

6

Solutions to State Equations

It was stated in Chapter 1 that the state of a system at a given time is sufficient information to determine the state at all future times, assuming that the system dynamics (the system matrices A , B , C , and D) and input are all known. Often, the reason for studying a linear system is that one or more of these quantities is not known. Sometimes the system matrices are unknown or only approximated, as with adaptive and robust control. Random signals are present in the input and output equation, requiring stochastic control. In the most common case, only a desired state or output is specified, and it is the task of the engineer to “design” the input by introducing compensators.

In this chapter, we consider only the simplest situations, wherein all terms of the state equations are known, and we seek an analytical solution to these state equations. This process is useful for our understanding of the behavior and properties of state equations, but often we cannot write such solutions in practice, because we usually have insufficient information to do so.

6.1 Linear, Time-Invariant (LTI) Systems

Recall the state equations for a linear, time-invariant (LTI) state space system^M:

SS (A, B, C, D)

$$\begin{aligned} \dot{x} &= Ax + Bu, & x(t_0) &= x_0 \\ y &= Cx + Du \end{aligned} \tag{6.1}$$

The difficulty in solving this system is the first equation, $\dot{x} = Ax + Bu$, because it is a differential equation. When we determine $x(t)$, it becomes a straightforward matter to substitute it into the second equation to determine $y(t)$. Note that by denoting our input, output, and feed-forward matrices as capital letters (B , C , and D), we are implying that these manipulations hold for MIMO as well as SISO situations.

We will use the method of introducing an integrating factor in the solution of first order differential equations of the form:

$$\dot{x}(t) - Ax(t) = Bu(t) \quad (6.2)$$

Multiplying (6.2) by the factor e^{-At} will result in a “perfect” differential on the left side:

$$\begin{aligned} e^{-At}[\dot{x}(t) - Ax(t) &= Bu(t)] \\ e^{-At}\dot{x}(t) - e^{-At}Ax(t) &= e^{-At}Bu(t) \\ \frac{d}{dt}[e^{-At}x(t)] &= e^{-At}Bu(t) \end{aligned} \quad (6.3)$$

(Note that A commutes with e^{At} .)

Integrating both sides of this equation over dummy variable τ from t_0 to t ,

$$e^{-At}x(t) - e^{-At_0}x(t_0) = \int_{t_0}^t e^{-A\tau}Bu(\tau) d\tau \quad (6.4)$$

Finally, moving the initial condition term to the right-hand side and multiplying both sides of the result by e^{At} gives:

$$\begin{aligned} x(t) &= e^{At}e^{-At_0}x(t_0) + e^{At} \int_{t_0}^t e^{-A\tau}Bu(\tau) d\tau \\ &= e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau) d\tau \end{aligned} \quad (6.5)$$

Actually, although we have noted that this result is based upon an assumption of time-invariance, we require only that the matrix A be time-invariant, because the integrating factor would not have worked properly otherwise. If B were time-varying, we would simply write:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}B(\tau)u(\tau) d\tau \quad (6.6)$$

This is the familiar *convolution integral* solution from basic linear systems. Its derivation here for vectors emphasizes one of the primary motivations for state

space analysis: many of the procedures and results for vectors are direct extensions of first-order (scalar) cases.

Completing the problem by computing $y(t)$, we obtain^M:

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lsim(sys,u,
T,X0)
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$$y(t) = C(t)e^{A(t-t_0)}x(t_0) + C(t)\int_{t_0}^t e^{A(t-\tau)}B(\tau)u(\tau)d\tau + Du(t) \quad (6.7)$$

Here again, we have allowed C and D to be functions of time because they do not interfere with the actual solution of the differential equation.

The solution (6.7) makes apparent the importance of the matrix exponential e^{At} studied in the last chapter. For LTI systems, this will become known as the state transition-matrix for reasons that will become apparent in Section 6.4.2.

Example 6.1: Simple LTI System

A simple free-floating object, moving in a straight line, might be described by Newton's law with the equation $F = m\ddot{x}$. Express the system as a pair of state equations and solve using $F(t) = e^{-at}$, $x(0) = x_0$, and $\dot{x}(0) = 0$.

Solution:

Using phase variables, we can define $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$, giving the state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This system is known as a double-integrator. The A -matrix is observed to be in Jordan form, where the two eigenvalues are obviously both zero. We can use the methods of the previous chapter to compute e^{At} , but for this case, a Taylor series expansion has only two nonzero terms since $A^k = 0$ for $k \geq 2$. Therefore,

$$e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Using (6.7), we compute

$$\begin{aligned}
y(t) &= [1 \ 0]e^{A(t-0)}x(0) + [1 \ 0]\int_0^t e^{A(t-\tau)} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} e^{-a\tau} d\tau \\
&= [1 \ 0] \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ 0 \end{bmatrix} + [1 \ 0] \int_0^t \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} e^{-a\tau} d\tau \\
&= x_0 + [1 \ 0] \int_0^t \begin{bmatrix} (t-\tau)1/m \\ 1/m \end{bmatrix} e^{-a\tau} d\tau \\
&= x_0 + \frac{1}{m} \int_0^t (t-\tau)e^{-a\tau} d\tau \\
&= x_0 + \frac{1}{m} \left[-\frac{1}{a}te^{-a\tau} + \frac{1}{a}\tau e^{-a\tau} + \frac{1}{a^2}e^{-a\tau} \right] \Big|_{\tau=0}^{\tau=t} \\
&= \left(x_0 - \frac{1}{a^2m} \right) + t \frac{1}{am} + \frac{1}{a^2m} e^{-at}
\end{aligned}$$

One may verify that this is the correct solution via the usual procedures for solving simple differential equations.

6.2 Homogeneous Systems

In a homogeneous system, of course, the only component of a system's response is the zero-input, or initial condition response. This part of the total response is not easily seen in frequency-domain analysis. Without an input, of course, the state equations are

$$\begin{aligned}
\dot{x} &= Ax \\
y &= Cx
\end{aligned}$$

and the state vector solution for an LTI system is simply

$$x(t) = e^{A(t-t_0)}x(t_0) \quad (6.8)$$

or

$$y(t) = Ce^{A(t-t_0)}x(t_0)$$

Obviously, the matrix exponential e^{At} plays a role in both the zero-state and the zero-input parts of the total response. To see the effect of this matrix exponential, it is useful to draw sketches of the solutions in the space, beginning with various initial conditions. This is most easily done in two dimensions, for which such plots are common and are called *phase portraits*.

6.2.1 Phase Portraits

A phase portrait is strictly defined as a graph of several zero-input responses on a plot of the *phase-plane*, $\dot{x}(t)$ versus $x(t)$, these being known as phase variables. However the term has become commonly used to denote any sketch of zero-input solutions on the plane of the state variables, regardless of whether they are phase variables or not.

To create a phase portrait, one simply chooses initial conditions to represent wide areas of interest in the $x_1 - x_2$ plane, solves the system^M according to (6.8), and sketches the result as a function of time, starting from t_0 . For example, consider the system

`initial(sys,X0)`

$$\dot{x} = A_1 x = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} x \quad (6.9)$$

A phase portrait for this system is given in Figure 6.1. In the figure, sufficient initial conditions are chosen around the edge of the graph to accurately interpolate the solution for any initial condition in between. The solution drawn from each initial condition is sketched as a directed curve, as indicated by the arrow showing the progression of positive time. These curves are known as *phase trajectories*.

With experience, the general shape of the trajectories can be imagined without a detailed plot. Generally, it is the qualitative shape of the trajectories that is important in a phase portrait. (This is particularly true with nonlinear systems, whose phase portraits can sometimes be constructed entirely qualitatively or with piecewise analysis of their dynamics.) For example, certain qualitative features of Figure 6.1 might be predicted from our knowledge of the state space.

Consider the A -matrix given in (6.9) as a linear operator, taking vectors x into vectors \dot{x} . Thus, given any position $x(t)$ on the plot, the equation $\dot{x} = Ax$ gives us the *tangent vector* to the phase trajectory at that point. This is the direction in which the trajectory is evolving. In the figure, we have illustrated this point by indicating a vector, $x(t) \approx [-3 \ -1]^T$, and the tangent to the curve at that point, $\dot{x}(t) = A_1 x = [3 \ 4]^T$. Of course, this *velocity* will change as the chosen point x changes.

We can compute that this system has eigenvalues $\sigma_1 = \{-1, -4\}$ (a set known as the *spectrum* of A), and corresponding eigenvectors of

$$\{e_1, e_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad (6.10)$$

Knowing that these two vectors are invariant subspaces, we conclude that if an initial condition $x(t_0)$ lies on either of these lines, then so will the vector

$\dot{x}(t_0) = Ax(t_0)$ at any t_0 . Therefore, if a chosen point lies on one of the eigenvectors, identified in the graph, the trajectory emanating from that point will lie on that same line forever. Thus, we can always get a start at constructing a phase portrait by drawing in the invariant subspaces, i.e., the straight lines, on the plot.

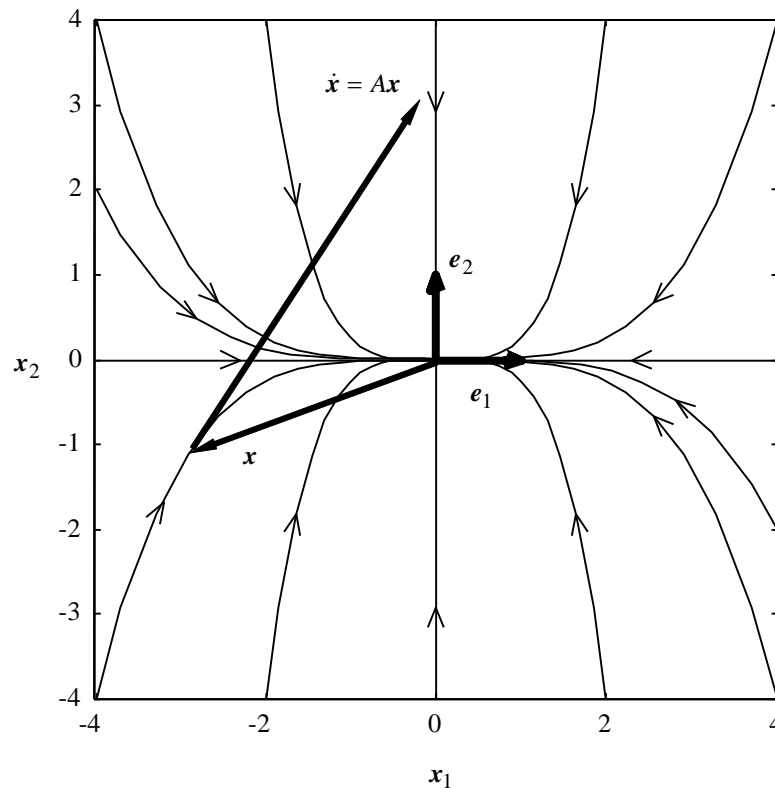


Figure 6.1 Phase portrait for the homogeneous system given in Equation (6.9). For this type of portrait, wherein all trajectories asymptotically approach the origin without encircling it, the origin is known as a *stable node*.

More will be said about the phase trajectories on these portraits in the next section. Until then we will present further examples of phase portraits for the homogeneous equation $\dot{x} = Ax$.

In order to relate such phase trajectories to the more familiar step responses, Figure 6.2 shows the step response^M for the system

step(sys)

$$\begin{aligned}\dot{x} &= A_1 x + b_1 u = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 4 \end{bmatrix} u \\ y &= [1 \quad 1]x\end{aligned}\quad (6.11)$$

where the output equation was selected arbitrarily and the matrix b_1 in (6.11) was chosen to give unity DC gain to each state variable, i.e., so that they both asymptotically approach 1. This figure further illustrates the relative speed of the two state variables. Note how the “faster” of the two variables converges to its final value sooner than the “slower” one, and of course the output is simply the sum of the two inputs.

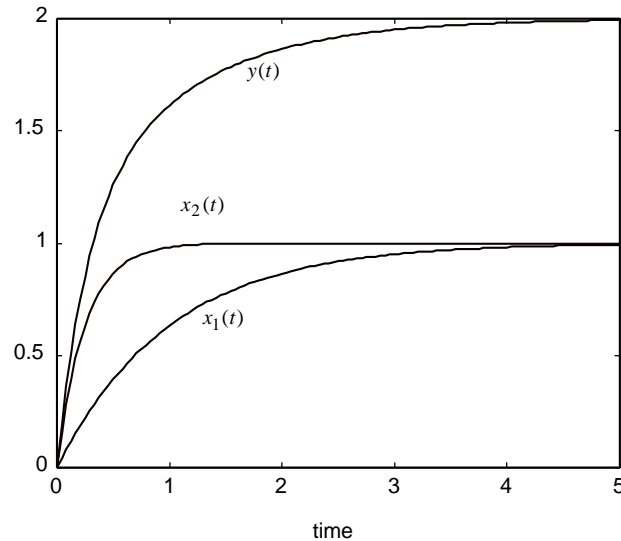


Figure 6.2 Step response for the system described by Equation (6.11). The state variables and output are plotted as functions of time.

Figure 6.3 shows a portrait that is similar to Figure 6.1 except that it appears rotated and distorted. This is the portrait for the homogeneous (no input) system with A -matrix

$$A_2 = \begin{bmatrix} -2 & -1 \\ -2 & -3 \end{bmatrix}\quad (6.12)$$

with the following spectrum and eigenvectors (expressed as columns in a modal matrix):

$$\sigma_2 = \{-1 \quad -4\} \quad M_2 = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

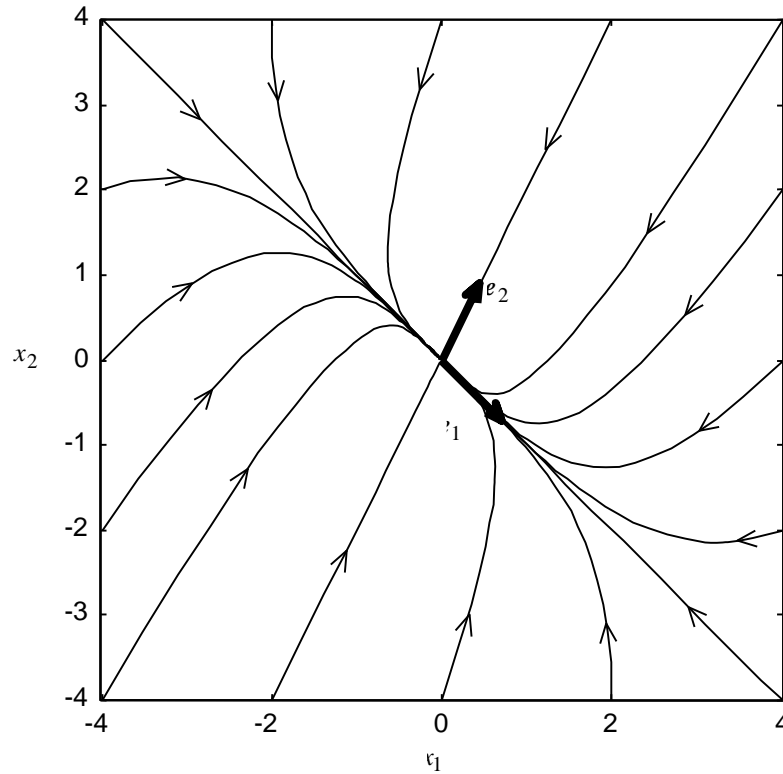


Figure 6.3 Phase portrait for the homogeneous system given by the matrix in (6.12). Note that this origin is also a *stable node*.

As before, we see that the eigenvectors represent invariant subspaces on which trajectories stay forever. This portrait was produced by performing the similarity transformation

$$A_2 = M_2 A_1 M_2^{-1} \quad (6.13)$$

From our knowledge of similarity transformations, we recognize this as nothing but a change of basis of the state space. We can therefore regard Figure 6.3 as containing the same information as Figure 6.1, but in different coordinates. When this same similarity matrix is applied to Equation (6.11), the result is

$$\dot{x} = M_2 A_1 M_2^{-1} x + M_2 b_1 u = \begin{bmatrix} -2 & -1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 5 \\ 7 \end{bmatrix} u \quad (6.14)$$

$$y = C M_2^{-1} x = [1 \quad 0] x$$

which has the step response shown below in Figure 6.4. Note that the state variables no longer reach unity asymptotically. This is because the similarity transformation affects the DC gain of the state variables. However, as we would expect, the output signal $y(t)$ remains exactly the same as in Figure 6.2, because similarity transformations do not affect the input/output performance, only its internal representation.

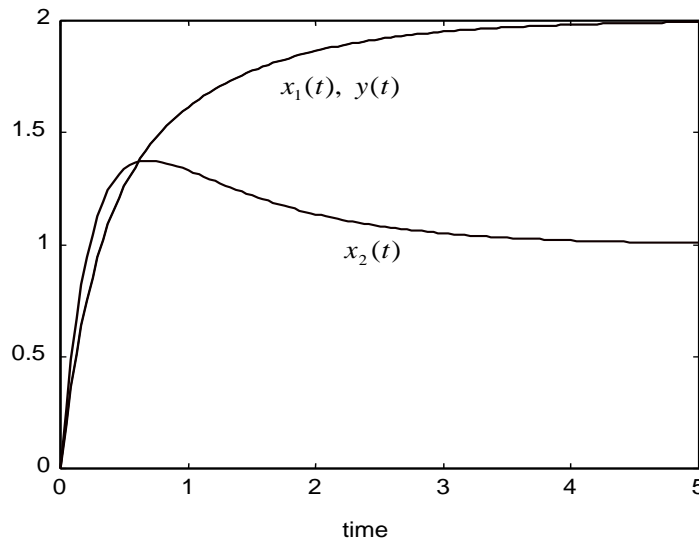


Figure 6.4 Step response for the system described by Equation (6.14). The state variables and output are plotted as a function of time.

Figure 6.5 is somewhat different in the sense that the trajectories do not all tend toward the origin. This portrait is based upon the following system:

$$A_3 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_3 = \{-2 \quad 1\} \quad M_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (6.15)$$

It is again a diagonal system with the eigenvectors constituting the standard basis. However, in this system, one of the corresponding eigenvalues is positive. Being diagonal, it is easy to decompose the system into decoupled parts:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ x_2 \end{bmatrix} \quad (6.16)$$

which of course has solutions that can be found independently of one another:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-2(t-t_0)} & 0 \\ 0 & e^{(t-t_0)} \end{bmatrix} \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} e^{-2(t-t_0)} x_1(t_0) \\ e^{(t-t_0)} x_2(t_0) \end{bmatrix}$$

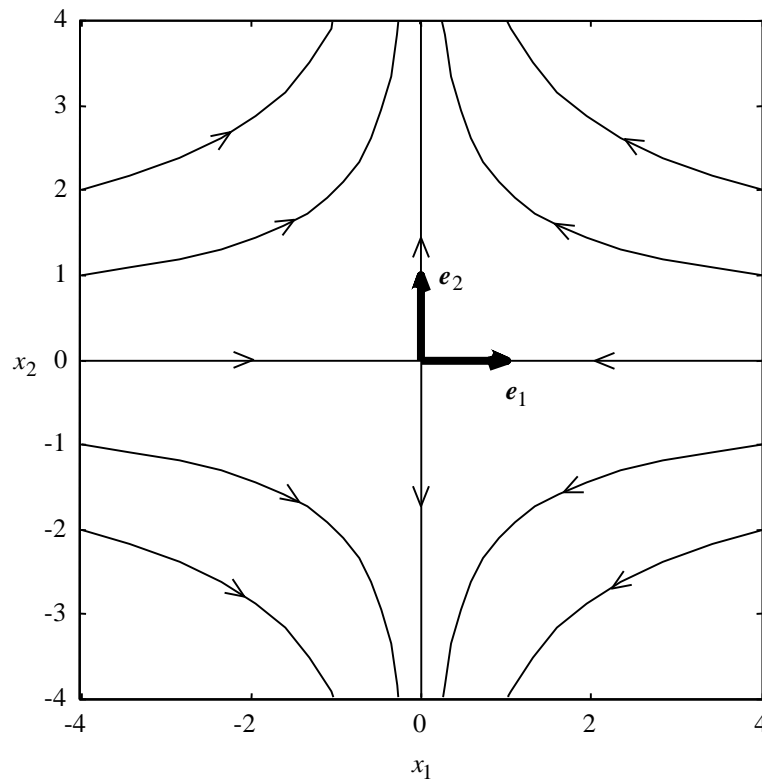


Figure 6.5 Phase portrait for the homogeneous system given by (6.15). In cases such as this wherein the trajectories approach the origin from one direction but diverge from it in the other, the origin is known as a *saddle point*.

The positive exponent in $x_2(t)$ indicates that a solution starting on the invariant space e_2 , while it will stay there, will nevertheless diverge from the origin. On the other hand, the negative exponent makes $x_1(t)$ tend toward zero.

The trajectories in between, while not lying on invariant subspaces, must interpolate the invariant trajectories, giving the shapes seen in the figure.

Performing a similarity transformation on the system described by (6.15), which is similar to (6.13) but with a different modal matrix, we can arrive at a form of (6.15) with a new basis:

$$A_4 = \begin{bmatrix} -3 & -2 \\ 2 & 2 \end{bmatrix} \quad \sigma_4 = \{-2 \ 1\} \quad M_4 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \quad (6.17)$$

The phase portrait of this transformed system is shown in Figure 6.6. Again, motion along the two invariant subspaces tends toward different directions, and the remaining trajectories interpolate smoothly.

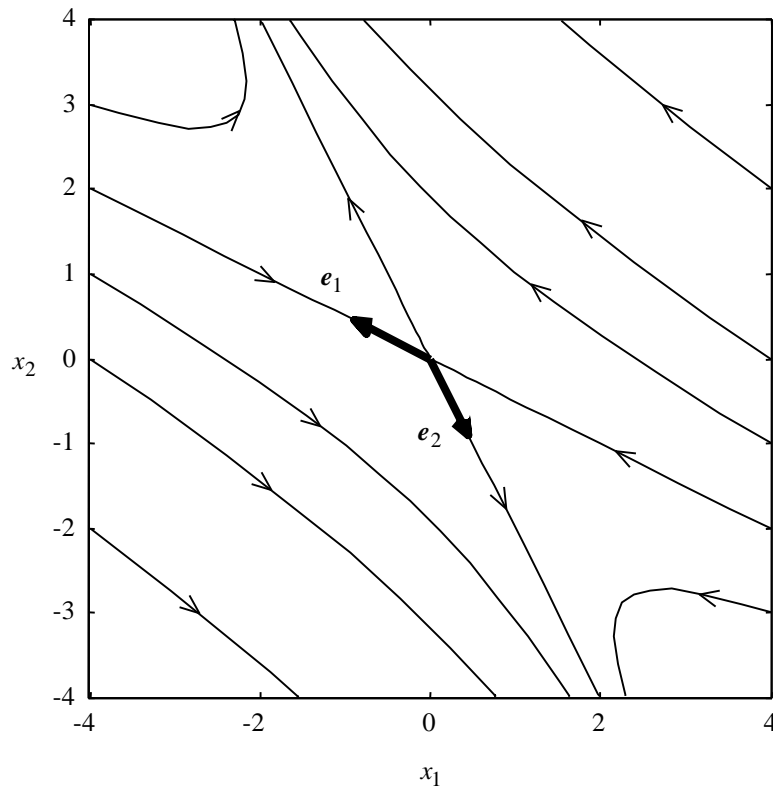


Figure 6.6 Phase portrait for the homogeneous system given by (6.17). The origin in this figure is a *saddle point*.

It should be pointed out that except at the singular points (for LTI systems, the origin only), two trajectories cannot cross or meet at a point. This would imply

that at that point, there are two independent solutions to the differential equation, which is prohibited by the uniqueness theorem. In our linear case, we also expect the trajectories to be smooth, which is seen in all the plots. Smoothness is guaranteed by the existence of a unique tangent vector given by the equation $\dot{x} = Ax$.

The nature of the phase portrait when there is only a single eigenvector, as we might expect in a system having a generalized eigenvector, is investigated next. An example of such a system is:

$$A_5 = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix} \quad \sigma_5 = \{-2 \quad -2\} \quad e_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad (6.18)$$

This system, represented by the phase portrait in Figure 6.7, indeed shows only a single invariant subspace. This eigenvector, which corresponds to a negative eigenvalue of -2 , tends toward the origin as expected.

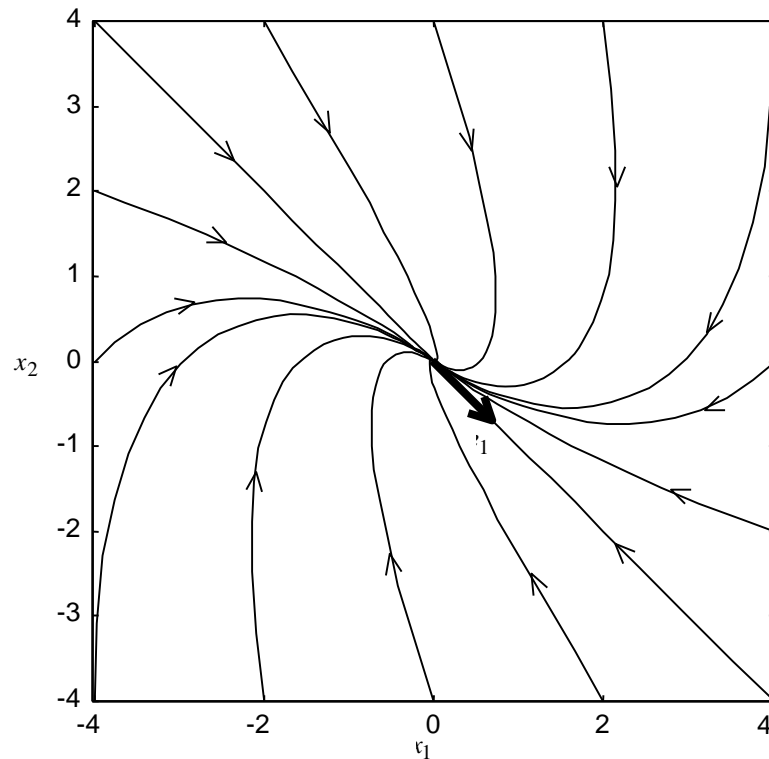


Figure 6.7 Phase portrait for the homogeneous system given by (6.18). The origin in this system is called a *stable node* because again the trajectories approach the origin without encircling it.

The other trajectories also tend toward the origin, but do so by partially spiraling inward. Because trajectories cannot cross, none of the spiraling trajectories may rotate more than 180° before asymptotically reaching the origin.

Figure 6.8 shows the initial condition response (zero inputs) for the same system, i.e., from (6.18). Note that qualitatively, the plot appears to be similar to previous time responses, e.g., Figure 6.4, except that now *both* curves show one inflection point.

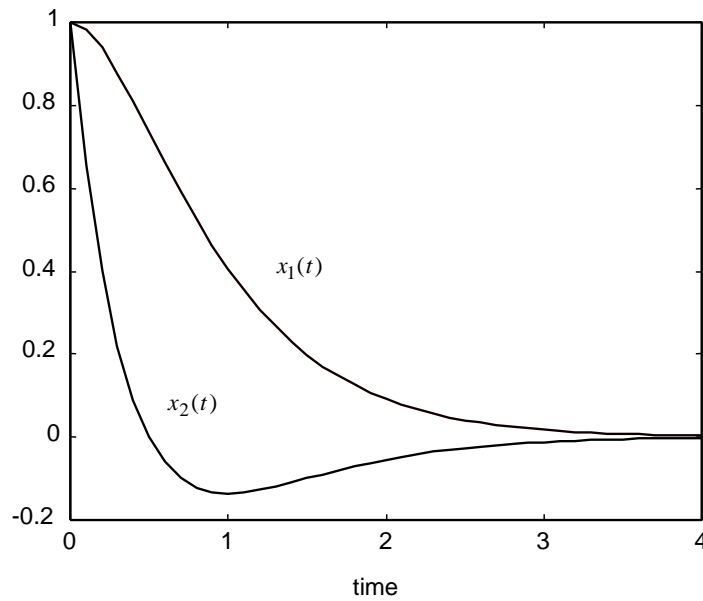


Figure 6.8 Initial condition response for the system described by Equation (6.14). The state variables are plotted as a function of time.

Transforming the system described by (6.18) into its Jordan form, we get

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \quad \sigma_6 = \{-2 \quad -2\} \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (6.19)$$

The phase portrait is given in Figure 6.9. As we might guess, the single regular eigenvector in this portrait lies along a coordinate axis because the Jordan form produces a system of differential equations, one of which is decoupled from the other.

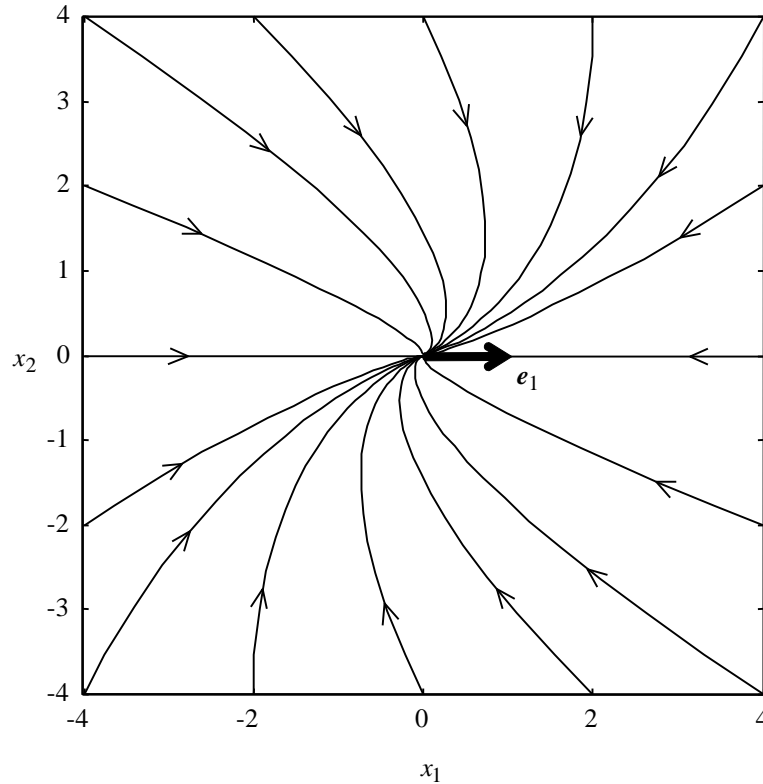


Figure 6.9 Phase portrait for the homogeneous system given by (6.19). The origin is a *stable node*.

In the case of complex eigenvectors, we mentioned in Section 4.3 that the geometric interpretation of invariant subspaces does not directly apply because we are interested in only the real solutions to the system. Therefore, for a system such as

$$A_7 = \begin{bmatrix} 2 & -3 \\ 6 & -4 \end{bmatrix} \quad \sigma_7 = \{-1 + j3 \quad -1 - j3\} \quad M_7 = \begin{bmatrix} -1 + j1 & -1 - j1 \\ j2 & -j2 \end{bmatrix} \quad (6.20)$$

we see in the portrait in Figure 6.10 that no invariant subspaces appear. This being the case, the spiraling trajectories rotate around the origin forever, asymptotically approaching it. (The trajectories are spiraling *inward* because of the negative real part of the eigenvalues; a system with eigenvalues containing positive real parts would spiral outward.)

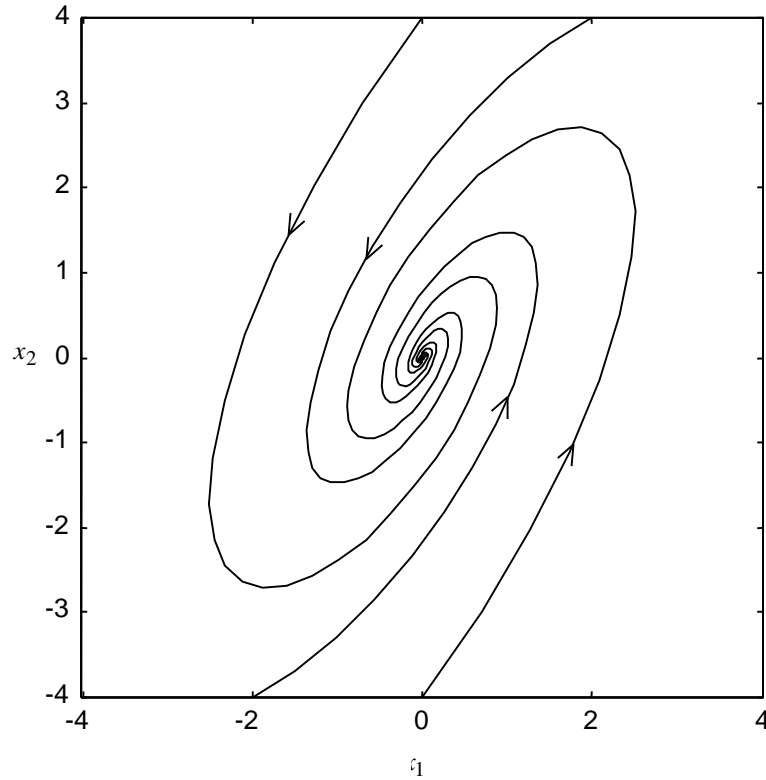


Figure 6.10 Phase portrait for the homogeneous system given by (6.20). This type of portrait shows an origin that is known as a *stable focus*.

The time-domain response of the system described by (6.20) is shown in the initial condition response of Figure 6.11. The oscillations shown in the phase portrait of Figure 6.10 are clearly seen.

As a final example, consider the system given by

$$A_8 = \begin{bmatrix} -1 & 2 \\ -5 & 1 \end{bmatrix} \quad \sigma_8 = \{j3 \quad -j3\} \quad M_8 = \begin{bmatrix} 3+j & 3-j \\ j5 & -j5 \end{bmatrix} \quad (6.21)$$

Relative to the previous example, we predict that this system will have no real invariant subspaces, and the trajectories will therefore be free to encircle the origin (without crossing each other). However, because there are *no* real parts of the eigenvalues, we cannot have solutions that decay to zero or tend to infinity. The resulting solutions are depicted in Figure 6.12. We know from our conventional solutions to differential equations or from our explicit computation

of e^{At} in Chapter 5 that the solutions given by (6.8) for (6.21) will be nondecaying sinusoids, just as those of (6.20) were decaying sinusoids. These periodic solutions appear as ellipses in phase portraits, as in Figure 6.12. A time response of such a system would show non-decaying oscillations, as opposed to the decaying oscillations of Figure 6.11. It is perhaps interesting to note that the principal axes of these ellipses, the directions of extremal displacement, occur along the *singular vectors* of matrix A . Refer to Examples 4.13 and 5.1 for a discussion of singular vectors and ellipse geometry.

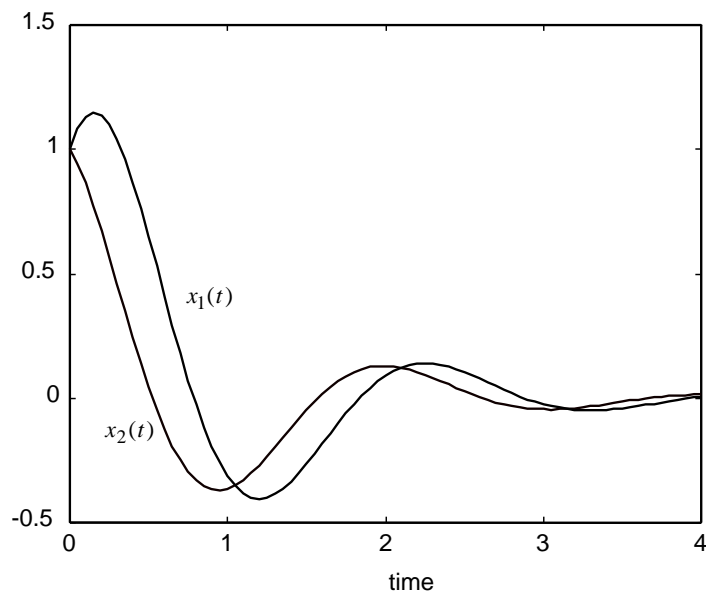


Figure 6.11 Initial condition response for the system described by Equation (6.20). The state variables are plotted as a function of time.

Without a decay or exponential growth in the solution, one might ask how it is possible to determine the directions of the arrows in Figure 6.12. The simplest way is to choose a single sample point, such as $x(t_0) = [1 \ 0]^T$, shown in the figure, and determine the direction of the tangent

$$\dot{x}(t_0) = A_8 x(t_0) = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$$

also shown in the figure. Given the direction of rotation of this single trajectory, the direction of all the others must be the same.

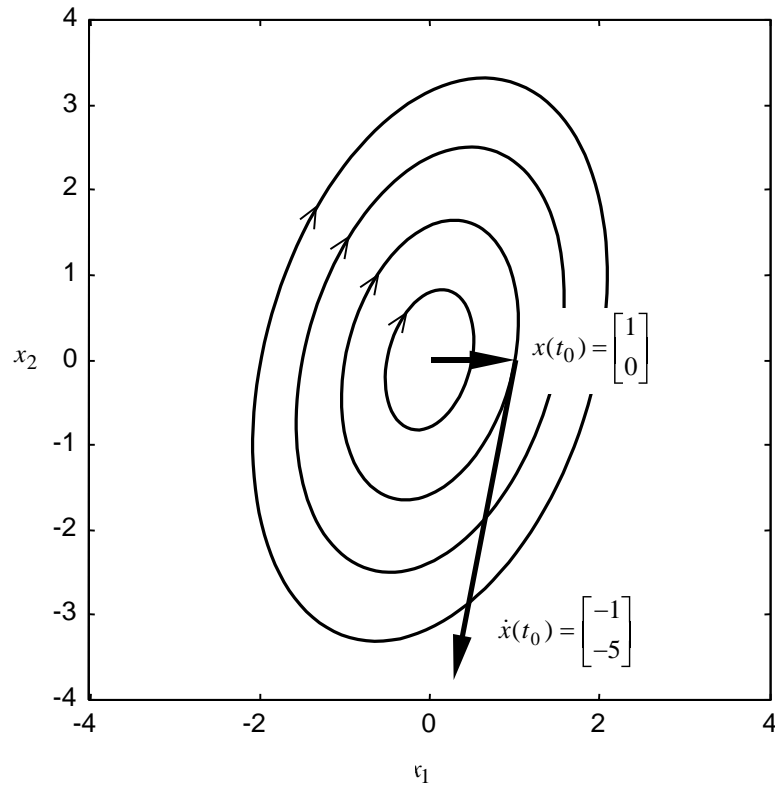


Figure 6.12 Phase portrait for the homogeneous system given by (6.21). The vector $\dot{x}(t_0)$ is shortened to fit on the plot. The origin in this portrait is called a *center*.

Although we have been presenting two-dimensional phase portraits here, the phase trajectory can be drawn in three dimensions, and the concept extends to arbitrary dimensions. Although our ability to draw in these higher dimensions diminishes, the intuitive geometric picture they provide does not. Our discussion of solutions using phase trajectories will continue in the next section and we will refer to them again in the next chapter on stability.

6.3 System Modes and Decompositions

In this section we will again consider LTI systems that are solved using Equation (6.6). Of course, for any t , vector $x(t)$ is an element in the (linear) state space, a vector space with all the attendant rules and properties. What we demonstrate in this section is the utility of considering the basis of eigenvectors for the state

space when computing $x(t)$. The decomposition of $x(t)$ in this way is widely used in engineering analysis, especially for large-scale problems such as the deflection of flexible structures, vibrations, and the reduction of large state space to smaller approximations.

Let $\{e_i\}$ be the set of n linearly independent eigenvectors for a system, including, if necessary, generalized eigenvectors. Because this set may be used as a basis for the state space, we can uniquely decompose $x(t)$ as

$$x(t) = \sum_{i=1}^n \xi_i(t) e_i \quad (6.22)$$

where the coefficients in this expansion $\xi_i(t)$, $i = 1, \dots, n$ are functions of time. We can perform the same decomposition on the input terms in (6.1), allowing B to vary with time:

$$B(t)u(t) = \sum_{i=1}^n \beta_i(t) e_i \quad (6.23)$$

Substituting these expansions into the original state equation:

$$\sum_{i=1}^n \dot{\xi}_i(t) e_i = \sum_{i=1}^n \xi_i(t) A e_i + \sum_{i=1}^n \beta_i(t) e_i$$

or, by rearranging and supposing for now that $\{e_i\}$ consists of a complete set of n regular eigenvectors, we can obtain

$$\sum_{i=1}^n (\dot{\xi}_i(t) - \xi_i(t)A - \beta_i(t)) e_i = \sum_{i=1}^n \dot{\xi}_i(t) e_i - \xi_i(t) \lambda_i e_i - \beta_i(t) e_i \quad (6.24)$$

$$= 0$$

where λ_i is the eigenvalue corresponding to regular eigenvector e_i . Because the set $\{e_i\}$ is linearly independent, the terms in parentheses in (6.24) must also be identically equal to zero, or

$$\dot{\xi}_i(t) = \xi_i(t) \lambda_i + \beta_i(t) \quad i = 1, \dots, n \quad (6.25)$$

Equation (6.25) constitutes a set of n independent, first-order LTI differential equations. In the case that $\{e_i\}$ contains some generalized eigenvectors, (6.25) would take the same form for some i (those i to which no generalized eigenvectors

are chained). However, when the generalized eigenvector e_{i+1} is chained to (regular or generalized) eigenvector e_i , (6.25) would change to

$$\dot{\xi}_i(t) = \xi_i(t)\lambda_i + \xi_{i+1}(t) + \beta_i(t) \quad i = 1, \dots, n \quad (6.26)$$

There will be a total of $n - \sum_j g_j$ such coupled equations, where the summation of geometric multiplicities g_j is taken over all distinct eigenvalues.

The terms $\xi_i(t)e_i$ in (6.22) and (6.24) are known as system *modes*, and (6.22) is the *modal decomposition* of the solution $x(t)$. The modes are equivalent to “new” state variables. They are, in fact, the same state variables we obtained when we make the change of basis $x = M\xi$, using the modal matrix M that results in our familiar similarity transformation, i.e.,

$$\begin{aligned} \dot{\xi} &= M^{-1}AM\xi + M^{-1}Bu \\ y &= CM\xi + Du \end{aligned} \quad (6.27)$$

This fact can be seen by expressing Equation (6.22) as

$$\begin{aligned} x(t) &= \sum_{i=1}^n \xi_i(t)e_i \\ &= M \begin{bmatrix} \xi_1(t) \\ \vdots \\ \xi_n(t) \end{bmatrix} \end{aligned} \quad (6.28)$$

This is the source of the name “modal” in the term *modal matrix*. It is the matrix that decomposes a system so that decoupled equations are solved to produce system modes. Matrix $M^{-1}AM = J$ is simply the Jordan form discussed in Chapter 4.

As we did in Chapter 4, if we first convert the system to its basis of eigenvectors, then we can exploit the simpler form of (6.27) to generate the state-vector solution:

$$\begin{aligned} \xi(t) &= e^{J(t-t_0)}\xi(t_0) + \int_{t_0}^t e^{J(t-\tau)} M^{-1}B(\tau)u(\tau) d\tau, \\ \xi(t_0) &= M^{-1}x(t_0) \end{aligned} \quad (6.29)$$

where $\xi \triangleq [\xi_1 \ \dots \ \xi_n]^T$. Equation (6.29) will be easier to solve because of the

simple structure of the Jordan form (see Section 5.4.2). After obtaining such a solution, the solution in the original basis may be computed via

$$x(t) = M\xi(t)$$

Modal Decompositions in Infinite-Dimensional Spaces

Modal decompositions are often useful in infinite-dimensional spaces, such as those produced by models of distributed-parameter systems. For example, the time- and space-varying displacement of flexible beams, strings, and plates are often expressed as an infinite summation of shape-functions multiplied by time-functions. These are sometimes called *eigenfunctions*, which are simply the modes as described above. For example, the displacement u of a point on a beam might then be expressed as a summation of products of time- and space-dependent functions:

$$u(x, t) = \sum_{i=1}^{\infty} X_i(x)T_i(t) \quad (6.30)$$

The infinite series arises because the describing equation for beam displacement is a partial differential equation, rather than an ordinary differential equation such as the ones with which we have been working. When ordered in decreasing magnitude of $X_i(x)$, it is common practice to truncate the series after the few most significant terms, thereby approximating an infinite dimensional system with a finite dimensional one that can be analyzed via matrix arithmetic.

For example, the equations that describe the displacement $u(x, t)$ of a stretched string are

$$\frac{\partial^2 u}{\partial x^2} - \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2} = 0 \quad (6.31)$$

where ρ is the density of the string and T is its tension. If the string is of length ℓ , is initially displaced at a location $x = c$ to a height $u(c, 0) = h$, and is subsequently released, it can be shown [6] that

$$u(x, t) = k \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi c}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{n\pi\sqrt{T}t}{\ell\sqrt{\rho}}\right) \quad (6.32)$$

where k is a constant that depends on the system parameters ℓ , c , and h . Although there are infinitely many modes [or “shapes”, i.e., $\sin(n\pi x/\ell)$] that contribute to

this solution, their magnitudes decrease as $1/n^2$, so after some finite number, a suitable approximation can be obtained by truncating the series.

Because we are not able to delve into the solution techniques for (6.31), we cannot further discuss infinite-dimensional systems. However even in finite-dimensional linear systems, this kind of modal expansion helps us understand solutions by considering them one component at a time.

6.3.1 A Phase Portrait Revisited

In light of our knowledge of system modes, we now reconsider the information provided by the phase portraits and provide further clues to their qualitative construction. Consider again the first example system, given by

$$\dot{x} = A_1 x = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} x \quad (6.33)$$

with eigenvalues and eigenvectors computed in (6.10). An enlarged view of the upper-right quadrant of the phase portrait for that system is given again in Figure 6.13, where the trajectories on the eigenvectors are drawn with arrows pointing in the direction of positive time. The natural question arises when using phase portraits as a qualitative solution tool, “How do we know which direction the trajectories will take, i.e., how do we know they appear concave up as in Figure 6.13, rather than facing toward the sides? Could not trajectories approach the origin from the vertical direction instead of seeming to flatten out and approach horizontally, as the graphs show?”

To resolve these questions and provide a tool for guessing the trajectory’s shapes, we show an initial condition, $x(t)$, and the tangent to the trajectory at that point, $\dot{x}(t) = Ax(t)$. This tangent vector is decomposed along the two eigenvectors, e_1 and e_2 . Analytically, we know from the preceding section that

$$\begin{aligned} \dot{x}(t) &= Ax(t) = A \sum_{i=1}^2 \xi_i(t) e_i \\ &= \sum_{i=1}^2 \xi_i(t) A e_i \\ &= \sum_{i=1}^2 \xi_i(t) \lambda_i e_i \\ &= -e^{-t} e_1 - 4e^{-4t} e_2 \end{aligned} \quad (6.34)$$

where the last line easily results from our A -matrix being already diagonal. From this expansion, the lengths of the components along each of the eigenvector directions are clear. For the time selected, the component along (negative) e_2 is

larger than the component along (negative) e_1 . The trajectory at this time, then, is evolving more along e_2 than e_1 .

Taking a macroscopic view of the plot, we can see that the eigenvalue $\lambda_2 = -4$ is larger in magnitude (“faster” in time constant terms) than the eigenvalue $\lambda_1 = -1$. The result is that, until it decays to negligible proportions relative to the first mode (the component along e_1), the second mode (the component of motion along e_2) is larger. When sketching the plot, then, we naturally expect the vertical component of motion to dominate for small time, and the reverse for large time. Hence, we expect the curvature of the graph to be the concave up (and down) shape as observed.

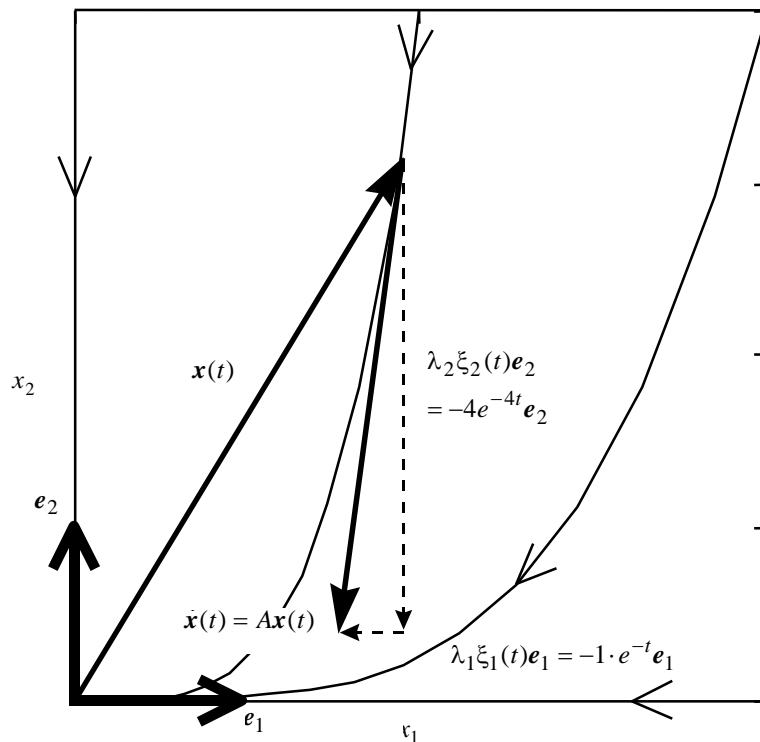


Figure 6.13 Detail of phase portrait for the homogeneous system given by (6.9).

Example 6.2: Sketching a Phase Portrait Using Qualitative Analysis

A two-dimensional LTI homogeneous system is known to have eigenvalues $\sigma = \{-10, -2\}$ and the corresponding eigenvectors

$$\{e_1, e_2\} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\} \quad (6.35)$$

Sketch the phase portrait for the system.

Solution:

The first step in determining the nature of the trajectories is to sketch the invariant subspaces, i.e., the straight lines that lie along the given eigenvectors. Noticing that both eigenvalues are negative real numbers, we expect all trajectories to approach the origin from any location. Because the eigenvalue λ_1 is faster than λ_2 , we expect the trajectories for small time to experience more change in the direction of e_1 than in the direction of e_2 . After a long time, the first mode will have decayed, and the trajectories' direction will be dominated by the second mode, i.e., along e_2 . We then have sufficient information to sketch the phase portrait shown in Figure 6.14.

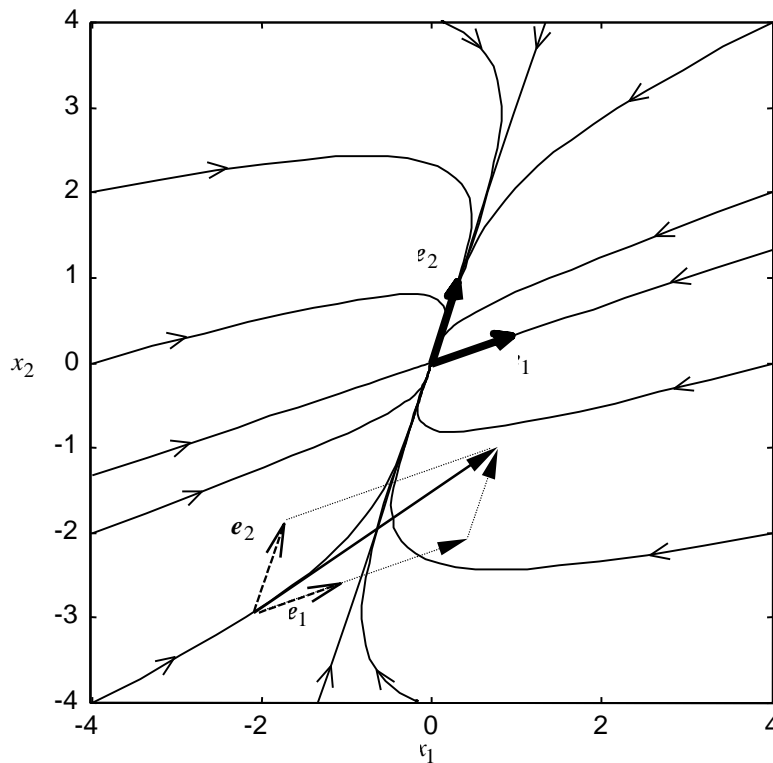


Figure 6.14 Phase portrait for the example system given by the eigenvectors in (6.35).

The figure also shows the decomposition of a trajectory into its modes. For the sample time selected, it can be seen that there is more motion along the first mode than along the second. At a later time (farther along that trajectory), there will be more motion along the second mode.

6.4 The Time-Varying Case

The matrix exponential integrating factor method, used in Equation (6.3), will not work when the A -matrix is a function of time. When we consider the linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (6.36)$$

we must either solve the system using a different technique or else find a different integrating factor. In the next two sections, special matrices are derived, called the fundamental solution matrix and the state transition matrix. The fundamental solution matrix performs the function of the integrating factor, while the overall solution is in terms of the state transition matrix. As we will see, though, the state transition matrix is difficult to compute.

6.4.1 State Fundamental Solution Matrix

Consider a homogeneous version of (6.36):

$$\dot{x}(t) = A(t)x(t) \quad (6.37)$$

We know from Chapter 2 that the set of solutions to (6.37) constitutes a linear vector space. It can be observed that this space is n -dimensional by considering a basis set of n linearly independent initial condition vectors $\{x_{i0}\}$, $i = 1, \dots, n$. If we restrict our attention to matrices $A(t)$ that are smooth, then it is possible to guarantee that the solutions to (6.37) given an arbitrary initial condition, $x(t_0)$, are unique. Then the set $\{x_i(t)\}$, where $\dot{x}_i(t) = A(t)x_i(t)$ and $x_i(t_0) = x_{i0}$, defines n solutions for (6.37) that are linearly independent on $[t_0, t]$. Any additional solution $\xi(t)$ must be a linear combination of the $x_i(t)$ because $\xi(t_0) = \sum_{i=1}^n \alpha_i x_{i0}$ implies that $\xi(t) = \sum_{i=1}^n \alpha_i x_i(t)$ for some set of scalars α_i , $i = 1, \dots, n$.

Organizing these linearly independent solutions, we construct a matrix $X(t)$ as follows:

$$X(t) = [x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)]$$

Clearly, $\dot{X}(t) = A(t)X(t)$. Such a matrix $X(t)$ is known as a *fundamental solution matrix*.

Using the matrix identity revealed in Problem 3.2:

$$\begin{aligned} \frac{dX^{-1}(t)}{dt} &= -X^{-1}(t) \frac{dX(t)}{dt} X^{-1}(t) \\ &= -X^{-1}(t) \underbrace{A(t)X(t)}_{=I} X^{-1}(t) \\ &= -X^{-1}(t)A(t) \end{aligned}$$

From this result, we can now show that the matrix $X^{-1}(t)$ qualifies as a valid integrating factor for the state equations in (6.37).

We may use this integrating factor in the nonhomogeneous equations in (6.36):

$$\begin{aligned} X^{-1}(t)[\dot{x}(t) - A(t)x(t) &= B(t)u(t)] \\ X^{-1}(t)\dot{x}(t) - \underbrace{X^{-1}(t)A(t)}_{=I}x(t) &= X^{-1}(t)B(t)u(t) \\ X^{-1}(t)\dot{x}(t) + \frac{dX^{-1}(t)}{dt}x(t) &= X^{-1}(t)B(t)u(t) \\ \frac{d}{dt}[X^{-1}(t)x(t)] &= X^{-1}(t)B(t)u(t) \end{aligned}$$

Now integrating both sides of the bottom line above yields

$$X^{-1}(t)x(t) - X^{-1}(t_0)x(t_0) = \int_{t_0}^t X^{-1}(\tau)B(\tau)u(\tau) d\tau$$

or

$$x(t) = X(t)X^{-1}(t_0)x(t_0) + \int_{t_0}^t X(t)X^{-1}(\tau)B(\tau)u(\tau) d\tau \quad (6.38)$$

This appears to be a general solution to the time-varying state equations, but, unfortunately, computing this solution is not so easy. After all, if we could easily find the n linearly independent solutions necessary for the construction of $X(t)$, we would not have needed to derive (6.38). In most cases, we have no general method for computing the fundamental solution matrix $X(t)$. The ease of computing components $x_i(t)$ depends a great deal on the exact form of the time functions within $A(t)$. Like nonlinear systems, such time-varying systems tend to be solved individually, as circumstances allow. One can see from any ordinary differential equations text that certain classes of time-varying equations are solved, e.g., Bessel, LeGendre, or Hermite equations, but certainly not all of them can be solved.

6.4.2 The State-Transition Matrix

If $X(t)$ is not easily calculated, then of what useful purpose is Equation (6.38)? First, we will use its existence to prove the existence of a different matrix. Note that in (6.38), the fundamental solution matrix only appears as a product of the form $X(t)X^{-1}(\tau)$. This product

$$\Phi(t, \tau) \triangleq X(t)X^{-1}(\tau) \quad (6.39)$$

is known as the *state-transition matrix*, and it lends insight into the solutions of time-varying systems and time-invariant systems. With this notation, (6.38) becomes

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau) d\tau \quad (6.40)$$

Computing the State-Transition Matrix

While $X(t)$ can indeed be difficult to find, there does exist an iterative method for the computation of $\Phi(t, \tau)$, known as the Peano-Baker integral series, which we provide here without proof [7]:

$$\Phi(t, \tau) = I + \int_{\tau}^t A(\sigma_1) d\sigma_1 + \int_{\tau}^t A(\sigma_1) \int_{\tau}^{\sigma_1} A(\sigma_2) d\sigma_2 d\sigma_1 + \dots \quad (6.41)$$

A difficulty with this technique is the necessary repeated integrations. In addition, it is unlikely that the series will converge to a closed form. Often, this method is

convenient when the matrix $A(t)$ is inherently *nilpotent*. A nilpotent matrix is one such that $A^p \equiv 0$ for all $p > q$. An example of a nilpotent matrix is a triangular matrix with zeros on the diagonal. As we saw in Section 5.4.2, repeated multiplications of such matrices by themselves eventually results in the zero matrix, not because of the numbers in the matrix, but because of its inherent physical structure.

A second method applies only in the special circumstance that $A(t)A(\tau) = A(\tau)A(t)$. If this is the case, then

$$\Phi(t, \tau) = \exp \left[\int_{\tau}^t A(\sigma) d\sigma \right] \quad (6.42)$$

Although it is rare for this condition to be satisfied for general $A(t)$, it does hold when A is constant or when $A(t)$ is diagonal, in which cases there are more direct methods for finding the solution of the system. One should be warned that the expression in (6.42) is *not* equivalent to e^{At} , and must often be computed with a Taylor series expansion of the matrix exponential.

Example 6.3: State-Transition Matrix Using Series Expansion

For a system with the A -matrix given in Example 6.1, find the state-transition matrix $\Phi(t, \tau)$ using the Peano-Baker series.

Solution.

It should first be noted that the state transition matrix for Example 6.1 is apparent from the solution. Computing it explicitly,

$$\begin{aligned} \Phi(t, \tau) &= I + \int_{\tau}^t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} d\sigma_1 + \int_{\tau}^t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \int_{\tau}^{\sigma_1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} d\sigma_2 d\sigma_1 + \dots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \sigma_1 \\ 0 & 0 \end{bmatrix} \Big|_{\tau}^t + \int_{\tau}^t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} \Big|_{\tau}^{\sigma_1} d\sigma_1 + \dots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t - \tau \\ 0 & 0 \end{bmatrix} + 0 + \dots \\ &= \begin{bmatrix} 1 & t - \tau \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Here, it is apparent that the second- and higher-order integrations will yield zero.

Note the agreement between this result and the matrix seen in the integrand of the solution of Example 6.1.

Properties of the State-Transition Matrix

By expressing an arbitrary solution as a linear combination of fundamental solutions, $x(t) = X(t)x(t_0)$, we can perform some algebraic manipulations to find that

$$\begin{aligned} x(t) &= X(t)X^{-1}(\tau)X(\tau)x(t_0) \\ &= X(t)X^{-1}(\tau)x(\tau) \\ &= \Phi(t, \tau)x(\tau) \end{aligned} \quad (6.43)$$

Therefore, the state-transition matrix $\Phi(t, \tau)$ is a linear operator that takes a vector $x(\tau)$ and produces a vector $x(t)$ (i.e., a solution at time t). Hence the name for the matrix; it performs the *transition* from a state at one time to a state at another. This property can be extended to successive time instants by realizing that

$$\begin{aligned} \Phi(t_2, t_1)\Phi(t_1, t_0) &= X(t_2)X^{-1}(t_1)X(t_1)X^{-1}(t_0) \\ &= X(t_2)X^{-1}(t_0) \\ &= \Phi(t_2, t_0) \end{aligned} \quad (6.44)$$

Similarly, it is easily shown that $\Phi^{-1}(t, \tau) = \Phi(\tau, t)$.

Furthermore, by differentiating,

$$\begin{aligned} \frac{d\Phi(t, \tau)}{dt} &= \frac{d[X(t)X^{-1}(\tau)]}{dt} = \frac{dX(t)}{dt}X^{-1}(\tau) \\ &= A(t)X(t)X^{-1}(\tau) \\ &= A(t)\Phi(t, \tau) \end{aligned} \quad (6.45)$$

The state transition matrix and the fundamental solution matrix $X(t)$ are solutions for the same homogeneous differential equation, (6.37). This suggests a numerical method of solving (6.45), numerically integrating (over time t), subject to the initial condition that $\Phi(\tau, \tau) = I$, in order to estimate $\Phi(t, \tau)$. This is not done in practice very often, mostly because the system input $u(t)$ is itself not known a priori. Therefore, such an *open-loop* solution of (6.40) is not useful.

When the system is time-invariant, we know from a comparison of (6.6) and (6.38) that $X(t) = e^{At}$, yielding

$$\Phi(t, t_0) = e^{At} e^{-At_0} = e^{A(t-t_0)}$$

Remember from the definition that for a time-invariant system, the solution depends only on the difference $t - t_0$. Fortunately, in this situation we have already demonstrated analytical methods for generating the state-transition matrix e^{At} (see Chapter 5). Note that this situation is entirely consistent with the properties of state-transition matrices derived for the more general time-varying case above. In particular, we have already shown that in a homogeneous system,

$$x(t) = e^{A(t-t_0)}x(t_0)$$

which is the time-invariant counterpart to (6.43). If we returned to the phase plane depictions, we could verify that for an initial condition $x(t_0)$, multiplication by the state transition matrix $e^{A(t-t_0)}$ would produce a different vector $x(t)$ for whatever time t we choose.

We should remark at this point that phase portraits for time-varying systems are of little use. Recall that the phase portrait for a system depends on the eigenvalues and eigenvectors of the A -matrix. When the A -matrix is time-varying, the eigenvalues and eigenvectors will therefore also depend on time. In phase portraits, time is a parameter on the trajectories. The plots, therefore, cannot be constructed for a time-varying A -matrix. In fact, we will observe in the next chapter that the eigenvalues of time-varying systems are themselves of limited use. One cannot always even predict the stability of a system by calculating the eigenvalues at a particular instant in time.

6.5 Solving Discrete-Time Systems

As we have mentioned, there are natural systems that are modeled in discrete-time. In a linear systems context, this means that the inputs are applied, and the states change, at discrete intervals of period T . Such a system may be modeled with the equations

$$\begin{aligned} x(k+1) &= A_d(k)x(k) + B_d(k)u(k) \\ y(k) &= C_d(k)x(k) + D_d(k)u(k) \end{aligned} \tag{6.46}$$

Here we have implicitly assumed by our notation that the discrete-time system

matrices A_d , B_d , C_d , and D_d may be functions of time k . In a time-invariant system, they will not.

However, often, because the universe is modeled (at least by most engineers) as evolving in continuous-time, the equations in (6.46) more often result from the discretization of a continuous-time system such as

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned}\tag{6.47}$$

We can arrive at the equivalent (6.46) from (6.47) by considering a discrete-time approximation to the state-equation solution (6.40).

6.5.1 Discretization

Consider the k^{th} time instant, wherein $t = kT$. At T units of time later, $t = (k+1)T$. In order to get an accurate discrete-time equivalent, we will assume that period T is much smaller than the Shannon period for the input signal $u(t)$ (see [4], pp. 79-82). If this is the case, we can approximate the input as $u(t) \approx u(kT)$, or simply $u(k)$, over the entire interval $kT \leq t < (k+1)T$. Making the input constant over the sampling interval allows us to remove the input term from the integrand in the solution for (6.40):

$$x(k+1) = \Phi(k+1, k)x(k) + \int_k^{k+1} \Phi(k+1, \tau)B(\tau) d\tau u(k)\tag{6.48}$$

In this form, if $\Phi(t, \tau)$ and $B(\tau)$ are known, the solution for the state vector could be computed each time an input is applied. The matrices

$$A_d(k) \triangleq \Phi(k+1, k) \quad B_d(k) \triangleq \int_k^{k+1} \Phi(k+1, \tau)B(\tau) d\tau$$

can be computed from knowledge of the system, and (6.46) is the result. The output equation, being an algebraic equation, follows from a knowledge of $C(t)$ and $D(t)$. Also note that (6.48) provides the values of the state vector at the chosen sample instants. At intermediate times, the state variables may take on other values, resulting in a “ripple” effect if the variables are plotted as functions of continuous time. For discrete-time analysis, though, it is only the values at the sample instants that we are interested in.

Note that (6.48) is not the solution to the state equations in (6.46). Rather, (6.48) is a discretization^M of the state equations in (6.47). As we saw in Examples 3.10 and 3.11, the solutions of discrete-time systems are often inductively obtained by iterating over a few times intervals on (6.46).

6.5.2 Discrete-Time State-Transition Matrix

Consider the first equation of (6.46). If we write a few terms in the computation of $x(k)$, given an initial point $x(j)$, we get:

$$\begin{aligned}
 x(j+1) &= A_d(j)x(j) + B_d(j)u(j) \\
 x(j+2) &= A_d(j+1)x(j+1) + B_d(j+1)u(j+1) \\
 &= A_d(j+1)[A_d(j)x(j) + B_d(j)u(j)] + B_d(j+1)u(j+1) \\
 &= A_d(j+1)A_d(j)x(j) + A_d(j+1)B_d(j)u(j) + B_d(j+1)u(j+1) \\
 x(j+3) &= A_d(j+2)A_d(j+1)A_d(j)x(j) + A_d(j+2)A_d(j+1)B_d(j)u(j) \\
 &\quad + A_d(j+2)B_d(j+1)u(j+1) + B_d(j+2)u(j+2) \\
 &\quad \vdots
 \end{aligned}$$

until, by induction,

$$x(k) = \left(\prod_{i=j}^{k-1} A_d(i) \right) x(j) + \sum_{i=j+1}^k \left(\prod_{q=i}^{k-1} A_d(q) \right) B(i-1)u(i-1) \quad (6.49)$$

where it is necessary to define

$$\prod_{q=k}^{k-1} A_d(q) \triangleq I$$

Consider the situation in which the system is homogeneous, i.e., $u(k) \equiv 0$ for all k . Then we would have

$$x(k) = \left(\prod_{i=j}^{k-1} A_d(i) \right) x(j) \quad (6.50)$$

This formula implicitly defines the state-transition matrix for discrete-time systems:

$$\Psi(k, j) = \prod_{i=j}^{k-1} A_d(i) \quad (6.51)$$

As in the continuous-time case, Equation (6.50) makes it apparent that the state-transition matrix $\Psi(k, j)$ may be interpreted as the linear operator that takes a state vector at time j and returns the state vector at time k . This matrix is defined only for $k \geq j$ and shares most of the properties of the continuous-time state transition matrix, except for invertibility. From the structure of (6.51), it is clear

that if any $A_d(i)$ is not invertible, which is entirely possible, then $\Psi(k, j)$ itself will not be invertible.

Using the notation of the discrete-time state-transition matrix for time-varying systems, i.e., Equation (6.51) above, then the expression (6.49) for the general solution of discrete-time systems is

$$x(k) = \Psi(k, j)x(j) + \sum_{i=j+1}^k \Psi(k, i)B(i-1)u(i-1) \quad (6.52)$$

where the initial condition is taken at $t = j$.

6.5.3 Time-Invariant Discrete-Time Systems

Suppose now that in the discrete-time system of (6.46), the matrices A_d and B_d were independent of time k . Then (6.48) would become

$$x((k+1)T) = e^{A(k+1)T-kT} x(kT) + \int_{kT}^{(k+1)T} e^{A(k+1)T-\tau} B(\tau) d\tau u(kT)$$

or

$$x(k+1) = e^{AT} x(k) + \int_{kT}^{(k+1)T} e^{A(k+1)T-\tau} B(\tau) d\tau u(k) \quad (6.53)$$

From this, we have the definitions

$$A_d \triangleq e^{AT} \quad B_d \triangleq \int_{kT}^{(k+1)T} e^{A(k+1)T-\tau} B(\tau) d\tau$$

again giving a formula for obtaining (6.46). Once again, matrices A_d and B_d can be computed “off-line,” i.e., without knowledge of the input. Note that matrix A_d is independent of k .

lsim(sys, u, t)

Furthermore, for a time-invariant system, (6.49) will become^M

$$x(k) = (A_d)^{k-j} x(j) + \sum_{i=j+1}^k (A_d)^{k-i} B(i-1)u(i-1) \quad (6.54)$$

yielding the time-invariant, discrete-time state transition matrix

$$\Psi(k, j) = \Psi(k-j) = (A_d)^{k-j}$$

We should point out here that the discussion of modes and modal decompositions as presented in Section 6.3 applies here as well. Because the eigenvalues and eigenvectors of a matrix A or A_d are calculated in the same manner regardless of whether the system is discrete-time or continuous-time, the modal matrix M functions in the same way for both systems. If we use it to define a new state vector $x(k) = M\xi(k)$, then

$$\begin{aligned}\xi(k+1) &= M^{-1}A_d M\xi(k) + M^{-1}B_d(k)u(k) \\ &\stackrel{\Delta}{=} \hat{A}_d \xi(k) + \hat{B}_d(k)u(k) \\ y(k) &= C_d M\xi(k) + D_d u(k) \\ &\stackrel{\Delta}{=} \hat{C}_d \xi(k) + \hat{D}_d u(k)\end{aligned}\tag{6.55}$$

The transformed matrix \hat{A}_d will again be in its Jordan form.

Example 6.4: Discretization of a System

For the A and B matrices given below, discretize the system to get a discrete-time state variable description using the formula (6.53). Assume the system is sampled at $T = 0.1$ s.

$$A = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution

The A -matrix was also used in Example 5.4, so we know from that example that

$$e^{At} = \begin{bmatrix} e^{-3t} & -e^{-3t} + e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

Therefore,

$$A_d = e^{AT} = \begin{bmatrix} e^{-3T} & -e^{-3T} + e^{-2T} \\ 0 & e^{-2T} \end{bmatrix} = \begin{bmatrix} 0.741 & 0.0779 \\ 0 & 0.819 \end{bmatrix}$$

As for B_d , we have

$$\begin{aligned}
B_d &= \int_{kT}^{(k+1)T} e^{A(k+1)T-\tau} B d\tau \\
&= \int_{kT}^{(k+1)T} \begin{bmatrix} e^{-3(k+1)T-\tau} & -e^{-3(k+1)T-\tau} + e^{-2(k+1)T-\tau} \\ 0 & e^{-2(k+1)T-\tau} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} d\tau \\
&= \int_{kT}^{(k+1)T} \begin{bmatrix} e^{-2(k+1)T-\tau} \\ e^{-2(k+1)T-\tau} \end{bmatrix} d\tau = \frac{1}{2} \begin{bmatrix} e^{-2(k+1)T-\tau} \\ e^{-2(k+1)T-\tau} \end{bmatrix} \Big|_{kT}^{(k+1)T} \\
&= \frac{1}{2} \begin{bmatrix} e^{-2(k+1)T-(k+1)T} & -e^{-2(k+1)T-kT} \\ e^{-2(k+1)T-(k+1)T} & -e^{-2(k+1)T-kT} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 - e^{-2T} \\ 1 - e^{-2T} \end{bmatrix} = \begin{bmatrix} 0.0906 \\ 0.0906 \end{bmatrix}
\end{aligned}$$

Therefore, the discrete-time approximation of the system, sampled at 10Hz, is

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.741 & 0.0779 \\ 0 & 0.819 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.0906 \\ 0.0906 \end{bmatrix} u(k)$$

6.6 Summary

In this chapter, we have investigated the analytical solution of state space equations. In doing so, we have demonstrated one of the important features of state variable analysis, namely, that the first-order analysis methods of scalar differential equations can help us solve (first-order) vector differential equations. This is first apparent in the integrating factor technique used to solve the LTI systems. Because any linear system can be represented in state space, these solution methods are presented to give the reader a set of tools for solving state space equations that are similar to those used for solving scalar differential equations.

However, as we have stated, the solution methods developed here are not necessarily the most useful tools for control system design and analysis. This is because, as we have shown, we must know the applied input to find the solution of a system. In control systems design, the input signal is the goal and is not generally available a priori. Nevertheless, the solution technique has provided insight into the evolution of state variables, and provided a technique by which discrete-time systems can be generated as approximations of sampled continuous-time systems.

Other highlights of the chapter are:

- The solution of an LTI system has the same form, i.e., a convolution integral, as the solution of scalar systems. Only the computation of the matrix exponential e^{At} complicates the matter.
- For homogeneous systems, phase portraits can be useful visualization tools. It is recommended that the reader practice generating phase portraits for systems with different dynamic characteristics. As we have mentioned, the evolution of a system's solution trajectories is often understood using this visual imagery, even in higher dimensions. In nonlinear problems as well, phase portraits are indispensable tools and can be used to predict the existence of limit cycles (nonlinear oscillations), switching times, final values, and stability characteristics that are very often difficult to determine analytically.
- We have introduced the notion of a system mode. This is another state space tool that is often generalized into higher dimensions. Heat conduction problems, bending plates and beams, and many electromagnetic phenomena are described by partial differential equations that result in infinite-dimensional state spaces. By describing the solution of such systems as sums of modes, we can retain only the most dominant and significant components of the overall solutions. Others, because they are less significant, might be simply ignored. The modal decomposition of a system is also a convenient method for generating a qualitative picture of a phase portrait.
- For time-varying systems, state variable solutions are quite difficult to obtain. The state-transition matrix, while symbolically providing a simple integral solution, can be as elusive as the solution of a nonlinear equation. Only limited techniques are available for its construction, such as the Peano-Baker series, which itself can be difficult to compute, especially in closed-form
- In discrete-time, we have shown that the state variable solution and the state transition matrix are similar in appearance to their continuous-time counterparts.

Part of the discussion of the solutions generated in this chapter has hinted at the stability properties of systems. For example, in the generation of the phase portraits, we mentioned the tendency of a system mode to approach or diverge from the origin. We sometimes find that such modes are unstable, for obvious reasons. However, such a notion of stability ignores the fact that phase portraits, by their definition, have not accounted for the presence of the input signal. Indeed, there are a number of different perspectives on the stability of a system, and these will be discussed in the next chapter.

6.7 Problems

6.1 Solve the state variable system given in Problem 1.9 for $x(t)$, $t \geq 0$, given that $u(t) = e^{-3t}$.

6.2 Draw phase portraits for the systems with the following A-matrices:

$$\text{a) } \begin{bmatrix} -8 & -6 \\ 0 & -2 \end{bmatrix} \quad \text{b) } \begin{bmatrix} -8 & -6 \\ 0 & 2 \end{bmatrix} \quad \text{c) } \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

$$\text{d) } \begin{bmatrix} 4 & -4 \\ 4 & 4 \end{bmatrix} \quad \text{e) } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{f) } \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\text{g) } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{h) } \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

6.3 Determine state-transition matrix $\Phi(t, \tau)$ for the following A-matrices:

$$\text{a) } \begin{bmatrix} 0 & 1 & t^2 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} \quad \text{b) } \begin{bmatrix} 0 & 0 \\ 2t & 2t \end{bmatrix} \quad \text{c) } \begin{bmatrix} e^{-2t} & 0 \\ 0 & 2t \end{bmatrix}$$

6.4 Show how systems of the form

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 1 & f(t) \end{bmatrix} x(t)$$

can be considered as decoupled scalar equations, the second of which has the solution of the first as an input term. Use this fact to determine an expression for $\Phi(t, \tau)$ for the second-order system.

6.5 Show that if $\Phi(t, t_0)$ is the state-transition matrix for A , then $A(t) = \dot{\Phi}(t, t_0) \Big|_{t=t_0}$. Derive an analogous result for the discrete-time state transition matrix, i.e., given $\Psi(k, j)$, determine $A(k)$.

- 6.6 Show for a time-invariant system, that $e^{At} = L^{-1}\{(sI - A)^{-1}\}$, where L is the LaPlace transform operator.
- 6.7 Suppose $\Phi_A(t, \tau)$ is the state-transition matrix for $A(t)$. Define a (nonsingular) change of variables by $x(t) = M\xi(t)$ such that $\dot{\xi}(t) = M^{-1}AM\xi(t) \triangleq \hat{A}\xi(t)$. Determine an expression for a new state-transition matrix $\Phi_{\hat{A}}(t, \tau)$ in terms of $\Phi_A(t, \tau)$. Does the result depend on whether M is time varying?
- 6.8 For every system $\dot{x}(t) = Ax(t)$ there is a defined system $\dot{p}(t) = -A^T p(t)$, which is called the adjoint system. Show that if $\Phi_A(t, \tau)$ is the state-transition matrix for the original system, then the state-transition matrix for the adjoint system is $\Phi_A^T(\tau, t)$.
- 6.9 Use Equation (6.7) to find an expression for the impulse response matrix, i.e., the solution to a LTI system whose input is the Dirac delta function $\delta(t)$.
- 6.10 Find the state-transition matrix $\Psi(k, 0)$ for the following system:

$$x(k+1) = \begin{bmatrix} 3.5 & -2 \\ 6 & -3.5 \end{bmatrix} x(k)$$

- 6.11 For the following system, find an expression for $y(n)$ if the input $u(k) = 1$, $k = 0, 1, \dots, \infty$

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \\ y(k) &= [1 \quad 0] x(k) \end{aligned}$$

6.12 Discretize the system below using a sample time of $T = 1$ s.

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = [0 \quad 1] x(t) + 2u(t)$$

6.8 References and Further Reading

Finding solutions to the time-invariant linear state space system is fairly easy and can be found in any linear systems or control systems text. For time-varying systems, which require the state-transition matrix, finding solutions is more difficult. Good discussions of such systems, including the computation of the state-transition matrix, may be found in [2], [5], and [7]. References [2] and [7] offer some particularly interesting problems. For the discrete-time case, see [4] and [7].

The stretched-string problem and many other modal system problems, such as deformable beams and plates, electromagnetic, and thermal gradient systems can be found in [6], which contains a large number of worked problems. Other infinite dimensional systems are discussed in [3].

More details on the construction and interpretation of phase portraits, particularly for nonlinear systems, can be found in [1].

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