A

Mathematical Tables and Identities

Throughout this book, references have been made to certain matrix identities, definitions, and properties. These, as well as other formulae such as selected integrals, are gathered here as a reference for the reader.

Basic Matrix Operations

An $n \times m$ matrix A consists of the nm elements a_{ij}, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. To represent a matrix by the array of its entries we use the notation:

$$A = \left[a_{ij} \right] \tag{A.1}$$

The sum of two matrices gives a third matrix of the same dimension: $A + B = C$, where

$$c_{ij} = a_{ij} + b_{ij} \tag{A.2}$$

The product of two matrices AB is only allowed if the number of columns of A is the same as the number of rows of B. In that case, $A_{n \times m} B_{m \times p} = C_{n \times p}$, where

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} \tag{A.3}$$

This multiplication operation may be performed block-by-block such that if matrices A and B are partitioned into compatibly dimensioned blocks, i.e., if

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

then

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \tag{A.4}$$

Except for special examples, matrix multiplication is not commutative, i.e.,

$$AB \neq BA \tag{A.5}$$

but other elementary properties of matrix algebra do hold:

1. Associativity over addition and multiplication:

$$\begin{aligned} (AB)C &= A(BC) \\ (A+B)+C &= A+(B+C) \end{aligned} \tag{A.6}$$

2. Distributivity of addition and multiplication:

$$\begin{aligned} A(B+C) &= AB+AC \\ (A+B)C &= AC+BC \end{aligned} \tag{A.7}$$

3. Multiplication of a matrix by a scalar is performed in an element-wise fashion: if $B = \alpha A$, then

$$b_{ij} = \alpha a_{ij} \tag{A.8}$$

The *Kronecker* product of two matrices *A* and *B*:

$$A \otimes B = \left[a_{ij} B \right] \tag{A.9}$$

This means that if *A* is $m \times n$ and *B* is $p \times q$, then $A \otimes B$ is $mp \times nq$ and the $(i, j)^{\text{th}}$ $p \times q$ block of $A \otimes B$ is $a_{ij} B$.

The *stacking* of a matrix is reshaping it in order to make it a single column. If we denote the j^{th} column of matrix *A* as $a_{\cdot j}$, then the stacking operator can be written as:

$$s(A) \stackrel{\Delta}{=} \begin{bmatrix} a_{\cdot 1} \\ a_{\cdot 2} \\ \vdots \\ a_{\cdot n} \end{bmatrix} \tag{A.10}$$

Transposes

The *transpose* of a matrix *A* is denoted A^{T}. If $B = A^{\mathrm{T}}$ and $A = \left[a_{ij}\right]$, then

$$B = \left[a_{ji}\right] \qquad \text{or} \qquad b_{ij} = a_{ji} \tag{A.11}$$

The complex-conjugate transpose of a matrix is denoted A^* or sometimes A^{H}. If $B = A^*$, then

$$b_{ij} = \bar{a}_{ji} \tag{A.12}$$

where the overbar indicates complex-conjugation.

For matrix products and sums,

$$(A_1 + A_2 + \cdots + A_n)^{\mathrm{T}} = (A_1^{\mathrm{T}} + A_2^{\mathrm{T}} + \cdots + A_n^{\mathrm{T}}) \tag{A.13}$$

$$(A_1 A_2 \cdots A_n)^{\mathrm{T}} = (A_n^{\mathrm{T}} \cdots A_2^{\mathrm{T}} A_1^{\mathrm{T}}) \tag{A.14}$$

A real-valued matrix *A* is said to be *symmetric* (sometimes called *self-adjoint*) if $A = A^{\mathrm{T}}$ and *skew-symmetric* if $A = -A^{\mathrm{T}}$. A complex-valued matrix *A* is said to be *hermitian* if $A = A^*$. It is *skew-hermitian* if $A = -A^*$. Regardless of the symmetry of matrix *A*, the products AA^{T} and $A^{\mathrm{T}}A$ are both hermitian.

Determinants

The determinant of a matrix *A* is defined only if it is square and is denoted $|A|$ or $\det(A)$. For any square matrix *A*,

$$\left|A^{\mathrm{T}}\right| = |A| \tag{A.15}$$

and for products of square matrices,

$$|AB| = |A||B| \tag{A.16}$$

The elementary properties of determinants are:

1. If any two rows or columns of a matrix are interchanged, its determinant changes sign.
2. If any single row or column of a matrix is multiplied by a scalar α, the determinant of the result is also multiplied by α.
3. If the entire matrix is multiplied by α, then

$$|\alpha A| = \alpha^n |A| \tag{A.17}$$

4. If A is an $n \times m$ matrix and B is an $m \times n$ matrix, then

$$|I_{n\times n} - AB| = |I_{m\times m} - BA| \tag{A.18}$$

The $(i, j)^{\text{th}}$ *cofactor* of matrix A is denoted $\text{cof}_{ij}(A)$ and is the determinant obtained by eliminating the i^{th} row and j^{th} column from A. It is the *negative* of such a determinant if $i + j$ is an odd number. The determinant of $n \times n$ matrix A can be defined in terms of cofactors:

$$|A| = a_{i1}\text{cof}_{i1}(A) + a_{i2}\text{cof}_{i2}(A) + \cdots + a_{in}\text{cof}_{in}(A) \tag{A.19}$$

or

$$|A| = a_{1j}\text{cof}_{1j}(A) + a_{2j}\text{cof}_{2j}(A) + \cdots + a_{nj}\text{cof}_{nj}(A) \tag{A.20}$$

for *any* row i or column j.

Inverses

The *inverse* of square matrix A is denoted A^{-1} and is a matrix such that

$$AA^{-1} = A^{-1}A = I \tag{A.21}$$

where I is the identity matrix. The inverse of A exists if and only if $|A| \neq 0$. The inverse may be computed from the cofactors and determinant. If $B = A^{-1}$, then

$$b_{ij} = \frac{\text{cof}_{ij}^{\text{T}}(A)}{|A|} \tag{A.22}$$

Some properties of matrix inverses are:

$$
\begin{aligned}
(A_1 A_2 \cdots A_n)^{-1} &= (A_n^{-1} \cdots A_2^{-1} A_1^{-1}) \\
(A^{-1})^{\mathrm{T}} &= (A^{-\mathrm{T}})^{-1} \stackrel{\Delta}{=} A^{-\mathrm{T}} \\
(\alpha A)^{-1} &= \frac{1}{\alpha} A^{-1} \\
\left|A^{-1}\right| &= \frac{1}{|A|}
\end{aligned}
\tag{A.23}
$$

When matrices are written in block form, further identities result. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad A^{-1} = B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{A.24}$$

Then $AB = I$, giving

$$A_{11}B_{11} + A_{12}B_{21} = I \tag{A.25}$$

$$A_{11}B_{12} + A_{12}B_{22} = 0 \tag{A.26}$$

$$A_{21}B_{11} + A_{22}B_{21} = 0 \tag{A.27}$$

$$A_{21}B_{12} + A_{22}B_{22} = I \tag{A.28}$$

If it is known that $|A_{11}| \neq 0$ and $|A_{22}| \neq 0$, then these four equations can be solved to give

$$B_{11} = \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} \tag{A.29}$$

$$B_{22} = \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \tag{A.30}$$

$$B_{12} = -A_{11}^{-1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \tag{A.31}$$

$$B_{21} = -A_{22}^{-1}A_{21}\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} \tag{A.32}$$

The first two of these, (A.29) and (A.30), are known as the *Schur* complements of A_{11} and A_{22}, respectively. If $|A_{22}| = 0$, then the solutions of (A.29) through (A.32) change to

$$B_{11} = A_{11}^{-1} + A_{11}^{-1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1}A_{21}A_{11}^{-1} \tag{A.33}$$

$$B_{22} = \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \tag{A.34}$$

$$B_{12} = -A_{11}^{-1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \tag{A.35}$$

$$B_{21} = -\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1}A_{21}A_{11}^{-1} \tag{A.36}$$

A similar result can be obtained if $|A_{22}| \neq 0$ and $|A_{11}| = 0$.

Comparing (A.29) and (A.33) gives

$$\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1}A_{21}A_{11}^{-1} \tag{A.37}$$

or, more generally,

$$\left(A + BCD\right)^{-1} = A^{-1} - A^{-1}B\left(C^{-1} + DA^{-1}B\right)^{-1}DA^{-1} \tag{A.38}$$

This is known as the *matrix inversion lemma*. It can be used for any set of dimensionally compatible matrices, and is also useful for reducing the size of certain matrix inversion operations:

$$\left(I_{n\times n} + BD\right)^{-1} = I_{p\times p} - B\left(I_{p\times p} + DB\right)^{-1}D \tag{A.39}$$

where B is $n \times p$ and D is $p \times n$. Alternatively, it is used for deriving a recursive matrix inversion formula useful in recursive estimation:

$$(A + xy^{\mathrm{T}})^{-1} = A^{-1} - \frac{\left(A^{-1}x\right)\left(y^{\mathrm{T}}A^{-1}\right)}{1 + y^{\mathrm{T}}A^{-1}x} \tag{A.40}$$

A similar formula applies to determinants:

$$|A| = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = |A_{11}|\left|A_{22} - A_{21}A_{11}^{-1}A_{12}\right| = |A_{22}|\left|A_{11} - A_{12}A_{22}^{-1}A_{21}\right| \tag{A.41}$$

The *resolvent* of a matrix *A* is $(sI - A)^{-1}$. The *resolvent identity* used in the derivation of the Bass-Gura formula of Chapter 10 is

$$(sI-A)^{-1} = \frac{\left[s^{n-1}I + (A + a_{n-1}I)s^{n-2} + \cdots + (A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I)\right]}{\left|(sI-A)\right|} \tag{A.42}$$

where the coefficients a_i come from the characteristic polynomial:

$$|sI - A| \stackrel{\Delta}{=} s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \tag{A.43}$$

The *generalized inverse* (or *pseudoinverse*) of a matrix *A* is a matrix denoted A^+ such that

$$AA^+A = A^+ \tag{A.44}$$

The *Moore-Penrose* generalized inverse of matrix *A* is a generalized inverse with the additional properties

$$A^+AA^+ = A \tag{A.45}$$

$$\left(A^+A\right)^* = A^+A \tag{A.46}$$

$$\left(AA^+\right)^* = AA^+ \tag{A.47}$$

Trace

The *trace* of an $n \times n$ matrix *A* is defined as the sum of all elements on the diagonal:

$$tr(A) = \sum_{i=1}^{n} a_{ii} \tag{A.48}$$

For compatible nonsquare matrices *A* and *B*,

$$tr(AB) = tr(BA) \tag{A.49}$$

or for three compatible matrices *A*, *B*, and *C*,

$$tr(ABC) = tr(CAB) = tr(BCA) \tag{A.50}$$

Special Matrices and Matrix Forms

A square matrix A is *idempotent* if

$$A^2 = AA = I \tag{A.51}$$

A square matrix A is *nilpotent* if there exists a positive integer r, called the *index of A*, such that

$$A^r = 0 \tag{A.52}$$

A *Vandermonde* matrix is a matrix of the form

$$A = \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix} \tag{A.53}$$

Matrices whose transposes are of this form are also called Vandermonde.

A *Toeplitz* matrix is a matrix of the form

$$A = \begin{bmatrix} a & b & c & \cdots & x \\ \beta & a & b & c & \vdots \\ \gamma & \beta & a & b & \ddots \\ \vdots & \gamma & \beta & a & \ddots \\ \chi & \cdots & \ddots & \ddots & \ddots \end{bmatrix} \tag{A.54}$$

One way to describe such a matrix is that entry a_{ij} is the same for every element for which $i - j$ is the same.

A matrix is said to be in *Hankel* form if

$$A = \begin{bmatrix} a & b & c & \cdots \\ b & c & \iddots & \\ c & \iddots & \ddots & \\ \vdots & & & \end{bmatrix} \tag{A.55}$$

A matrix A is said to be upper (or lower) *triangular* if it has the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \ddots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{A.56}$$

A matrix A is said to be *diagonal* if it has the form

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} \tag{A.57}$$

A matrix A is said to be *orthogonal* if $A^{\mathrm{T}} A$ is diagonal. In that case, the inner product of any two different columns is zero, i.e., the columns are pair-wise orthogonal.

A matrix A is said to be *orthonormal* if $A^{-1} = A^{\mathrm{T}}$.

Matrices in *companion form* are discussed in the text.

Matrix Decompositions

Matrices may be decomposed into factors in several ways, each convenient for different purposes. We will give the notation here but will not present the methods by which such decompositions may be computed.

A *Cholesky* decomposition of matrix A is a factorization

$$A = B^* B \tag{A.58}$$

where B is a nonsingular upper triangular matrix.

A *QR factorization* of matrix A is a factorization

$$A = QR \tag{A.59}$$

such that Q is orthonormal and R is upper triangular. If matrix A is any $n \times m$ and has rank m, then the decomposition will be of the form

$$A = Q \begin{bmatrix} R \\ \hdashline 0 \end{bmatrix} \tag{A.60}$$

where R is again upper triangular. QR factorizations may be used to determine the eigenvalues or rank of a matrix.

An *LU decomposition* of matrix A is a factorization

$$A = LU \tag{A.61}$$

such that matrix L is lower triangular and matrix U is upper triangular.

Singular value decompositions are discussed in the text.

Matrix Calculus

For matrices whose elements are functions of a scalar parameter t, matrix differentiation and integration is performed in an element-wise fashion:

$$\frac{d}{dt} A(t) = \left[\frac{d}{dt} a_{ij}(t) \right] \tag{A.62}$$

and

$$\int A(t) = \left[\int a_{ij}(t)\, dt \right] \tag{A.63}$$

In the following definitions, $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}$ and $y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^{\mathrm{T}}$ are vectors of length n, $g(x)$ is a scalar function of such a vector, and

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix}$$

is a vector function of length m.

The differential of g is

$$dg = \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 + \cdots + \frac{\partial g}{\partial x_n} dx_n$$
$$= \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & \frac{\partial g}{\partial x_n} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix} \qquad \text{(A.64)}$$

This implies the vector derivative

$$\frac{dg}{dx} = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & \frac{\partial g}{\partial x_n} \end{bmatrix} \qquad \text{(A.65)}$$

Note that this is properly written as a row vector rather than as a column vector as often presented. We define the *gradient* of *g* as the column-vector counterpart:

$$\text{grad}(g) = \nabla g = \frac{d^{\mathrm{T}} g}{dx} = \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \vdots \\ \frac{\partial g}{\partial x_n} \end{bmatrix} \qquad \text{(A.66)}$$

The derivative of a vector with respect to another vector is then consistently defined as the *jacobian* matrix

$$\frac{df}{dx} = \begin{bmatrix} \frac{\partial f_i}{\partial x_j} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \qquad \text{(A.67)}$$

which of course is an $m \times n$ matrix. Note that this is consistent with the definition of the derivative of a scalar function with respect to a vector, Equation (A.65).

If A is an $n \times n$ matrix

$$\frac{\partial}{\partial y}(Ay) = A \tag{A.68}$$

$$\frac{\partial}{\partial y}\left(x^{\mathrm{T}} Ay\right) = x^{\mathrm{T}} A \tag{A.69}$$

$$\frac{\partial}{\partial x}\left(x^{\mathrm{T}} Ay\right) = (Ay)^{\mathrm{T}} = y^{\mathrm{T}} A^{\mathrm{T}} \tag{A.70}$$

$$\frac{\partial}{\partial x^{\mathrm{T}}}\left(x^{\mathrm{T}} Ay\right) = \frac{\partial^{\mathrm{T}}}{\partial x}\left(x^{\mathrm{T}} Ay\right) = A^{\mathrm{T}} y \tag{A.71}$$

As a result of these,

$$\frac{\partial}{\partial x}\left(x^{\mathrm{T}} Ax\right) = x^{\mathrm{T}} A + x^{\mathrm{T}} A^{\mathrm{T}} = x^{\mathrm{T}} (A + A^{\mathrm{T}}) \tag{A.72}$$

or

$$\frac{\partial^{\mathrm{T}}}{\partial x}\left(x^{\mathrm{T}} Ax\right) = \nabla\left(x^{\mathrm{T}} Ax\right) = (A + A^{\mathrm{T}})x \tag{A.73}$$

A result of this is that if A is symmetric,

$$\frac{\partial^{\mathrm{T}}}{\partial x}\left(x^{\mathrm{T}} Ax\right) = \nabla\left(x^{\mathrm{T}} Ax\right) = 2Ax \tag{A.74}$$

When performing matrix differentiation, the chain rule and product rule apply, but we must always be careful to preserve the proper order of matrix multiplication. If A and B are compatible matrices and x, y, and z are vectors, then

$$\begin{aligned} \frac{d}{dt}(A(t)B(t)) &= \frac{dA(t)}{dt} B(t) + A(t) \frac{dB(t)}{dt} \\ &\neq \frac{dA(t)}{dt} B(t) + \frac{dB(t)}{dt} A(t) \end{aligned} \tag{A.75}$$

$$\begin{aligned} \frac{d\left[x(y(z))\right]}{dt} &= \frac{\partial x}{\partial y}\frac{\partial y}{\partial z} \\ &\neq \frac{\partial y}{\partial z}\frac{\partial x}{\partial y} \end{aligned} \tag{A.76}$$

The differentiation of trace operators provides several useful results for optimizing the squared-error criterion and error covariances:

$$\frac{\partial}{\partial A} tr(A) = I \tag{A.77}$$

$$\frac{\partial}{\partial A} tr(CAB) = (CB)^{\mathrm{T}} = B^{\mathrm{T}} C^{\mathrm{T}} \tag{A.78}$$

$$\frac{\partial}{\partial A} tr\left(ABA^{\mathrm{T}}\right) = AB + AB^{\mathrm{T}} \tag{A.79}$$

Integrals

Sometimes integral tables are helpful in the computation of vector inner products and norms, and in the projection operations such as those performed during the Gram-Schmidt orthogonalization process of Chapter 3.

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax \tag{A.80}$$

$$\int \sin^2 ax\, dx = \frac{1}{2}x - \frac{1}{4a}\sin 2ax \tag{A.81}$$

$$\int \sin^n ax\, dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax\, dx \tag{A.82}$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax \tag{A.83}$$

$$\int \cos^2 ax\, dx = \frac{1}{2}x + \frac{1}{4a}\sin 2ax \tag{A.84}$$

$$\int \cos^n ax\, dx = \frac{\sin ax \cos^{n-1} ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\, dx \tag{A.85}$$

$$\int (\sin ax)(\cos ax)\; dx = \frac{1}{2a}\sin^2 ax \tag{A.86}$$

$$\int xe^{ax}\, dx = \frac{e^{ax}}{a^2}(ax-1) \tag{A.87}$$

$$\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx \tag{A.88}$$

$$\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} \tag{A.89}$$

$$\int_0^{\infty} x e^{-x^2}\, dx = \frac{1}{2} \tag{A.90}$$

$$\int_0^{\infty} x^2 e^{-x^2}\, dx = \frac{\sqrt{\pi}}{4} \tag{A.91}$$

$$\int_0^{\infty} x^{2n} e^{-x^2}\, dx = \frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2^{n+1}}\sqrt{\pi} \tag{A.92}$$

$$\int_0^{\infty} x^{2n+1} e^{-x^2}\, dx = \frac{n!}{2} \tag{A.93}$$

$$\frac{d}{dt}\left[\int_{g(t)}^{h(t)} f(x,t)\, dx\right] = \int_{g(t)}^{h(t)} \frac{\partial}{\partial t} f(x,t)\, dx + f(h(t),t)\frac{dh(t)}{dt} - f(g(t),t)\frac{dg(t)}{dt} \tag{A.94}$$