 \end{bmatrix} x + \begin{bmatrix} 0 \\ 3 \\ -1 \\ 4 \end{bmatrix} u$$

10.2 A two-input linear system has the following matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 7 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 \end{bmatrix} \qquad D = 0$$

a) Find two different state-feedback gain matrices such that the poles of the closed-loop system are at $s = -4$ and $s = -6$.

b) Change the *B*-matrix as given above to $b = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$. Then apply *output* feedback $u = -ky$ to the system. Find a range of *k* such that the closed-loop system is asymptotically stable.

10.3 A discrete-time system with transfer function

$$\frac{y(z)}{u(z)} = \frac{5}{z-3}$$

is to be controlled through state feedback. The closed-loop pole should be placed at $z = 0.6$, and the observer should have pole $z = 0.3$. Find the state feedback gain, observer gain, and the closed-loop augmented state variable description of the controller and observer pair.

10.4 A permanent magnet DC motor with negligible armature inductance has transfer function

$$G(s) = \frac{\theta(s)}{v(s)} = \frac{-50}{s(s+5)}$$

where $\theta(s)$ is the shaft angle and $v(s)$ is the applied voltage.

a) Find a state variable representation of the system, where the two state variables are $x_1(t) = \theta(t)$ and $x_2(t) = \dot{\theta}(t)$.

b) Design a full-state feedback matrix *K* such that the closed-loop transfer function of the system has damping $\zeta = 0.707$ and undamped natural frequency $\omega_n = 10 \text{ rad s}^{-1}$ (i.e., the characteristic polynomial has the form $s^2 + 2\zeta\omega_n s + \omega_n^2$).

c) Find a full-order observer with a gain matrix *L* such that the observer itself has $\zeta_o = 0.5$ and $\omega_{n_o} = 20 \text{ rad s}^{-1}$.

d) Compute the transfer function of the combined system with the observer and controller.

10.5 The dynamics of a wind turbine drive train are given in [12] as:

$$\dot{x} = \begin{bmatrix} -0.94 & 0.43 & 7.14\times10^{-6} \\ 0.98 & -0.98 & 1.64\times10^{-5} \\ 1.2\times10^{7} & -1.2\times10^{7} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1.64\times10^{-5} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x$$

where the state vector $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathrm{T}}$ consists of, respectively, the turbine angular velocity, the generator angular velocity, and the shaft torque due to torsional flexing. The input u represents the generator torque reference. Design a full-order observer and state-feedback controller combination that places the closed-loop poles at $s_1 = -10$ and $s_{2,3} = -5 \pm j5$. Notice how poorly conditioned this model is. How does this affect your procedure?

10.6 Find a state feedback gain and an observer such that the following system is stabilized.

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} x$$

10.7 Show that for the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = Ax + Bu = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = Cx + Du = \begin{bmatrix} I_{q \times q} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du = x_1 + Du$$

where each block submatrix is of consistent dimensions, the system (A_{22}, A_{12}) is observable if and only if the system (A, C) is observable.

10.8 Define the *observability* index of a system analogous to the controllability index, and show that when a system is in the form of Equation (10.84), if the observability index of the system (A, C) is ν, then the observability index of the system (A_{22}, A_{12}) is $\nu - 1$. Show that when the *individual* observability indices of (A, C) are ν_i, then the observability indices of (A_{22}, A_{12}) are $\nu_i - 1$.

10.9 For the following SISO continuous-time system:

$$\dot{x} = \begin{bmatrix} 3 & 5 & 4 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 1 & 0 & 1 & 0 \end{bmatrix} x$$

a) Determine the controllability and observability.

b) Design a full-order observer and a reduced-order (fourth-order) observer. Give the system suitable initial conditions and simulate each observer. For each, compare to the true states. Place the observer poles in the region $[-15, -10]$ in the complex plane. Compare each observer's output with the true states.

c) Design and implement full-state feedback $u = K\hat{x} + v$ such that the closed-loop poles are placed on the set $\{-2, -2, -3, -5, -5\}$. Verify by determining the closed-loop eigenvalues of the controller and observer (i.e., augmented) system.

10.10 An inverted pendulum on a driven cart is modeled in [15] as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 30.66 & 0 & 0 & 20.27 \\ 0 & 0 & 0 & 1 \\ -1.63 & 0 & 0 & -7.56 \end{bmatrix} x + \begin{bmatrix} 0 \\ -24.1 \\ 0 \\ 8.99 \end{bmatrix} u$$

$$y = \begin{bmatrix} 37.57 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & -4.5 \end{bmatrix} x$$

where the state vector $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ consists of, respectively, the pendulum angle, the pendulum's angular velocity, the cart position, and the cart's velocity. The input represents the voltage applied to the wheels on the cart.

a) Design a stabilizing state-feedback controller with a full-order observer and a reduced-order (first order) observer. Compare the performance of each.

b) Can the system be observed through only one of the outputs? Which one?

10.11 Design a state feedback controller to place the closed-loop eigenvalues of the system below at $z = 0.25 \pm 0.25j$:

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0.5 & 1 \end{bmatrix} x(k)$$

10.12 A discrete-time system is given as

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(k)$$

Determine a state feedback $u(k) = Kx(k)$ such that the closed-loop poles are located at $z = 0, 0$. Is it possible to do this with feedback to only $u_1(k)$? Is it possible to do this with feedback to only $u_2(k)$? In each case, find the feedback if possible.

10.13 The discrete-time system below is unstable. Design a state-feedback controller that places its poles at $z = 0.5 \pm 0.5j$. Also design both a full-order observer and a reduced-order (first-order) observer to estimate the system state vector. Specify suitable observer poles for each. Combine both types of observer with the feedback and simulate the zero-input response of each system. Compare and comment.

$$x(k+1) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(k)$$

10.14 For the system given in Problem 10.11, design and simulate a current estimator, and compare the results with that problem.

10.15 Show that a valid representation for a current estimator for a system with state feedback K is with the equations

$$\xi(k+1) = A_c \xi(k) + B_c y(k)$$
$$u(k) = C_c \xi(k) + D_c y(k)$$

where

$$\xi(k) \stackrel{\Delta}{=} \hat{x}(k) - Ly(k)$$

$$A_c \stackrel{\Delta}{=} (I - LC_d)(A_d - B_d K) \qquad B_c \stackrel{\Delta}{=} A_d L$$
$$C_c \stackrel{\Delta}{=} -K \qquad D_c \stackrel{\Delta}{=} -KL$$

10.16 Show that if the output matrix C is $n \times n$ and invertible, then for every *state* feedback gain matrix K that results in a particular set of closed-loop eigenvalues, there exists an *output* feedback gain matrix K_o that results in the same set of eigenvalues.

10.6 References and Further Reading

Fundamental procedures and properties of static state feedback can be found in the texts [2], [3], and [7]. The basic principles of state feedback depend of course on controllability. However, it was not originally obvious that controllability and the ability to arbitrarily place a system's eigenvalues were synonymous. That result can be found in [4] and [16].

Although not discussed here, the concept of *output* feedback is perhaps more basic (in the sense that a student will encounter it first in classical controls courses), but is actually a more difficult problem. Results on the possibility of arbitrary eigenvalue placement through output feedback can be found in [5] and [13]. A discussion of *dynamic* output feedback is given in [1], which includes a discussion of graphical techniques such as Bode and Nyquist plots, with which classical controls students are probably familiar.

The concept of the observer is attributable to Luenberger, [10] and [11]. This is interestingly a result of his Ph.D. thesis of 1963 from Stanford University. This work includes most of the basic aspects of observers, including the reduced order observer, canonical transformations, and even functional observers. Observers have since become a mainstay of state-feedback control theory and can be used as the basis for the *optimal estimators* [8] and [14] that we will consider in the next chapter.

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