1

Models of Linear Systems

Linear systems are usually mathematically described in one of two domains: time-domain and frequency-domain. The frequency-domain approach (s- or ω-domain) usually results in a system representation in the form of a *transfer function*. Transfer functions represent the ratio of a system's frequency-domain output to the frequency-domain input, assuming that the initial conditions on the system are zero. Such descriptions are the subject of many texts in *signals and systems*.

In time-domain, the system's representation retains the form of a differential equation. However, as any student of engineering will attest, differential equations can be difficult to analyze. The mathematics gets more burdensome as the order of the equations increases, and the combination of several differential equations into one single system can be difficult.

In this chapter, we will introduce a time-domain representation of systems that alleviates some of the problems of working with single, high-order differential equations. We will describe a system with *state variables*, which collectively can be thought of as a vector. Using the language of vector analysis, we will demonstrate that state variables provide a convenient time-domain representation that is essentially the same for systems of all order. Furthermore, state variable descriptions do not assume zero initial conditions, and allow for the analysis and design of system characteristics that are not possible with frequency-domain representations. We will begin with some elementary definitions and a review of mathematical concepts. We will give a number of examples of state variable descriptions and introduce several of their important properties.

1.1 Linear Systems and State Equations

To define what we mean by a *linear system*, we will categorize the types of systems encountered in nature. First, a *system* is simply the mathematical description of a relationship between externally supplied quantities (i.e., those coming from outside the system) and the dependent quantities that result from the action or effect on those external quantities. We use the term "input" or u to refer to the independent quantity, although we indeed may have no control over it at all. It merely represents an excitation for the system. The response of the system will be referred to as the output y. These input and output signals may be constant, defined as functions over continuous-time or discrete-time, and may be either deterministic or stochastic. The system that relates the two may be defined in many ways, so for the time being, we depict it as in Figure 1.1, simply a block that performs some mathematical operation.

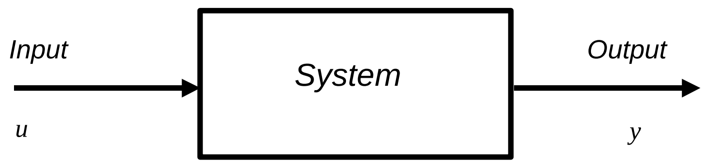


Figure 1.1 Elementary representation of a system acting on an input and producing an output.

1.1.1 Definitions and Review

In this section, we present some definitions for systems that will be useful in subsequent chapters. It is expected that the reader already has some familiarity and practice with these concepts from signals and systems studies.

> ***Memory:*** A system with *memory* is one whose output depends on itself from an earlier point in time. A system whose output depends only on the current time and the current input is *memoryless*. (1.1)

Systems with memory most often occur as differential equations (continuous-time), or as difference equations (discrete-time) because closed-form solutions of such systems require integration (or summation) of a quantity over past time. Systems with hysteresis are also examples of systems with memory because the portion of curve on which they operate depends on the past state and the direction of change of the input. For our purposes, we will have systems we call *algebraic*, which are memoryless, and *differential* or *difference*, which represent differential equations or difference equations. Furthermore, our treatment of algebraic systems will serve as a tool for the more detailed discussion of differential systems in the latter chapters.

> ***Causality:*** A system is said to be *causal* if the value of the output at time t_0 depends on the values of the input and output

for all t up to time t_0 but no further, i.e., only for $t \le t_0$
. (1.2)

Systems that are *not* causal are sometimes called *anticipatory*, because they violate the seemingly impossible condition that they can anticipate future values of a signal, predicting it at some future time for use at the current time. Anticipatory systems are often used in data filtering and image processing applications, wherein an entire data set is first acquired, then processed in batch mode. In such situations, the “next data” is already available for processing at any given time.

It is known that for a system to be causal, its transfer function (if it has one) must be proper. That is, the degree of its numerator polynomial must be no greater than its denominator polynomial. This is true in both continuous-time systems (*s*-domain) and discrete-time systems (*z*-domain).

Time Invariance: Given an initial time t_0, the output of a system will in general depend on the current time as well as this initial time, $y = y(t, t_0)$. A *time-invariant system* is one whose output depends only on the difference between the initial time and the current time, $y = y(t - t_0)$. Otherwise, the system is *time-varying*. (1.3)

Time-varying systems are typically systems in which time appears as an explicit variable in the differential, difference, or algebraic equation that describes the system. Thus, a time-invariant differential equation must, by necessity, be one with constant coefficients. Time-varying systems have outputs that depend, in a sense, on the actual “clock” time at which they were “turned on.” Time-invariant systems have outputs that depend on time only to the extent that they depend on how long it has been since they were “turned on.” Thus if the input were shifted in time, the output would be simply shifted in time as well. Time-varying equations are very difficult to solve, rivaling *nonlinear* equations.

To define linearity, we consider the action of the system to be represented by the symbol *S*, i.e., using our previous notation, $y = S(u)$. If we consider two inputs, u_1 and u_2, and a scaling factor, *a*, we introduce the definition:

Linearity: A *linear system* is one that satisfies *homogeneity* and *additivity*. A *homogeneous system* is one for which $S(au) = aS(u)$ for all a and u, and an *additive system* is one for which $S(u_1 + u_2) = S(u_1) + S(u_2)$ for all u_1 and u_2. (1.4)

Linear systems are thus systems for which the principle of superposition holds. We will later consider so-called multivariable systems, which have more than one input and more than one output. When such systems are linear, the effect of each input can be considered independently of one another. In systems with memory, the term *linear* refers to systems that are linear in *all* of the variables on which they depend. Therefore, for example, a linear n^{th} order differential equation is one whose n^{th} derivative depends in a linear way on each of the lower derivatives, and also in a linear way on the forcing function, if any.

Nonlinear systems are notoriously difficult to analyze and solve, partly because they exist in such an infinite variety of forms, preventing any cohesive theory for analysis.

In the next section, we will review the process by which models of linear systems are derived, followed by some examples for practical physical systems.

1.1.2 Physical System Modeling

The underlying motivation for all the analysis tools presented in this book is the understanding of physical systems. Whether the system is mechanical, electrical, or chemical, a mathematical description must be written in a unified way so that a single theory of stability, control, or analysis can be applied to the model. This is often the first task of an engineer in any design problem. In this section, we will introduce linear modeling principles for electrical, mechanical, and some fluid systems, and we will attempt to illustrate the unity in such models.

Physical Variables

We start by categorizing the physical quantities of interest to us. The first quantities available in any problem specification are the *constants*. These are, of course, constant numerical values specifying the dimensions, ranges, amounts, and other physical attributes of the masses in the system. These are often available as known quantities, but are sometimes unknown or poorly known and are subject to a process known as *system identification*. System identification is also useful when the physical attributes of a system are not constant, but vary with time. For example, the weight of a vehicle may change as its fuel is consumed, and the resistance of a resistor may change with temperature. In this chapter and for most of this book, we will not consider time-varying quantities in much detail.

The second class of number to be considered is the *variables*, which are of interest to us because they do usually vary with time. Some variables, i.e., those considered *inputs*, are known a priori, while others (the outputs) are to be determined. We separate these into two broad categories: *flow* variables and *potential* variables. Flow variables are quantities that must be measured through the cross-section of the medium through which they are transmitted. The easiest flow variable to imagine is a fluid (nonviscous, incompressible). The flow of such a fluid through a conduit must be measured by breaking the pipe and "counting" the amount (mass) of fluid passing through the cross-section.* For electrical systems, the analogous flow variable can be considered to be the current. Because current is defined as the amount of charge per unit area flowing across a cross-section per unit time, we can equally well consider charge to be a flow variable. In mechanical systems, force is a flow variable. Although it may not conform to the fluid analogy, it is nevertheless a quantity that must be measured by breaking a connection and inserting a measurement device.

The second type of variable is the potential variable. Potential variables are physical quantities that must be measured at two locations; the value of the measurement is the relative difference between the locations. Pressure, voltage, and displacement (position or velocity) are all potential variables because their definitions all require a reference location. Although we speak of a voltage appearing at a particular location in a circuit, it is always understood that this measurement was taken relative to another point.

Physical Laws

For many simple systems, there are only a few basic physical laws that must be obeyed in order to generate a sufficient set of equations that describe a system. If we consider only the basic necessities (i.e., using finite-dimensional, lumped-parameter, Newtonian dynamics rather than relativistic mechanics), we can categorize these into two types of laws: *mass* conservation and *circuit* conservation laws. Mass conservation laws are defined on nodes, and circuit conservation laws are defined on closed paths. A *node* is an interconnection point between two or more conduits transmitting a flow variable. In mechanical systems, nodes are associated with masses so that applied forces are shown to be acting on something. These circuit laws are integrally related to the two types of variables above: flow and potential.

Mass conservation laws take the basic form:

$$\sum \begin{pmatrix} \text{all flow variables} \\ \text{entering a node} \end{pmatrix} = \sum \begin{pmatrix} \text{net equivalent flow} \\ \text{into node} \end{pmatrix} \tag{1.5}$$

For an electrical network, this type of law translates to Kirchoff's current law (KCL), which states that the sum of all currents entering a node must equal zero, $\sum i_i = 0$. For a mechanical system, the mass conservation law takes the form of Newton's law: $\sum F_i = ma$. Note that in Newton's law, the sum of flow variables need not equal zero but must be proportional to the net acceleration of the object on which the forces act. In electrical and fluid systems, the net equivalent flow is

* In reality, clever fluid flow measurement systems have been devised that can measure flow variables without interrupting them, e.g., ultrasonic flowmeters and inductive ammeters.

zero, because it is impossible for net charges to accumulate indefinitely in a wire connection, just as it would be for fluid molecules to accumulate in a conduit junction.

Circuit conservation laws take the following form:

$$\sum \begin{pmatrix} \text{signed changes in a potential} \\ \text{variable around a closed path} \end{pmatrix} = 0 \tag{1.6}$$

Such laws enforce the intuitive notion that if a potential variable is measured at one location relative to a fixed reference, and if relative changes are added as components are traversed in a closed path, then the potential measured at the original location upon returning should not have changed. Thus, Kirchoff's voltage law (KVL) specifies that around any closed path in a network, $\sum v_i = 0$, being careful to include the appropriate algebraic signs in the terms. In mechanical systems, circuit conservation allows us to measure what we consider absolute position by summing a sequence of relative displacements (although in truth all positions are relative to something). For fluids, the sum of pressure drops and rises throughout a closed network of pipes and components must equal zero.

These laws go a long way toward generating the equations that describe physical systems. We are ignoring a great many physical quantities and processes, such as deformation, compressibility, and distributed parameters, that usually provide a more complete and accurate model of a system. Usually, though, it is best to attempt a simple model that will suffice until its deficiencies can be discovered later.

Constitutive Relationships

The physical laws above are not by themselves sufficient to write complete equations. Flow variables and potential variables are not unrelated, but their relationship depends on the physical device being considered. Aside from sources, which provide, e.g., input forces, voltages, currents, and flows to systems, we also have *components*, which we assume to be lumped, i.e., their effects are modeled as being concentrated at a single location as opposed to being distributed over space. Each type of component has associated with it a *constitutive relationship* that relates the flow variable through it and the potential variable across it.

Electrical Components

The three most basic linear components common in electrical networks are the resistor (R, measured in ohms, Ω), capacitor (C, measured in farads, F), and inductor (L, measured in henrys, H). These are pictured in Figure 1.2.

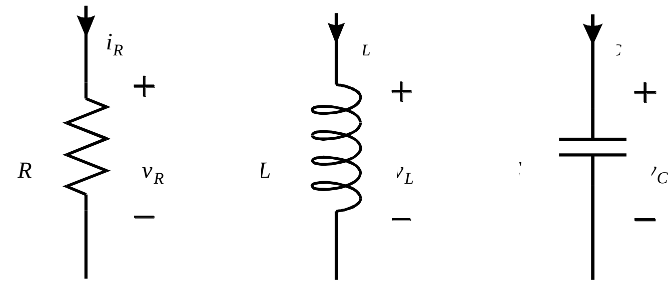


Figure 1.2 Electrical components: resistor, inductor, and capacitor (left to right).

For each component, we define reference directions for the flow variable (indicated by an arrow) and the potential variable (indicated by $+/-$ signs), so that the appropriate algebraic sign can be defined for the component.

For these three components, the constitutive relationships are:

$$v_R = i_R R \qquad v_L = L\frac{di_L}{dt} \qquad i_C = C\frac{dv_C}{dt} \tag{1.7}$$

At any such component, these relationships can be used to substitute a flow variable for a potential variable or vice versa. Note that with the differential relationships, an integration is necessary when expressing the reverse relationship.

Mechanical Components

For mechanical systems, the fundamental components are the mass, damper, and spring. These are pictured in Figure 1.3 below.

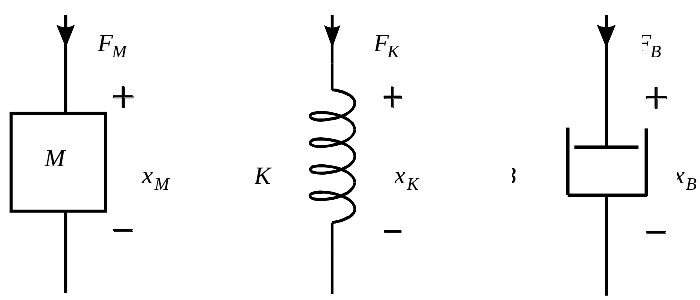


Figure 1.3 Mechanical components: mass, spring, and damper (left to right).

In the figure, a displacement is shown which indicates that one side of the component is displaced relative to the other side, except for the mass, which is displaced relative to a fixed reference defined elsewhere. The constitutive equations for these elements are

$$F_M = M\frac{d^2 x_M}{dt^2} \qquad F_K = K x_K \qquad F_B = B\frac{dx_B}{dt} \tag{1.8}$$

where F is the force applied to (or resulting from) the component. It should be noted that these equations do not appear entirely analogous to those of the electrical system (1.7). Most notably, these equations have a second derivative, and are all expressions of the flow variables in terms of the potential variable. This is partly based on convention so that the requisite computations are more convenient, and partly natural, since electrical and mechanical quantities are perceived differently by the systems on which they act. Nevertheless, the mathematical analogies remain. For fluid systems, the analogy becomes even weaker, with tanks and valves being the primary components. These components have somewhat more complex, sometimes nonlinear constitutive relationships. In truth, though, all constitutive relationships become nonlinear when the limits of their capacity are approached.

Example 1.1: Mechanical System Equations

Derive the equations of motion for the system of two masses shown in Figure 1.4. In the system, the two masses are connected by a spring with Hooke's law constant K and a damper with damping constant B, both initially unstretched and stationary. Their positions on the horizontal plane are measured as $x_1(t)$ and $x_2(t)$ from points on the masses such that they are initially equal. In addition, an externally applied force F pushes on the first mass.

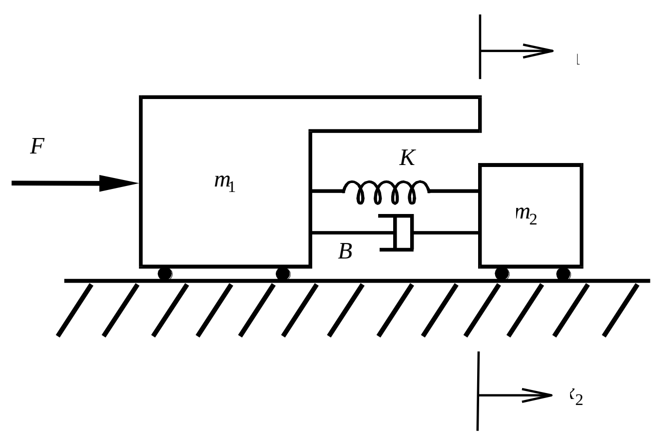


Figure 1.4 Two-mass system containing a spring and damper, and a forcing function.

Solution:

As mass 1 moves in the direction of positive x_1, the spring will compress and react with a force $K(x_1 - x_2)$ against the motion. Likewise, the damper will resist motion with viscous friction force $B(\dot{x}_1 - \dot{x}_2)$. The free-body diagram of the system in Figure 1.5 shows the two masses and all of the forces acting on them.

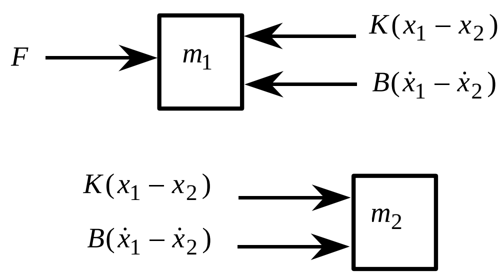


Figure 1.5 Free-body diagrams showing the masses in Figure 1.4 and the forces that act on them.

Applying Newton's law, $ma = \sum F$, we get

$$m_1\ddot{x}_1 = F - K(x_1 - x_2) - B(\dot{x}_1 - \dot{x}_2) \tag{1.9}$$

For the second mass, only the spring and damper provide forces, which are equal and opposite to the forces seen in (1.9). Therefore,

$$m_2\ddot{x}_2 = K(x_1 - x_2) + B(\dot{x}_1 - \dot{x}_2) \tag{1.10}$$

Rearranging these equations to a more convenient form,

$$\begin{aligned} m_1\ddot{x}_1 + B(\dot{x}_1 - \dot{x}_2) + K(x_1 - x_2) &= F \\ m_2\ddot{x}_2 + B(\dot{x}_2 - \dot{x}_1) + K(x_2 - x_1) &= 0 \end{aligned} \tag{1.11}$$

These simple linear equations will be used for further examples later.

Example 1.2: Electrical System Equations

For the circuit shown in Figure 1.6, derive differential equations in terms of the capacitor voltage $v_c(t)$ and the inductor current $i(t)$.

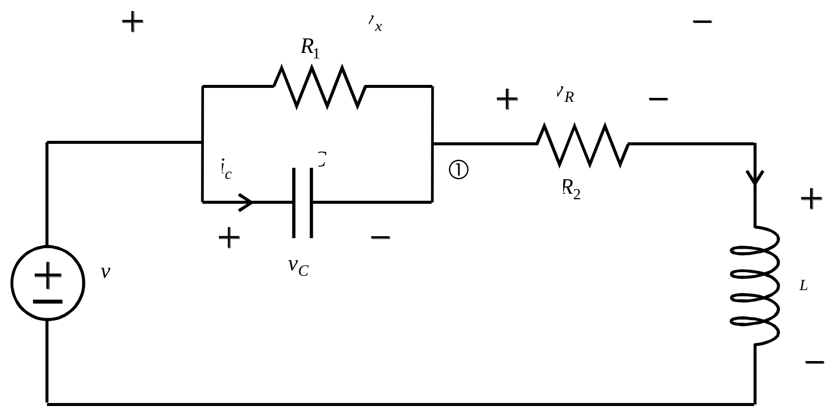


Figure 1.6 Electric circuit example with variables $v_c(t)$ and $i(t)$.

Solution:

In this example, the sum of the three currents entering node 1 must be zero. While Ohm's law $v = iR$ may be used for the resistor, the current through the capacitor is given by $i_c = C\,dv_c/dt$. Thus,

$$-\frac{v_c}{R_1} - C\frac{dv_c}{dt} + i = 0 \tag{1.12}$$

where care has been taken to use only the desired variables $v_c(t)$ and $i(t)$. For a second relationship, we note that the sum of all voltages around the main loop of the circuit must be zero. The voltage across the inductor is given by $v = L\,di/dt$, so

$$-v + v_c + R_2 i + L\frac{di}{dt} = 0 \tag{1.13}$$

where $v(t)$ is the forcing function provided by the voltage source in the circuit. Rewriting (1.12) and (1.13) in a more convenient form,

$$C\frac{dv_c}{dt} + \frac{1}{R_1}v_c - i = 0 \qquad L\frac{di}{dt} + R_2 i + v_c = v \tag{1.14}$$

1.1.3 State Equations

The sets of equations derived in (1.11) and (1.14) are coupled, in the sense that the variables in one appear in the other. This implies that they must be solved simultaneously, or else they must be combined into a single, larger-order differential equation by taking derivatives and substituting one into the other. The standard methods for solving differential equations are then applied.

However, such a process can be tedious, and the methods employed vary in complexity as the order of the differential equation increases. Instead, we prefer to write the dynamic equations of physical systems as *state equations*. State equations are simply collections of first-order differential equations that together represent exactly the same information as the original larger differential equation. Of course, with an n^{th}-order differential equation, we will need n first-order equations. However, the variables used to write these n first-order equations are not unique. These so-called "state variables" may be chosen for convenience, as one set of state variables may result in mathematical expressions that make the solution or other characteristic of the system more apparent.

In a strict sense, the collection of state variables at any given time is known as the *state* of the system, and the set of all values that can be taken on by the state is known as the *state space*. The state of the system represents complete information of the system, such that if we know the state at time t_0, it is possible to compute the state at all future times. We will model the state spaces for linear systems as linear vector spaces, which we begin to discuss in the next chapter.

State Variables

Consider a n^{th}-order linear, time-invariant differential equation:

$$\frac{d^n x(t)}{dt^n} + a_{n-1}\frac{d^{n-1}x(t)}{dt^{n-1}} + \cdots + a_1\frac{dx(t)}{dt} + a_0 x(t) = b_0 u(t) \tag{1.15}$$

The most straightforward method for choosing n state variables to represent this system is to let the state variables be equal to $x(t)$ and its first $(n-1)$ derivatives. Thus, if the state variables are denoted by ξ, then

$$\begin{aligned}
\xi_1(t) &= x(t) \\
\xi_2(t) &= \frac{dx(t)}{dt} \\
&\vdots \\
\xi_n(t) &= \frac{d^{n-1}x(t)}{dt^{n-1}}
\end{aligned}$$

These definitions of state variables are also called *phase variables*. The n differential equations resulting from these definitions become

$$\begin{aligned}
\dot{\xi}_1(t) &= \xi_2(t) \\
\dot{\xi}_2(t) &= \xi_3(t) \\
&\vdots \\
\dot{\xi}_{n-1}(t) &= \xi_n(t) \\
\dot{\xi}_n(t) &= -a_0\xi_1(t) - a_1\xi_2(t) - \cdots - a_{n-2}\xi_{n-1} - a_{n-1}\xi_n + b_0u(t)
\end{aligned} \tag{1.16}$$

We will find it convenient to express such a system of equations in vector-matrix form:

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} u(t) \tag{1.17}$$

If in addition, one of the state variables, say $\xi_1(t)$, is designated as the "output" of interest, denoted $y(t)$, then we can also write the so-called "output equation" in vector-matrix form as well:

$$y(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} \tag{1.18}$$

More generally, we may designate as the output a weighted sum of the state variables and sometimes also a sum of state variables and *input* variables. Defining $x(t) \overset{\Delta}{=} \begin{bmatrix} \xi_1 & \cdots & \xi_n \end{bmatrix}^{\mathrm{T}}$, the two equations (1.17) and (1.18) are together written[M] as

$$\begin{aligned}
\dot{x} &= Ax + bu \\
y &= cx + du
\end{aligned} \tag{1.19}$$

`ss(a,b,c,d)`
`ssdata(sys)`

where the matrices A, b, c, and d are the corresponding matrices[M] in (1.17) and

(1.18). These will be referred to as the *state matrix* (A), the *input matrix* (b), the *output matrix* (c), and the *feedthrough matrix* (d) (so named because it is the gain through which inputs feed directly into outputs). These equations are expressed for a single input, single output (SISO) system. For multi-input, multioutput (MIMO) or multivariable systems, the equations in (1.19) can be written exactly the same except that the input matrix is a capital B, the output matrix is a capital C, and the feedthrough matrix is a capital D. These changes indicate that they are matrices rather than simple columns (b), rows (c), or scalars (d). The equations in (1.19) will become quite familiar, as they are the format used for studying the properties of linear systems of equations throughout this book.

Of course, if the original Equation (1.15) were time-varying, then the coefficients might be functions of time, i.e., $a_i(t)$ and $b_i(t)$. In that case, (1.19) might contain $A(t)$, $b(t)$, $c(t)$ and $d(t)$.

Example 1.3: State Variables for the Mechanical System Example

Write a state variable expression for the differential equations already derived for the mechanical system of Example 1.1, using force F as the input and the difference $x_2 - x_1$ as the output.

Solution:

In the mechanical system, we derived two separate equations, each being second order (1.11). To generate state equations, we will introduce the variables $\xi_1 = x_1$, $\xi_2 = \dot{x}_1$, $\xi_3 = x_2$, and $\xi_4 = \dot{x}_2$. Then, by inspection of (1.11), the state equations are:

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/m_1 & -B/m_1 & K/m_1 & B/m_1 \\ 0 & 0 & 0 & 1 \\ K/m_2 & B/m_2 & -K/m_2 & -B/m_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ 0 \end{bmatrix} F(t)$$

$$y(t) = \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} \tag{1.20}$$

Example 1.4: State Variables for the Electrical System Example

Find a state variable expression for the electrical system equations in (1.14) from Example 1.2. As the system output, use the voltage across the inductor.

Solution:

In this system, the two equations in (1.14) are each first order. The total system is then second order. Using v_c and $i(t)$ as the two state variables, i.e., $\xi_1(t) = v_c(t)$ and $\xi_2(t) = i(t)$, we can immediately write the equations:

$$
\begin{aligned}
\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_1 \end{bmatrix} &= \begin{bmatrix} -1/CR_1 & 1/C \\ -1/L & -R_2/L \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} v(t) \\
y(t) &= \begin{bmatrix} -1 & -R_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + (1) v(t)
\end{aligned}
\tag{1.21}
$$

The output equation in this result comes from the KVL equation $v_L = v - v_c - iR_2$.

Alternative State Variables

If we had attempted to use the definitions of state variables in (1.16) to write state equations for the more general differential equation:

$$
\begin{aligned}
\frac{d^n x(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + \cdots + a_1 \frac{dx(t)}{dt} + a_0 x(t) \\
= b_n \frac{d^n u(t)}{dt^n} + \cdots + b_0 u(t)
\end{aligned}
\tag{1.22}
$$

we would have required derivatives of $u(t)$ in the state equations. According to the standard format of (1.19), this is not allowed. Instead, there are a number of commonly used formulations for state variables for equations such as (1.22). These are best represented in the simulation diagrams on the following pages.

In Figure 1.7, the state variables shown are similar to those found in (1.16), in the sense that $n-1$ of them are simply derivatives of the previous ones. The state equations that describe this diagram are given immediately following Figure 1.7, in Equation (1.25). We see that in this form, the feedback coefficients a_i appear in only the final state equation. Having feedforward connections with b_i coefficients allows for derivatives of $u(t)$ in (1.22) without appearing in (1.25) below, which is the state variable equivalent of (1.22). Note that in this form each state variable is assigned to be the output of an integrator, just as with the phase variables discussed above.

A second common choice of state variables can be generated from the following manipulations. Suppose that in an attempt to solve the second-order equation of the form:

$$x''(t)+a_1x'(t)+a_0x(t)=b_2u''(t)+b_1u'(t)+b_0u(t) \tag{1.23}$$

both sides of the equation are integrated twice with respect to the time variable. This would result in:

$$\begin{aligned}\int_{-\infty}^{t}\int_{-\infty}^{s}&\left[x''(\sigma)+a_1x'(\sigma)+a_0(\sigma)x(\sigma)\right]d\sigma\, ds\\ &=\int_{-\infty}^{t}\int_{-\infty}^{s}\left[b_2u''(\sigma)+b_1u'(\sigma)+b_0(\sigma)u(\sigma)\right]d\sigma\, ds\end{aligned}$$

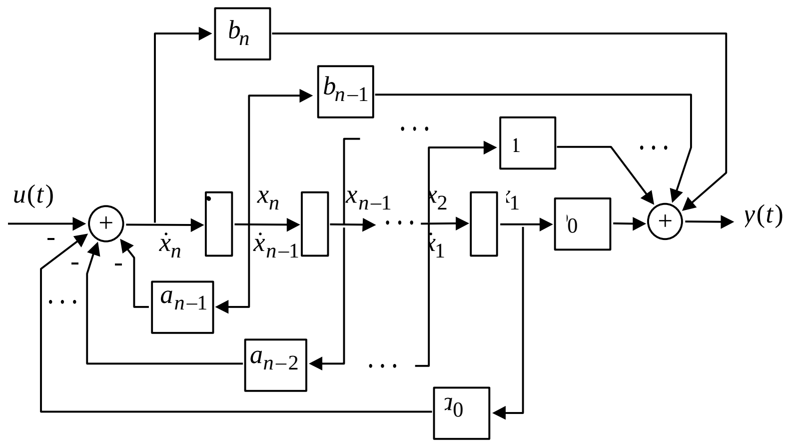
giving

$$\begin{aligned}x(t)+&\int_{-\infty}^{t}a_1x(s)\,ds+\int_{-\infty}^{t}\int_{-\infty}^{s}a_0x(\sigma)\,d\sigma\, ds\\ &=b_2u(t)+\int_{-\infty}^{t}b_1u(s)\,ds+\int_{-\infty}^{t}\int_{-\infty}^{s}b_0u(\sigma)\,d\sigma\, ds\end{aligned}$$

or

$$\begin{aligned}x(t)&=b_2u(t)+\int_{-\infty}^{t}\left[b_1u(s)-a_1x(s)\right]ds+\int_{-\infty}^{t}\int_{-\infty}^{s}\left[b_0u(\sigma)-a_0x(\sigma)\right]d\sigma\, ds\\ &=b_2u(t)+\int_{-\infty}^{t}\left\{\left[b_1u(s)-a_1x(s)\right]+\int_{-\infty}^{s}\left[b_0u(\sigma)-a_0x(\sigma)\right]d\sigma\right\}ds\end{aligned} \tag{1.24}$$

For higher-order systems, this process continues until an equation of the form of (1.24) is derived. From the form of (1.24), the simulation diagram shown in Figure 1.8 can be drawn, with the associated state equations appearing in Equation (1.26). Note that in this formulation, the integrators in the network are not connected end-to-end. Thus the state variables are not simply derivatives of one another as are phase variables. Instead, the state equations are written as in (1.26). The state variables are, however, still defined as the outputs of the integrators. This is commonly done, but is not necessary. Additional examples will be shown in Chapter 9.



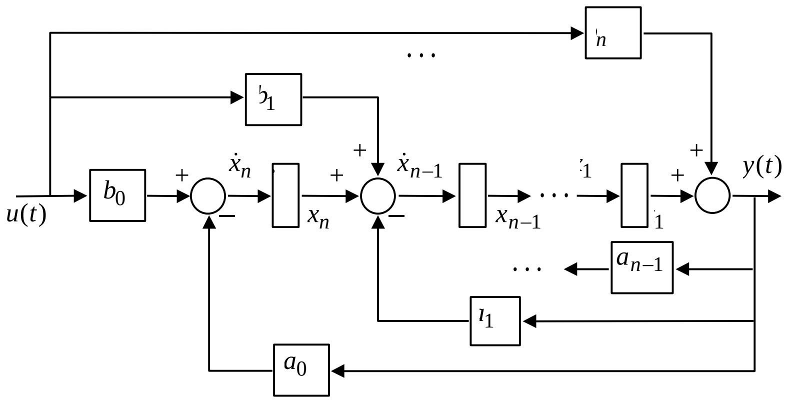
$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 - b_n a_0 & b_1 - b_n a_1 & \cdots & b_{n-1} - b_n a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_n u(t) \tag{1.25}$$

Figure 1.7 Simulation diagram and state equations for phase variable definitions of state variables.

These special choices are examples of different ways in which state variables can be assigned to a particular system. They have some convenient properties that we will examine in later chapters. Such special forms of equations are known as *canonical* forms.

It should be noted that there are an infinite number of other ways in which to derive definitions for state variables, a fact that will soon become readily apparent. It is also important to realize that in our simple physical system examples, the variables we choose are physically measurable or are otherwise meaningful quantities, such as voltages or displacements. Often, variables will be selected purely for the effect they have on the structure of the state equations, not for the physical meaning they represent. State variables need not be physically meaningful. This is one of the primary advantages to the state variable technique.



$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_{n-1} & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ -a_1 & \cdots & 0 & 1 \\ -a_0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_{n-1} - a_{n-1}b_n \\ \vdots \\ b_1 - a_1 b_n \\ b_0 - a_0 b_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_n u(t) \tag{1.26}$$

Figure 1.8 A second type of simulation diagram and state equations.

Changing State Variables

In the state variable description for the mechanical system, we might be inclined to write equations not for the independent variables, $x_1(t)$ and $x_2(t)$, but rather for the variables $x_1(t)$ and $(x_1(t) - x_2(t))$. Perhaps we make this choice because the device is being used as a seismograph and the motion of the second mass relative to the first is more important than the absolute position. For this system, the equations in (1.11) are relatively simple to rewrite with this change of variables. If we introduce $\hat{x}_1(t) = x_1(t)$ and $\hat{x}_2(t) = x_1(t) - x_2(t)$, then (1.11) becomes:

$$\begin{aligned} m_1\ddot{\hat{x}}_1 + B\dot{\hat{x}}_2 + K\hat{x}_2 &= F \\ m_2\left(\ddot{\hat{x}}_1 - \ddot{\hat{x}}_2\right) - B\dot{\hat{x}}_2 - K\hat{x}_2 &= 0 \end{aligned} \tag{1.27}$$

In order to get each equation to contain second derivatives in only one of the variables, we can solve the first equation of (1.27) for $\ddot{\hat{x}}_1$ and substitute it into the second equation. Performing this operation and simplifying, we obtain

$$
\begin{aligned}
m_1\ddot{\hat{x}}_1 + B\dot{\hat{x}}_2 + K\hat{x}_2 &= F \\
m_2\ddot{\hat{x}}_2 + B\left(1+\frac{m_2}{m_1}\right)\dot{\hat{x}}_2 + K\left(1+\frac{m_2}{m_1}\right)\hat{x}_2 &= \frac{m_2}{m_1}F
\end{aligned}
$$

Now using new state variables $\hat{\xi}_1 = \hat{x}_1$, $\hat{\xi}_2 = \dot{\hat{x}}_1$, $\hat{\xi}_3 = \hat{x}_2$ and $\hat{\xi}_4 = \dot{\hat{x}}_2$, the new state equations are readily apparent:

$$
\begin{bmatrix} \dot{\hat{\xi}}_1 \\ \dot{\hat{\xi}}_2 \\ \dot{\hat{\xi}}_3 \\ \dot{\hat{\xi}}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{K}{m_1} & -\dfrac{B}{m_1} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -K\left(\dfrac{1}{m_1}+\dfrac{1}{m_2}\right) & -B\left(\dfrac{1}{m_1}+\dfrac{1}{m_2}\right) \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \hat{\xi}_3 \\ \hat{\xi}_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m_1} \\ 0 \\ \dfrac{1}{m_1} \end{bmatrix} F
$$

$$
y = \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \hat{\xi}_3 \\ \hat{\xi}_4 \end{bmatrix}
$$

Here again we have let the output variable be $y(t) = x_2 - x_1$ $(= -\hat{x}_2(t) = -\hat{\xi}_3)$. Notice that the state equations above are significantly different from (1.20), although they are both valid equations representing the same physical system.

Clearly, redefining all the state variables and rederiving or manipulating the equations can be time consuming. To demonstrate an easier method that uses the vector-matrix notation we have adopted, we turn to the electrical system example (Example 1.2). Suppose we wish to write new state equations, where the relevant state variables are v_R and v_x (see the circuit diagram in Figure 1.6). Because we can write the new state variables as weighted sums of the old state variables

$$
\begin{aligned}
v_R &= R_2 i \\
v_x &= v_c + R_2 i
\end{aligned}
\tag{1.28}
$$

we can use vector-matrix notation immediately to write

$$\begin{bmatrix} v_R \\ v_x \end{bmatrix} = \begin{bmatrix} 0 & R_2 \\ 1 & R_2 \end{bmatrix} \begin{bmatrix} v_c \\ i \end{bmatrix} \tag{1.29}$$

Alternatively, using the inverse relationship,

$$\begin{bmatrix} v_c \\ i \end{bmatrix} = \begin{bmatrix} 0 & R_2 \\ 1 & R_2 \end{bmatrix}^{-1} \begin{bmatrix} v_R \\ v_x \end{bmatrix} = \frac{\begin{bmatrix} R_2 & -R_2 \\ -1 & 0 \end{bmatrix}}{-R_2} \begin{bmatrix} v_R \\ v_x \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1/R_2 & 0 \end{bmatrix} \begin{bmatrix} v_R \\ v_x \end{bmatrix} \tag{1.30}$$

If we use the notation $X = \begin{bmatrix} v_C & i \end{bmatrix}^T \triangleq \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix}^T$ for the state vector already introduced in (1.21) and $\hat{X} = \begin{bmatrix} v_R & v_x \end{bmatrix}^T \triangleq \begin{bmatrix} \hat{\xi}_1 & \hat{\xi}_2 \end{bmatrix}^T$ for the new state vector, then we can write (1.30) symbolically as

$$X = M\hat{X} \tag{1.31}$$

where matrix M is defined in (1.30). Likewise, the symbolic form for (1.21) is as given in (1.19), i.e.,

$$\begin{aligned} \dot{X} &= AX + bv \\ y &= cX + dv \end{aligned} \tag{1.32}$$

Equation (1.31) directly implies that $\dot{X} = M\dot{\hat{X}}$, so substituting into (1.32),

$$\begin{aligned} M\dot{\hat{X}} &= AM\hat{X} + bv \\ y &= cM\hat{X} + dv \end{aligned} \tag{1.33}$$

or

$$\begin{aligned} \dot{\hat{X}} &= M^{-1}AM\hat{X} + M^{-1}bv \triangleq \hat{A}\hat{X} + \hat{b}v \\ y &= cM\hat{X} + dv \triangleq \hat{c}\hat{X} + \hat{d}v \end{aligned} \tag{1.34}$$

where

$$\hat{A} \triangleq M^{-1}AM = \begin{bmatrix} 0 & R_2 \\ 1 & R_2 \end{bmatrix} \begin{bmatrix} -\dfrac{1}{CR_1} & \dfrac{1}{C} \\ -\dfrac{1}{L} & -\dfrac{R_2}{L} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ \dfrac{1}{R_2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\dfrac{R_2}{L} \\ \dfrac{1}{CR_1} + \dfrac{1}{CR_2} & -\dfrac{1}{CR_1} - \dfrac{R_2}{L} \end{bmatrix}$$

$$\hat{b} \triangleq M^{-1}b = \begin{bmatrix} 0 & R_2 \\ 1 & R_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1/L \end{bmatrix} = \begin{bmatrix} \dfrac{R_2}{L} \\ \dfrac{R_2}{L} \end{bmatrix}$$

$$\hat{c} \triangleq cM = \begin{bmatrix} -1 & -R_2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ \dfrac{1}{R_2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \end{bmatrix} \tag{1.35}$$

$$\hat{d} \triangleq d = 1$$

This gives a new system of state equations that are equivalent to those in (1.21) but appear quite different. Notice that the feedthrough matrix, d, is the same in each case.

The procedure we have just introduced will be considered from a different perspective in later chapters. The important points here are:

- n^{th}-order linear differential equations (or several coupled systems constituting an n^{th}-order system) can be written as n first-order state equations.
- State equations are entirely equivalent to the original differential equations but are not unique.
- State equations can be changed with matrix-vector operations, resulting in a new form for each of the four system matrices.

In future chapters, we will see what conveniences can be realized with different choices of state vector definition and what insight might be gained into the original physical system by using the state variable notation. There will be certain characteristics of differential equations and control systems that are more apparent in state space (time-domain) than in transfer function form (frequency-domain). In order to understand these properties, some details of linear vector spaces and linear algebras will be necessary.

1.1.4 Discrete-Time Systems

Throughout this book we will, on occasion, use discrete-time systems as examples or for other illustrative purposes. Perhaps more so than in frequency-domain, state space methods for discrete-time systems are very similar to those used in continuous-time systems. For example, most of the first part of the book on vector spaces, linear operators, and functions of matrices is indifferent to the time-domain, because the equations being considered are independent of time. In latter chapters, some concepts, such as controllability and pole placement are common to continuous and discrete-time, while other concepts, including stability and advanced controller design, can vary significantly in discrete-time. The basic concepts remain parallel, but the matrix equations may look different.

We do not give a detailed treatment here of digital filtering or *z*-domain methods, or of continuous filtering and *s*-domain methods. Nevertheless, it is useful to review the terminology.

Discrete-time systems may be inherently discrete, as, for example, in the equations that describe the balance of a bank account that undergoes withdrawals, deposits, and interest postings at regular intervals. Alternatively, they may be discretizations of continuous-time systems. A discretization is a conversion of a continuous-time equation into discrete-time. These discretizations may be performed in a number of different ways, e.g., by using integrator equivalence, pole-zero mapping, or hold equivalence [6]. In either case, we end up with a *difference* equation rather than a differential equation. Difference equations are expressed in terms of time delays rather than derivatives. If, for example, the sampling period of a discrete-time system is T, then a simple difference equation might appear as

$$x(kT+2T)+a_1x(kT+T)+a_0x(kT)=b_0u(kT) \tag{1.36}$$

where k is an integer. In such a situation, we will sometimes simplify the notation by dropping the (constant) sampling time T and using a subscript rather than an argument in parentheses, i.e.,

$$x_{k+2}+a_1x_{k+1}+a_0x_k=b_0u_k \tag{1.37}$$

With this difference equation, we may define a state vector in any of the ways discussed above for continuous-time systems. For example, if we let $\mathrm{X}\stackrel{\Delta}{=}\begin{bmatrix}x_k & x_{k+1}\end{bmatrix}^{\mathrm{T}}$, then we will obtain the discrete-time state equation

$$\begin{aligned}\mathrm{X}_{k+1}&=\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}\mathrm{X}_k+\begin{bmatrix}0\\ b_0\end{bmatrix}u_k\\ &\stackrel{\Delta}{=}A_d\mathrm{X}_k+b_du_k\end{aligned} \tag{1.38}$$

where the notation A_d and b_d are introduced to distinguish these discrete-time system matrices from their continuous-time counterparts. An output equation can be similarly defined.

Furthermore, if an arbitrary difference equation is specified:

$$x_{k+n} + a_{n-1}x_{k+(n-1)} + \cdots + a_1 x_{k+1} + a_0 x_k = b_n u_{k+n} + \cdots + b_0 u_k \tag{1.39}$$

then the simulation diagrams of Figures 1.7 and 1.8 (and all other possibilities) may also be used, with the exception that the integrators are replaced with unit-time delays, i.e., delay blocks representing one sample period, *T*, in length. When such a system is time-varying, the coefficients a_i and b_i above may be replaced with $a_i(k)$ and $b_i(k)$. (The inclusion of this extra time argument is the motivation for the subscript notation; writing $a_i(k, kT - T)$ would be cumbersome.)

The following examples illustrate a number of ways a difference equation can be used to model physical systems. The first two examples illustrate the derivation of difference equations by direct discrete-time modeling, and in the third example, a method is given for approximating a differential equation by a discrete-time system. Another method for representing discrete-time systems as approximations of continuous-time systems is given in Chapter 6.

Example 1.5: Direct Difference Equation Modeling of a Savings Account

The simplest model of an interest-bearing savings account provides an example of a first-order difference equation. Derive the difference equation model for the balance of an account earning *i*% per year, compounded monthly. Assume that interest is computed on the previous month's balance and that the account owner may make any number of deposits and withdrawals during that month.

Solution:

The balance, after compounding, in month *k* is denoted $x(k)$. Then $x(k)$ will be equal to the previous month's balance plus the interest for month *k* and the net total of the owner's deposits and withdrawals, which we will denote by $u(k)$. Then we have the difference equation

$$\begin{aligned} x(k) &= x(k-1) + {}^{i}\!/_{12}\, x(k-1) + u(k-1) \\ &= \left(1 + {}^{i}\!/_{12}\right) x(k-1) + u(k-1) \end{aligned}$$

This is a first-order difference equation and, hence, is already in linear state variable form. Note that time can be arbitrarily shifted to reflect that $k = 0$ is the time origin and that $x(1)$ is the first balance that must be computed. Because the system is time-invariant, this is merely a notational convenience:

$$x(k+1) = \left(1 + \tfrac{i}{12}\right)x(k) + u(k)$$

Example 1.6: A Difference Equation for a Predator-Prey System

A tropical fish enthusiast buys a new fish tank and starts his collection with $P_p(0)$ piranhas and $P_g(0)$ guppies. He buys an ample supply of guppy food (which the piranhas will not eat) and expects the piranhas to eat the guppies. He samples the populations of the two species each day, $P_p(d)$ and $P_g(d)$, and finds, as he expected, that they change. Generate a linear difference equation model for these "population dynamics."

Solution:

Several assumptions must be made to derive a suitable model. First, we will assume that the birthrate of the piranhas is directly proportional to their food supply, which of course is the population of guppies. Further, we assume that because piranhas are large, their death rate is proportional to overcrowding, i.e., to the level of their own population. Therefore, we can write the relationship

$$\begin{aligned} P_p(d+1) &= P_p(d) + k_1 P_g(d) - k_2 P_p(d) \\ &= (1-k_2)P_p(d) + k_1 P_g(d) \end{aligned} \tag{1.40}$$

where k_1 and k_2 are constants of proportionality. Of course, $P_p(0)$ and $P_g(0)$ are the initial conditions.

Now assume that the birthrate of the guppies is proportional to their food supply, which is the input of guppy food, $u(d)$. The death rate of the guppies will be proportional to the population of the piranhas. Therefore,

$$P_g(d+1) = P_g(d) + u(d) - k_3 P_p(d) \tag{1.41}$$

Together, these two equations can be combined into the state space difference equation model

$$\begin{bmatrix} P_p(d+1) \\ P_g(d+1) \end{bmatrix} = \begin{bmatrix} 1-k_2 & k_1 \\ -k_3 & 1 \end{bmatrix} \begin{bmatrix} P_p(d) \\ P_g(d) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(d) \tag{1.42}$$

Example 1.7: Discretized Differential Equations for a Harmonic Oscillator

By approximating the derivative as a finite difference, find a discrete-time version of the forced harmonic oscillator equations

$$\ddot{x}(t)+\omega^2 x(t)=u(t) \tag{1.43}$$

Solution:

At time $t=kT$, where T is a sampling time, the approximation of the first derivative is

$$\dot{x}(kT)\approx\frac{x(kT+T)-x(kT)}{T}$$

This implies that the approximation for the second derivative would be:

$$\begin{aligned}\ddot{x}(kT)&\approx\frac{\dot{x}(kT+T)-\dot{x}(kT)}{T}\\&=\frac{\dfrac{x(kT+2T)-x(kT+T)}{T}-\left(\dfrac{x(kT+T)-x(kT)}{T}\right)}{T}\\&=\frac{x(kT+2T)-2x(kT+T)+x(kT)}{T^2}\end{aligned}$$

Substituting these approximations into the original Equation (1.43), we get

$$\frac{x(kT+2T)-2x(kT+T)+x(kT)}{T^2}+\omega^2 x(kT)=u(kT)$$

or simplifying and dropping the T's in the time arguments,

$$x(k+2)-2x(k+1)+\left(\omega^2T^2+1\right)x(k)=T^2u(k) \tag{1.44}$$

This can be seen as a discrete-time equation of the form (1.39).

If we now choose state variables as $x_1(k)=x(k)$ and $x_2(k)=x(k+1)$, then the discrete-time state space description of the system becomes

$$\begin{bmatrix}x_1(k+1)\\x_2(k+2)\end{bmatrix}=\begin{bmatrix}0 & 1\\-(\omega^2T^2+1) & 2\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\end{bmatrix}+\begin{bmatrix}0\\T^2\end{bmatrix}u(k) \tag{1.45}$$

If we now wish to draw a simulation diagram for this system that is similar to the one in Figure 1.7, we must remember that instead of integrators, we must use the unit delay operator, z^{-1}. Because (1.45) has the same form as (1.39), which itself is similar in form to (1.22), the simulation diagram of Figure 1.9 is obtained.

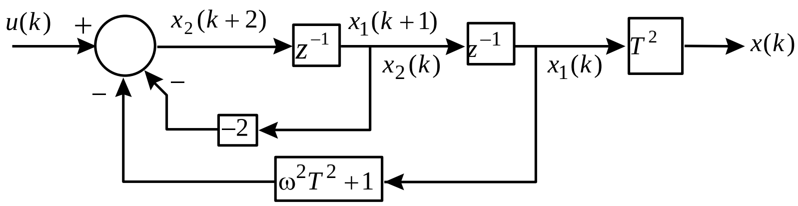


Figure 1.9 Simulation diagram for a discrete-time state space system. The unit delay operator is signified by the symbol z^{-1}.

1.1.5 Relationship to Transfer Functions

It may be reasonably asked at this point how the state variable description, which is in time-domain, relates to the transfer function representation of systems that is usually introduced in lower level signals and systems courses. The most fundamental answer to this question is that transfer functions are defined for systems with zero initial conditions only. State variable descriptions are therefore more appropriate when transients due to nonzero initial conditions are important.

However there are many other distinctions as well. The mathematical procedures for solving time- and frequency-domain systems differ greatly. From a control systems point of view, there exist entirely different design tools in time and frequency-domains, and certain aspects of system behavior are more obvious in one domain or the other. Design criteria can be specified in one domain that are not easily expressed in the other. Other concepts that will be introduced in later chapters include controllability, observability, and optimal compensator design, concepts that are all straightforward in time-domain but not necessarily in frequency-domain.

For the time being, though, we will give a flavor of the intrinsic relationship between the two domains. Consider the form of general SISO state equations:

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx + du \end{aligned} \tag{1.46}$$

If we assume that the initial condition is $x(t_0) = 0$ for this system, Equations (1.46) can be LaPlace transformed simply as

$$
\begin{aligned}
sX(s) &= AX(s) + bU(s) \\
Y(s) &= cX(s) + dU(s)
\end{aligned}
\tag{1.47}
$$

Simplifying the first of these equations,

$$
sX(s) - AX(s) = (sI - A)X(s) = bU(s)
$$

or

$$
X(s) = (sI - A)^{-1} bU(s) \tag{1.48}
$$

Substituting (1.48) into the output equation in (1.47), we get

$$
Y(s) = \left[c(sI - A)^{-1} b + d \right] U(s)
$$

or, for a SISO system in which $Y(s)$ and $U(s)$ are scalar quantities and the division operation exists,

$$
\frac{Y(s)}{U(s)} = \left[c(sI - A)^{-1} b + d \right] \stackrel{\Delta}{=} P(s) \tag{1.49}
$$

giving the transfer function, which is a ratio of polynomials.

It is worth noting that many textbooks pay little attention to the feedthrough term of the state equations, i.e., *d* (*D*). This is because its effect can be "factored out" such that it represents an algebraic relationship between the input and output, rather than a differential one. It occurs only in nonstrictly proper transfer functions wherein the order of the numerator is the same as the order of the denominator. By using polynomial division, such a transfer function can always be rewritten as the sum of a constant factor and a strictly proper transfer function. For example, consider the transfer function

$$
P(s) = \frac{2s + 4}{s + 1} = 2 + \frac{2}{s + 1} \tag{1.50}
$$

Converting this system to state space representation will always give $d = 2$ as the feedthrough term, regardless of the choice of variables. Recall that if the transfer function is *im*proper, then positive powers of *s* would divide out of such a fraction, and a noncausal system would be apparent.

Equation (1.49) shows how it is possible to obtain a transfer function from the matrices given in a state variable representation.[M] To get a state variable representation from a transfer function,[M] we usually return to the original

`tf2ss(num,den)`
`ss2tf(a,b,c,d)`

differential equation or use the simulation diagrams in Figures 1.7 and 1.8. This subject is complicated by controllability and observability issues introduced in Chapters 8 and 9.

We have asserted that no matter which (valid) state variables we choose, the state equations represent the original system exactly. One may than ask if a change of variables is performed as in (1.31), how is the transfer function (1.49) altered? Suppose in the equation for the transfer function in terms of system matrices in (1.49), we substitute the "transformed" matrices from (1.35). Then we obtain

$$\begin{aligned}\hat{P}(s) &= \hat{c}\left(sI-\hat{A}\right)^{-1}\hat{b}+\hat{d} \\ &= cM\left(sI-M^{-1}AM\right)^{-1}M^{-1}b+d \\ &= cM\left(sM^{-1}M-M^{-1}AM\right)^{-1}M^{-1}b+d \\ &= cMM^{-1}(sI-A)^{-1}MM^{-1}b+d \\ &= c(sI-A)^{-1}b+d \\ &= P(s)\end{aligned} \tag{1.51}$$

So we have shown that the same transfer function results regardless of the choice of state equations. Note that this same result holds for discrete-time systems in exactly the same form; simply replace the *s* operator in this section with the *z* operator.

1.2 Linearization of Nonlinear Equations

It is an unfortunate fact that most physical systems encountered in practice are not linear. It is almost always the case that when one encounters a linear model for a physical system, it is an idealized or simplified version of a more accurate but much more complicated nonlinear model. In order to create a linear model from a nonlinear system, we introduce a linearization method based on the Taylor series expansion of a function.

1.2.1 Linearizing Functions

Recall that the Taylor series expansion expresses a general function $f(x)$ as the infinite series

$$f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{d^n f(x)}{dx^n}\right|_{x=x_0}(x-x_0)^n \tag{1.52}$$

This series is said to be expanded about the point $x = x_0$. The point x_0 is interchangeably referred to as the *bias point*, *operating point*, or, depending on some stability conditions discussed in Chapter 7, the *equilibrium point*.

Any function for which such a series converges is said to be *analytic*. Writing out the first few terms of such a series,

$$f(x) = f(x_0) + \left.\frac{df(x)}{dx}\right|_{x=x_0}(x - x_0) + \frac{1}{2}\left.\frac{d^2 f(x)}{dx^2}\right|_{x=x_0}(x - x_0)^2 + \dots \tag{1.53}$$

For functions that are relatively smooth, the magnitudes of the terms in this series decrease as higher order derivatives are introduced, so an approximation of a function can be achieved by selecting only the low-order terms. Choosing only the first two, for example, we obtain

$$\begin{aligned} f(x) &\approx f(x_0) + f'(x_0)(x - x_0) \\ &= f'(x_0)x + \left[f(x_0) - f'(x_0)x_0 \right] \end{aligned} \tag{1.54}$$

It can be seen from (1.54) that by keeping only the first two terms of the Taylor series, an equation of a line results. The approximation of (1.54) will be referred to as the *linearization* of $f(x)$. This linearization is illustrated in Figure 1.10. In the figure, the original curve $f(x)$ appears as the wavy line. The point of expansion, $x = x_0$, is the point at which the curve is approximated as a straight line by (1.54). For most curves, it is important that the straight-line approximation not be used if the value of *x* strays too far from x_0. However if *x* remains close to x_0, the Taylor series approximation is sometimes a very good one for practical purposes. The accuracy and so-called “linear region” depend, of course, on the particular function $f(x)$.

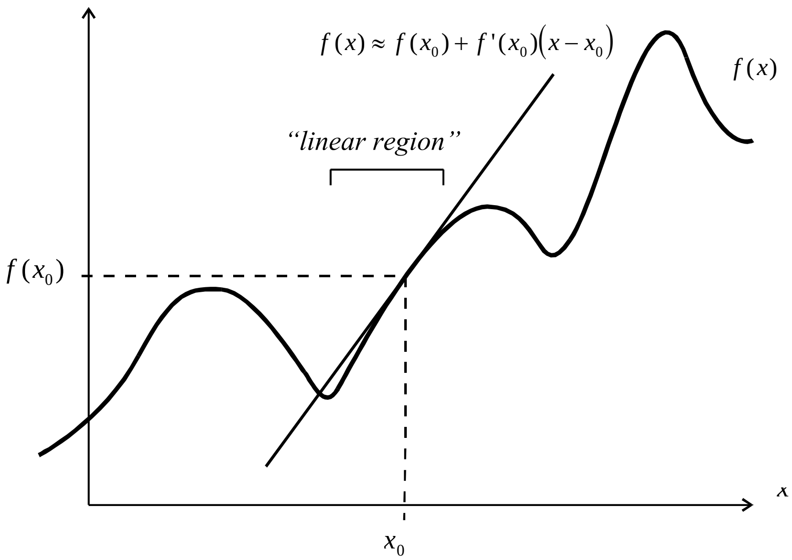


Figure 1.10 Taylor series linearization of a curve.

When the function *f* depends on several variables, such as $x_1, x_2, \ldots, x_n$, not only must all the partial derivatives of the individual variables x_i be used in (1.52), but all their cross-derivatives as well. That is, the Taylor series may be expressed as

$$
\begin{aligned}
f(x_1, x_2, \ldots x_n) &= f(x_{10}, x_{20}, \ldots x_{n0}) \\
&\quad + \left((x_1 - x_{10}) \frac{\partial}{\partial x_1} + \cdots + (x_n - x_{n0}) \frac{\partial}{\partial x_n} \right) f(x_1, x_2, \ldots, x_n) \bigg|_{x_i = x_{i0}} \\
&\quad + \frac{1}{2!} \left((x_1 - x_{10}) \frac{\partial}{\partial x_1} + \cdots + (x_n - x_{n0}) \frac{\partial}{\partial x_n} \right)^2 f(x_1, x_2, \ldots x_n) \bigg|_{x_i = x_{i0}} \\
&\quad + \cdots
\end{aligned}
$$

Of course, for a linear approximation, only the first two terms of this series need be retained.

It is also quite common that several functions of several variables need to be linearized, such as

$$\begin{matrix} f_1(x_1, x_2, \ldots x_n) \\ f_2(x_1, x_2, \ldots x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots x_n) \end{matrix} \tag{1.55}$$

In this case, each function $f_j(x_1, x_2, \ldots, x_n)$, $j = 1, \ldots, m$ can be expanded into a Taylor series and thus linearized separately. Alternatively, we can use *matrix-vector notation* and rewrite (1.55) as

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(x_1, x_2, \ldots x_n) \\ f_2(x_1, x_2, \ldots x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots x_n) \end{bmatrix} \tag{1.56}$$

where $\boldsymbol{f} = \begin{bmatrix} f_1 & f_2 & \cdots & f_m \end{bmatrix}^{\mathrm{T}}$ and $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}$. Using this notation, the linearized version of the nonlinear functions are (by taking the first two terms of the Taylor series):

$$\boldsymbol{f}(\boldsymbol{x}) \approx \boldsymbol{f}(\boldsymbol{x}_0) + \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_0} (\boldsymbol{x} - \boldsymbol{x}_0) \tag{1.57}$$

In this expression, the derivative of $\boldsymbol{f}(\boldsymbol{x})$ is a derivative of an $m \times 1$ vector with respect to an $n \times 1$ vector, resulting in an $m \times n$ matrix whose $(i, j)^{th}$ element is $\partial f_i / \partial x_j$. See Appendix A.

1.2.2 Linearizing Differential Equations

The Taylor series linearization process can be performed on differential equations as well as on functions. When each term of a nonlinear differential equation is linearized in terms of the variables on which it depends, we say that the entire equation has been *linearized*. When this has been done, all the linear analysis tools presented here and elsewhere may be applied to the equation(s), remembering that this linear equation is just an approximation of the original system. There are some situations, such as in chaotic systems, wherein linear approximations are not very good approximators of the true solution of the system, and still more, such as when the nonlinear equations are not analytic (e.g., systems with static friction), when the linearization does not apply at all. In many physical systems, though, the two-term Taylor series approximation provides a reasonably accurate representation of the system, usually good enough that linear controllers can be applied, provided that the approximation is taken sufficiently near the operating point.

Example 1.8: Linearization of a Differential Equation for an Inverted Pendulum

The equation of motion can be derived for the model of an inverted pendulum on a cart shown in Figure 1.11. In the model, $\theta(t)$ is the angle of the pendulum clockwise with respect to the vertical, $x(t)$ is the horizontal position of the cart relative to some arbitrary fixed reference location, 2ℓ is the length of the pendulum, M and m are the masses of the cart and the pendulum, respectively, and I is the moment of inertia of the pendulum about its center of gravity. F is a force applied to the body of the cart.

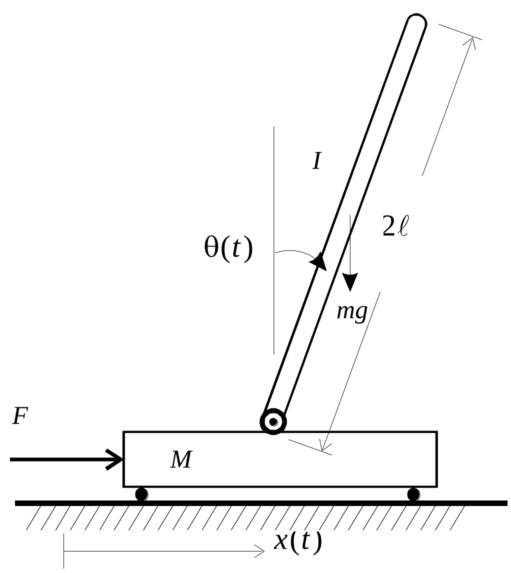


Figure 1.11 Physical model of an inverted pendulum on a cart.

It can be shown that the two coupled differential equations that describe the motion of this system are:

$$\begin{aligned} (m+M)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta &= F \\ \left(I + m\ell^2\right)\ddot{\theta} + m\ell\ddot{x}\cos\theta - mg\ell\sin\theta &= 0 \end{aligned} \tag{1.58}$$

Linearize these two equations using the first two terms of the Taylor series.

Solution:

Although only $x(t)$ and $\theta(t)$ are considered "coordinate variables," when linearizing we consider their respective derivatives to also constitute independent variables. Likewise, the input force *F* is a variable (although the equations already happen to be linear in terms of *F*). We therefore are linearizing *two* equations, each in terms of the *seven independent* variables $\theta(t)$, $\dot{\theta}(t)$, $\ddot{\theta}(t)$, $x(t)$, $\dot{x}(t)$, $\ddot{x}(t)$, and *F*.

To begin, we select an operating point. For convenience, we choose the "zero" position of $\theta_0(t) = \dot{\theta}_0(t) = \ddot{\theta}_0(t) = 0$, $x_0(t) = \dot{x}_0(t) = \ddot{x}_0(t) = 0$, and $F_0 = 0$. We should note that these values are not all independent of one another. If we arbitrarily select the first six values, then substitution into (1.58) allows us to determine $F_0 = 0$. If, for example, the cart and pendulum were moving up a wall, then a different F_0 (nonzero) would have to be chosen in order to counteract gravity. The equations in (1.58) must hold at all times.

We will apply (1.54) to the four terms in the first equation of (1.58) and leave the details of the second equation as an exercise.

For $(m + M)\ddot{x}$,

$$\begin{aligned}(m+M)\ddot{x} &\approx (m+M)\ddot{x}_0 + (m+M)(\ddot{x} - \ddot{x}_0)\\ &= 0 + (m+M)\ddot{x}\\ &= (m+M)\ddot{x}\end{aligned}$$

which of course returns the original expression since it was linear to begin with.

For $m\ell\ddot{\theta}\cos\theta$,

$$\begin{aligned}m\ell\ddot{\theta}\cos\theta &\approx m\ell\ddot{\theta}_0\cos\theta_0 + \frac{\partial}{\partial\theta}\left(m\ell\ddot{\theta}\cos\theta\right)\Big|_{\substack{\theta_0\\ \ddot{\theta}_0}}(\theta - \theta_0)\\ &\quad + \frac{\partial}{\partial\ddot{\theta}}\left(m\ell\ddot{\theta}\cos\theta\right)\Big|_{\substack{\theta_0\\ \ddot{\theta}_0}}(\ddot{\theta} - \ddot{\theta}_0)\\ &= 0 - m\ell\ddot{\theta}_0\sin\theta_0(\theta) + m\ell\cos\theta_0(\ddot{\theta})\\ &= m\ell\,\ddot{\theta}\end{aligned}$$

For $m\ell\dot{\theta}^2\sin\theta$,

$$\begin{aligned} m\ell\dot{\theta}^2 \sin\theta &\approx m\ell\dot{\theta}_0{}^2 \sin\theta_0 + \frac{\partial}{\partial\theta}\left(m\ell\dot{\theta}^2 \sin\theta\right)\Big|_{\substack{\theta_0 \\ \dot{\theta}_0}} (\theta - \theta_0) \\ &\quad + \frac{\partial}{\partial\dot{\theta}}\left(m\ell\dot{\theta}^2 \sin\theta\right)\Big|_{\substack{\theta_0 \\ \dot{\theta}_0}} \left(\dot{\theta} - \dot{\theta}_0\right) \\ &= 0 + m\ell\dot{\theta}_0{}^2 \cos\theta_0(\theta) + 2m\ell\dot{\theta}_0 \sin\theta_0\left(\dot{\theta}\right) \\ &= 0 \end{aligned}$$

It may at first seem somewhat surprising that this term should disappear entirely. However, recalling that the equilibrium point we have chosen has $\dot{\theta}_0 = 0$, the "linear" region in this problem will include only small velocities. Because the velocity in this term appears only as the square, and we are assuming that second order and higher terms are negligible, this entire term should be negligible near the equilibrium.

As for the final term, *F*, it is clear that this is already linear, and we will leave it as is. The "linearized" version of the first of Equations (1.58) is therefore:

$$\left(m + M\right)\ddot{x} + m\ell\ddot{\theta} = F \tag{1.59}$$

Note that this equation has two second-order derivatives in it, which are the highest derivatives of each variable $\theta(t)$ and $x(t)$. The second equation will linearize to

$$(I + m\ell^2)\ddot{\theta} + m\ell\ddot{x} - mg\ell\theta = 0 \tag{1.60}$$

It also has two second-order derivatives. In order to construct a state space representation for such a system, in both Equation (1.59) and (1.60) we will have solve for one such second derivative and substitute it into the other, so that each equation contains only one of the highest derivatives of any of the state variables. This procedure and the subsequent formulation of the state equations is left as Exercise 1.12.

The Choice of Operating Point

One should be careful in applying these methods with the goal of arriving at an equation of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}_0) + \frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\bigg|_{\boldsymbol{x}_0} (\boldsymbol{x} - \boldsymbol{x}_0) \tag{1.61}$$

If in (1.61) we have

$$\boldsymbol{f}(\boldsymbol{x}_0) - \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_0} (\boldsymbol{x}_0) \neq 0$$

then (1.61) is *not linear*! It has a constant offset and will therefore violate the homogeneity and additivity conditions required for linearity. Equation (1.61) will indeed look like a straight line on the $\dot{\boldsymbol{x}}$ versus $\boldsymbol{x}$ plane, but it will not pass through the origin of this plane. Such a system is called *affine*.

In order to make the system linear, one must choose an appropriate operating value for the variable $\boldsymbol{x}$. In particular, choose as the operating point the equilibrium value $\boldsymbol{x}_e$ such that $\boldsymbol{f}(\boldsymbol{x}_e) = 0$. Then to eliminate the constant term in (1.61), we must make a change of variable $\xi \stackrel{\Delta}{=} \boldsymbol{x} - \boldsymbol{x}_e$. Then, $\boldsymbol{x} = \xi + \boldsymbol{x}_e$ and $\dot{\boldsymbol{x}} = \dot{\xi}$. Substituting these values into (1.61), we get

$$\dot{\xi} = \boldsymbol{f}(\boldsymbol{x}_e) + \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_e} (\xi) = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_e} \xi \tag{1.62}$$

This equation is then truly *linear*. If after solving for ξ, we desire to know the value of $\boldsymbol{x}$, then the change of variables may be reversed.

1.3 Summary

In this chapter we have introduced the state space notation to simple models of physical systems. Our purpose has been to establish terms and definitions that are both common to and different from the transfer function descriptions with which the reader may be more familiar. Using the state variable technique, our main points have been as follows:

- State variables are simply regarded as a set of variables in which the behavior of a system may be mathematically modeled.
- Mathematical models for use in state variable systems may be derived, simplified, linearized, and discretized just as any other mathematical model is produced. It is only the choice of state variables that is new so far.
- State variables and state variable equations together constitute the same information about a physical system as do the original differential equations derived from familiar conservation and constitutive laws.

- State variables may be changed to represent either other physical variables or variables that have no discernible physical significance at all. This changes the structure of the matrix elements in the state equations but does not alter the fact that the equations describe the same physical system equally well.

We have tried to stress the need for matrix and vector analysis and linear algebra in the discussions of this chapter. In the next chapter, we will begin to study these mathematical tools. They will later lead us to see the benefits and drawbacks of the state variable techniques.

1.4 Problems

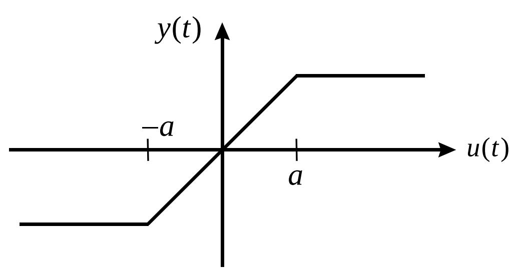
1.1 A system has an input $u(t)$ and an output $y(t)$, which are related by the information provided below. Classify each system as linear or nonlinear and time-invariant or time-varying.

a) $y(t) = 0$ for all t.

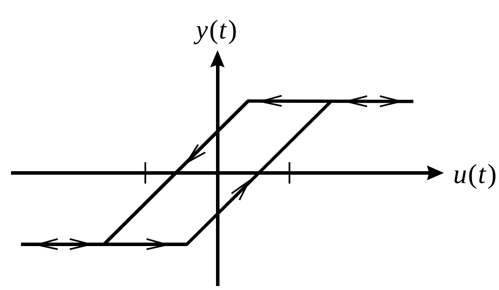
b) $y(t) = a$, $a \neq 0$, for all t.

c) $y(t) = -3u(t) + 2$.

d)



e)



f) $\ddot{y}(t) + 3e^{-t}\dot{y}(t) + y(t) = u(t)$.

g) $\ddot{y}(t) + (1 - y^2(t))\dot{y}(t) + \omega^2 y(t) = u(t)$, $\omega = \text{constant} \neq 0$.

h) $y(t) = u(t-3)$.

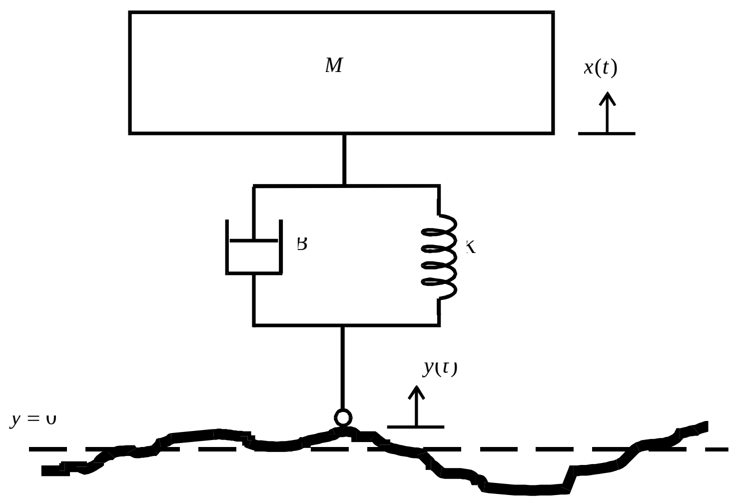
i) $\ddot{y}(t) + u(t)y(t) = 0$.

j) $y(t) = \int_0^t e^{-\tau} u(t-\tau)d\tau$.

k) $y(k+2) = -0.4y(k+1) - y(k) + 3u(k+2) - u(k)$.

l) $y(k) = \sum_{i=0}^{k} \sin(iT)e^{-iT} u(k-i)$, $T = \text{constant} > 0$.

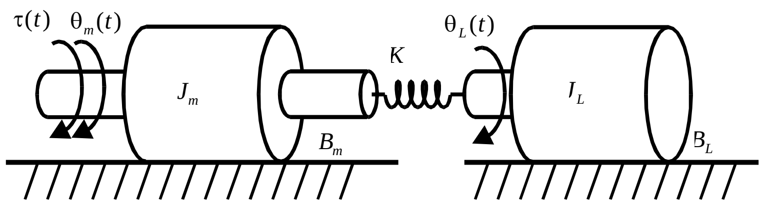
1.2 Figure P1.2 shows a model commonly used for automobile suspension analysis. In it, the uneven ground specifies the position of the wheel's contact point. The wheel itself is not shown, as its mass is considered negligible compared to the mass of the rest of the car. Write a differential equation and a state variable description for this system, considering the height of the car, $x(t)$, to be the output, and the road height, $y(t)$, to be the input.



P1.2

1.3 The motor in Figure P1.3 exerts a torque τ on the shaft of a motor, which has inertia J_m and bearing friction B_m (assumed viscous). This motor is attached to a load inertia J_L, which itself has a viscous friction B_L. The motor coupling is slightly flexible and is modeled as a torsional spring K

. Write the equations of motion and a set of state equations for the system, taking $\tau(t)$ as the input, and $\theta_L(t)$ as the output.

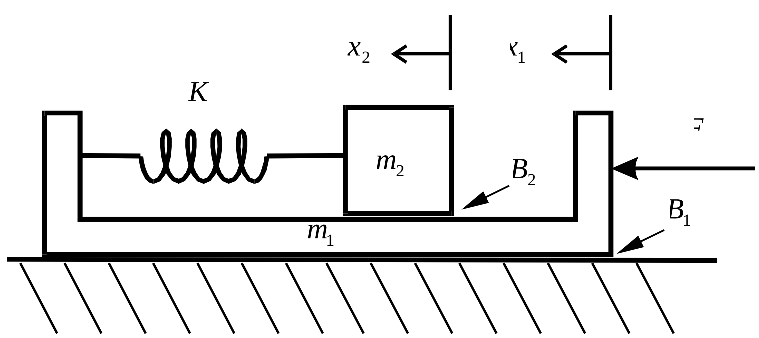


P1.3

1.4 For the mechanical system in Figure P1.4 with a spring, friction forces, and an external force, write state equations using the following sets of variables:

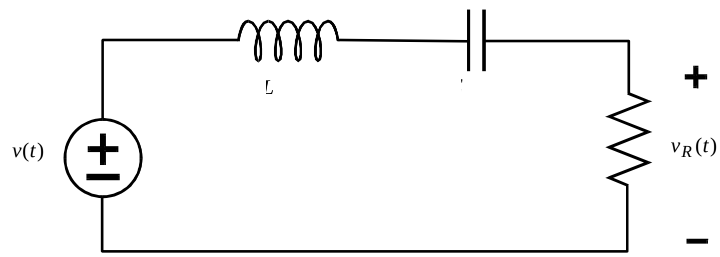
a) State variables $\xi_1 = x_1$, $\xi_2 = \dot{x}_1$, $\xi_3 = x_2$, and $\xi_4 = \dot{x}_2$; and output variable $y = x_2$.

b) State variables $\xi_1 = x_1$, $\xi_2 = \dot{x}_1$, $\xi_3 = x_2 - x_1$, and $\xi_4 = \dot{x}_2 - \dot{x}_1$; and output variable $y = x_2 - x_1$.



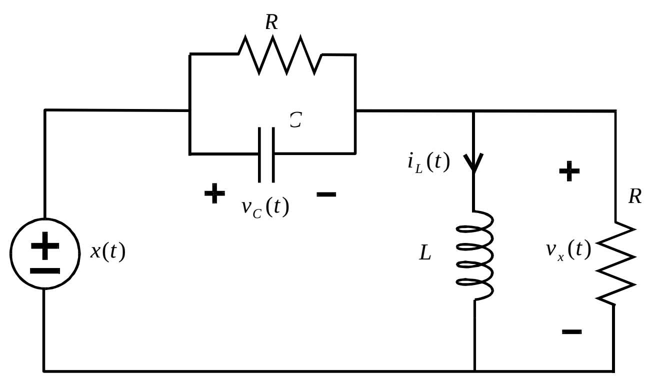
P1.4

1.5 For the circuit shown in Figure P1.5, choose appropriate state variables and write state equations, taking as output the voltage across the resistor.



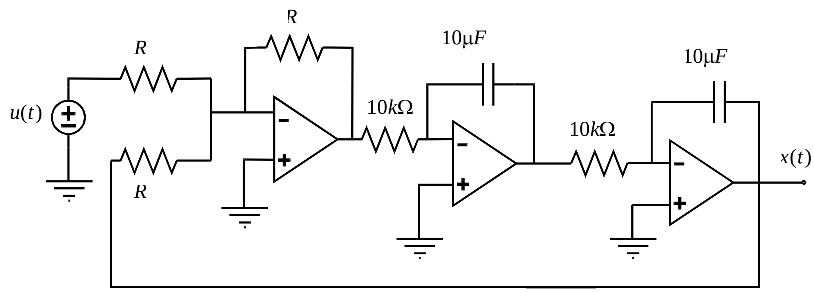
P1.5

1.6 For the circuit shown in Figure P1.6, write a single second-order differential equation in terms of the voltage $v_x(t)$ and the input $x(t)$. Then write state equations using state variables $v_C(t)$ and $i_L(t)$, where the output of the system is considered to be $v_x(t)$, and the input is $x(t)$.



P1.6

1.7 For the circuit shown in Figure P1.7, with ideal OP-AMPS, find a state variable representation, using the capacitor voltages as state variables, and the signal $x(t)$ as the output. What will be the zero-input solution for $x(t)$?



P1.7

1.8 Write two differential equations that describe the behavior of the circuit of Example 1.2, using $v_L(t)$ and $v_C(t)$ as the state variables.

1.9 Given the state variable description of the system in terms of $x_i(t)$ below, change the state variables and write new state equations for the variables $\xi_1(t) = 3x_1(t) + 2x_2(t)$, and $\xi_2(t) = 7x_1(t) + 5x_2(t)$.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 11 & 10 \\ -21 & -18 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + u(t)$$

1.10 For the state variable description of the system in terms of $x_i(t)$,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 18 & 9 & 13 \\ 50 & 23 & 35 \\ -65 & -31 & -46 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 5 & -5 & 5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

change the state variables and write new state equations for variables

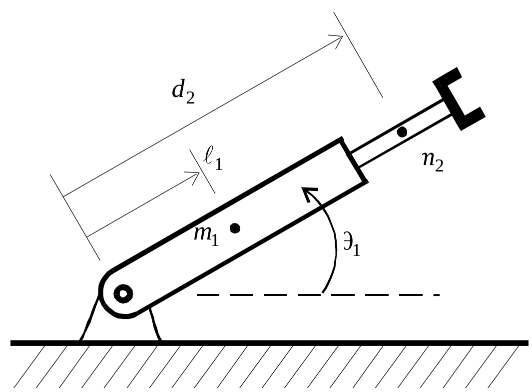
$$\xi_1(t) = -4x_1(t) - 2x_2(t) - 3x_3(t)$$

$$\xi_2(t) = 15x_1(t) + 7x_2(t) + 10x_3(t)$$

and

$$\xi_3(t) = -5x_1(t) - 2x_2(t) - 3x_3(t)$$

1.11 The robot shown in Figure P1.11 has the differential equations of motion given. Symbols $m_1, m_2, I_1, I_2, \ell_1$, and g are constant parameters, representing the characteristics of the rigid body links. Quantities θ_1 and d_2 are the coordinate variables and are functions of time. The inputs are τ_1 and τ_2. Linearize the two equations about the operating point $\theta_1 = \dot{\theta}_1 = \ddot{\theta}_1 = 0$, $d_2 = 3$, and $\dot{d}_2 = \ddot{d}_2 = 0$.

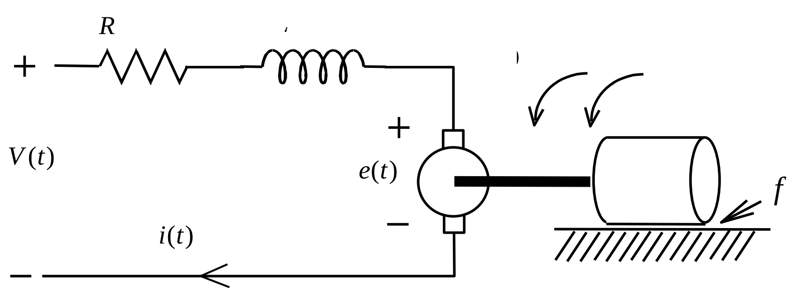


P1.11

$$(m_1\ell_1^2 + I_1 + I_2 + m_2 d_2^2)\ddot{\theta}_1 + 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 + (m_1\ell_1 + m_2 d_2) g \cos\theta_1 = \tau_1$$
$$m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + m_2 g \sin\theta_1 = \tau_2$$

1.12 For the system in Example 1.8, the nonlinear equations of motion are given in (1.58). Show that (1.60) is the linear approximation of the second of these equations, and combine the two linearized equations (1.59) and (1.60) into state space form.

1.13 A permanent-magnet DC motor with a connected inertial and friction load is depicted in Figure P1.13. The motor armature is driven with voltage *V*, and the motor turns through angle θ with torque τ. The armature has resistance *R* and inductance *L*, and the armature current is denoted *i*. The mechanical load is an inertia *J*, and the bearings have viscous friction coefficient *f*. The motor produces a back-emf of $e = k_b\dot{\theta}$ and a torque of $\tau = k_a i$, where k_a and k_b are constants. Determine a set of describing equations for the electrical and mechanical components of the system. Then express these equations as a set of linear state equations, using the voltage V as input and the angle θ as output.



P1.13

1.5 References and Further Reading

The formulation of the dynamic equations for physical systems can be found in any physics text. For information specific to mechanical systems, texts in statics and dynamics can be consulted, and for circuit analysis, there are a great many elementary texts, including [1], which focuses on state variable analysis of electrical networks. A reference that ties them together and introduces the unified modeling terminology of "through" variables and "across" variables is [4]. Other good introductions to the state space representation for physical systems can be found in [2], [7], [8], and [10]. In particular, [10] gives a very detailed introduction to linear system terminology and definitions. For systems described in frequency-domain, which we do not treat in much depth in this book, the student can consult [3] and [7].

Additional state variable models can be found in [5] and [8], both of which provide numerous examples from systems that engineering students do not traditionally encounter, such as genetics, populations, economics, arms races, air pollution, and predator-prey systems.

Further information on nonlinear systems and linearization is given in [9].

[1] Belevitch, V., *Classical Network Theory*, Holden-Day, 1968.

[2] Brogan, William L., *Modern Control Theory*, 3rd edition, Prentice-Hall, 1991.

[3] Callier, Frank M., and Charles A. Desoer, *Multivariable Feedback Systems*, Springer-Verlag, 1982.

[4] Cannon, Robert H. Jr., *Dynamics of Physical Systems*, McGraw-Hill, 1967.

[5] Casti, John L., *Linear Dynamical Systems*, Academic Press, 1987.

[6] Franklin, Gene, and J. David Powell, *Digital Control of Dynamic Systems*, Addison-Wesley, 1981.

[7] Kailath, Thomas, *Linear Systems*, Prentice-Hall, 1980.

[8] Luenberger, David G., *Introduction to Dynamic Systems*, John Wiley & Sons, 1979.

[9] Slotine, Jean-Jacques, and Weiping Li, *Applied Nonlinear Control*, Prentice-Hall, 1991.

[10] Zadeh, Lotfi A. and Charles A. Desoer, *Linear System Theory: The State Space Approach*, McGraw-Hill, 1963.