7

System Stability

During the description of phase portrait construction in the preceding chapter, we made note of a trajectory's tendency to either approach the origin or diverge from it. As one might expect, this behavior is related to the *stability* properties of the system. However, phase portraits do not account for the effects of applied inputs; only for initial conditions. Furthermore, we have not clarified the definition of stability of systems such as the one in Equation (6.21) (Figure 6.12), which neither approaches nor departs from the origin. All of these distinctions in system behavior must be categorized with a study of stability theory. In this chapter, we will present definitions for different kinds of stability that are useful in different situations and for different purposes. We will then develop stability testing methods so that one can determine the characteristics of systems without having to actually generate explicit solutions for them.

7.1 Lyapunov Stability

The first "type" of stability we consider is the one most directly illustrated by the phase portraits we have been discussing. It can be thought of as a stability classification that depends solely on a system's initial conditions and not on its input. It is named for Russian scientist A. M. Lyapunov, who contributed much of the pioneering work in stability theory at the end of the nineteenth century. So-called "Lyapunov stability" actually describes the properties of a particular point in the state space, known as the *equilibrium point*, rather than referring to the properties of the system as a whole. It is necessary first to establish the terminology of such points.

7.1.1 Equilibrium Points

Many of the concepts introduced in this chapter will apply equally well to nonlinear systems as to linear systems. Even though much of the control system design we will perform will be applied to linear models of perhaps nonlinear systems, it is a useful skill to analyze the stability properties of nonlinear systems also. So consider the general form of an arbitrary system, perhaps nonlinear, and perhaps time-varying:

$$\dot{x}(t) = f(x(t), u(t), t) \tag{7.1}$$

Whereas we were interested in the behavior of the phase trajectories relative to only the origin in the previous chapter, for general nonlinear systems the point of interest is not necessarily the zero vector, but rather a general equilibrium point:

> ***Equilibrium point:*** An equilibrium point x_e is a constant vector such that if $x(t_0) = x_e$ and $u(t) \equiv 0$ in (7.1), then $x(t) = x_e$ for all $t \geq t_0$. (7.2)

Because an equilibrium point is a constant vector, we have

$$\dot{x}(t) = 0 = f(x_e, 0, t) \tag{7.3}$$

Equilibrium points are sometimes also referred to as *critical points* or *stationary points*. They represent a steady-state constant solution to the dynamic equation, (7.1). In discrete-time systems, an equilibrium point is a vector x_e such that $x(k+1) = x(k)$ $(= x_e)$, rather than $\dot{x}(t) = 0$.

Considering now the special case of a linear system, an equilibrium point(s) can be computed from

$$\dot{x}(t) = 0 = Ax_e \tag{7.4}$$

From this expression, it is apparent that the origin $x_e = 0$ will always be an equilibrium point of a linear system. Other equilibrium points will be any point that lies in the null space[M] of the system matrix A. Sets of equilibrium points will therefore constitute subspaces of the state space, and we can have no isolated equilibrium points outside the origin.

`null(A)`

Example 7.1: Equilibrium Points for Linear and Nonlinear Systems

Find the equilibrium points for the following systems:

a) $\dot{x} = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ b) $x(k+1) = \begin{bmatrix} 2 & 0 \\ 0 & .5 \end{bmatrix} x(k)$ c) $\dot{x} = -\sin x$

Solution:

a) The *b*-matrix is irrelevant because we will assume a zero input. The *A*-matrix has a zero eigenvalue so it has a nontrivial null space, which can be found to lie along the line $x_e = [-3 \quad 1]^T$. This is the equilibrium set.

b) The discrete-time equilibrium implies that $x(k+1) = x(k) = Ax(k)$, so we seek the solutions to the system $(A - I)x_e = 0$. For the *A*-matrix given, $A - I$ has no zero eigenvalues, so it has no nontrivial null space. The only equilibrium solution is therefore $x_e = [0 \quad 0]^T$.

c) The third system is nonlinear. Setting $\dot{x} = 0 = -\sin x_e$, we see that $x_e = \pm n\pi$, where $n = 0,1,\ldots$. This constitutes an infinite sequence of isolated equilibria, which we cannot encounter in a linear system.

We are now prepared for the stability definitions pertinent to equilibrium points. For the linear systems we treat here, we will speak of only isolated equilibria at the origin, but sets of equilibria such as lines through the origin may be interpreted similarly.

> ***Stability in the Sense of Lyapunov***: An equilibrium point x_e of the system $\dot{x} = A(t)x$ is stable in the sense of Lyapunov, or simply *stable*, if for any $\varepsilon > 0$, there exists a value $\delta(t_0, \varepsilon) > 0$ such that if $\|x(t_0) - x_e\| < \delta$, then $\|x(t) - x_e\| < \varepsilon$, regardless of t ($t > t_0$). The equilibrium point is *uniformly stable* in the sense of Lyapunov if $\delta = \delta(\varepsilon)$, i.e., if the constant δ does not depend on initial time t_0. (7.5)

As one might expect, time-invariant systems that are stable are uniformly stable because the initial time t_0 should not affect the qualitative behavior of the system. In words, the above definition means that for a stable in the sense of Lyapunov system, anytime we desire that the state vector remain within distance ε of the origin, it is possible to accomplish this by simply giving it an initial condition away from x_e by an amount no larger than δ. This scenario is depicted in Figure 7.1. Clearly, we must have $\delta < \varepsilon$ or else the system will have an initial condition that has already violated the desired bounds of the state vector.

Lyapunov stability is sometimes referred to as *internal stability* because its definition describes the behavior of the state variables rather than the output variables, which are by contrast considered *external*. We will see stability definitions related to system inputs and outputs in Section 7.3.

Note that the notion of Lyapunov stability does not require that the system in question approach the equilibrium (usually, the origin), only that it remain within the bounding radius ε. For a system that is Lyapunov stable, we further qualify the equilibrium point as *asymptotically stable* if the state vector tends toward zero, i.e., if $\|x(t)-x_e\| \to 0$ as $t \to \infty$. In addition, if we can find constants $\gamma > 0$ and $\lambda > 0$ such that for all $t > t_0$,

$$\|x(t)-x_e\| \le \gamma e^{-\lambda(t-t_0)} \|x(t_0)-x_e\|$$

then the equilibrium is said to be *exponentially stable*.

As one might suspect, an LTI system that is asymptotically stable is also exponentially stable because all solutions of LTI systems that approach the origin do so by exponential decay. This is not necessarily the case for time-varying systems and is certainly not always the case for nonlinear systems.

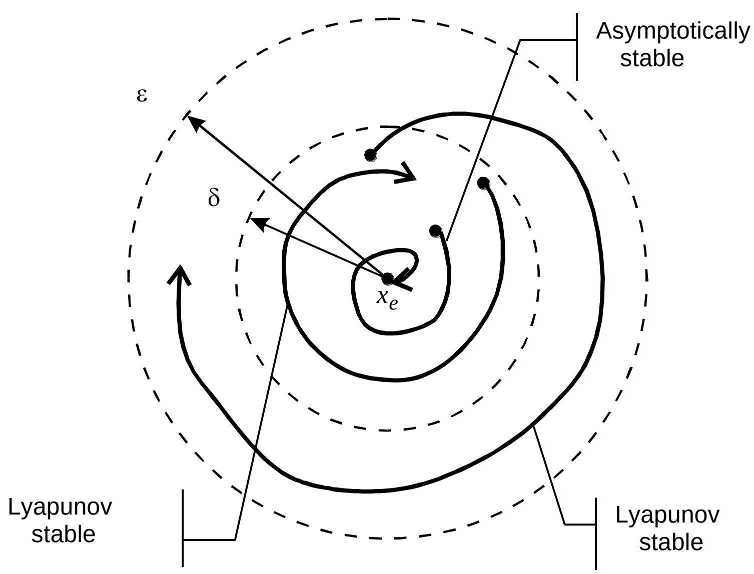

Figure 7.1 Representative trajectories illustrating Lyapunov stability. The picture shows a two-dimensional space such that the dotted circles indicate bounds on the 2-norm of the state vector.

We can also speak of systems that are stable *locally* or stable *globally* (also, stable *in the large*). An equilibrium that is stable globally is one such that the constant δ can be chosen arbitrarily large. This implies that if an equilibrium is asymptotically stable, then the trajectory from *any* initial condition will asymptotically approach it. This in turn implies that there can be only one equilibrium that is asymptotically stable in the large.

For linear systems, wherein equilibria are either isolated points or subspaces, all stability is global. There is no other equilibrium that a large initial condition might approach (or diverge from). However, for nonlinear systems, wherein multiple isolated equilibria might exist, one sometimes has to define regions of attraction around attractors, i.e., stable equilibria. These are sometimes called *basins of attraction* or *insets*, denoting the region in the state space from which an initial condition will approach the equilibrium point. Outside that region of attraction, the trajectory might go elsewhere. Again, that will not happen in linear systems.

As an illustration, the phase portrait of the nonlinear system

$$\ddot{\theta} + 0.4\dot{\theta} + \sin\theta = 0 \tag{7.6}$$

is shown in Figure 7.2.

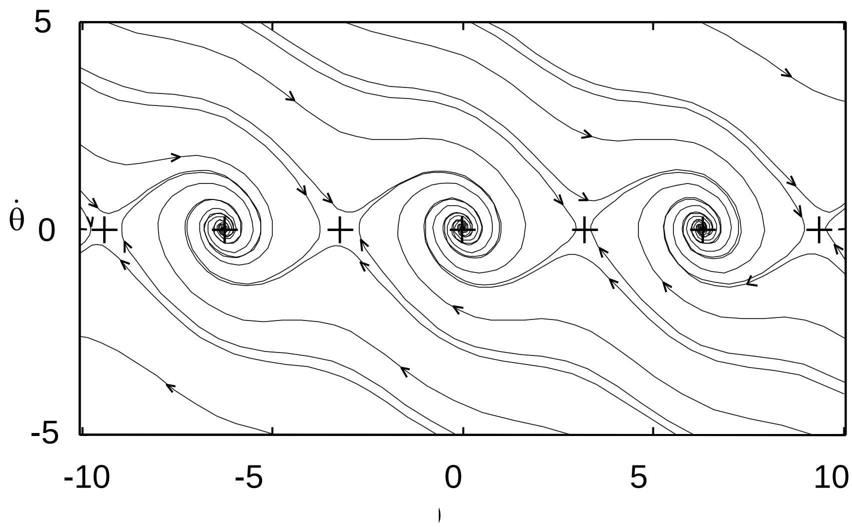

Figure 7.2 Phase portrait of the damped pendulum of Equation (7.6). The symbols (+) indicate the equilibria. (What would the phase portrait of an undamped pendulum look like?)

The graph plots trajectories on the $\dot{\theta}$ versus θ axis. The system (7.6) represents the equation of a damped pendulum, i.e., with viscous friction. If we compute the equilibria of this system by solving $\sin\theta_e = 0$, which is similar to Example 7.1c), we find equilibrium points at $x_e = \pm n\pi$, where $n = 0, 1, \ldots$ Of these, it is apparent from the figure that zero and the even multiples of $\prec$ are stable, whereas the odd multiples of $\prec$ are unstable. Note, in particular, that each stable equilibrium has a well-defined region of attraction, which defines the set from which an initial condition will produce a trajectory that approaches that equilibrium. For the equilibria that are stable, they are not *globally* stable. This being the equation for a pendulum, we can identify the stable equilibria as points at which the pendulum hangs directly down. The unstable equilibria are points at which the pendulum is balancing straight up, which are intuitively clearly unstable.

7.1.2 Classification of Equilibria

Given our definition of Lyapunov stability, we can reconsider the phase portraits of Chapter 6 in terms of these definitions. Because it is the eigenvalues of the *A*-matrix that determine the directions of the phase trajectories, it will be the eigenvalues that determine the type of stability exhibited.

In the captions to the phase portraits in Chapter 6, the origins of each plot were given labels, e.g., stable node, or focus. Each such label applies to any equilibrium of a phase portrait, and is a characteristic of the stability type of that equilibrium. Recalling the system examples from Chapter 6 that had two eigenvalues with negative real parts, i.e., the systems given by Equations (6.9), (6.12), (6.18), (6.19), and (6.20), we can immediately see from the corresponding figures that all trajectories asymptotically approach the origin (in fact, they do so exponentially). Such equilibria are clearly Lyapunov stable. Those that have at least one real invariant subspace (i.e., eigenvector), namely (6.9), (6.12), (6.18), and (6.19), have equilibria known as *stable nodes*. The equilibrium in the asymptotically stable system with complex conjugate eigenvalues (6.20) is known as a *stable focus*. The origin of the system described by (6.21) is stable but not asymptotically. It is known as a *center*. For systems with one negative and one positive eigenvalue, such as in (6.15) and (6.17), the origin is known as a *saddle point*.

Most of the phase portraits drawn (with the exception of the saddle points) show stable systems. Unstable systems, for example, those with *unstable nodes*, can easily be imagined by reversing the arrowheads on the portraits with stable nodes. Likewise, an unstable focus spirals out of the origin, the opposite of Figure 6.10. In the case of the center, there is no real part to the eigenvalues, so no such distinction needs to be made. However, with an appropriate choice of the *A*-matrix, the trajectories can be made to progress in the opposite direction from those shown (i.e., counterclockwise rather than clockwise).

7.1.3 Testing For Lyapunov Stability

While the definitions above give us the *meaning* of stability, they do little to help us *determine* the stability of a given system (except that it is now easy to anticipate the stability conditions for a time-invariant system based on the locations of its eigenvalues). In this section, we will give some results that will constitute stability tests, useful for examining a system at hand.

Considering a general time-varying zero-input system,

$$\dot{x}(t) = A(t)x(t) \tag{7.7}$$

we present the first result as a theorem:

THEOREM: The zero-input system in (7.7) is stable in the sense of Lyapunov if and only if there exists a constant $\kappa(t_0) < \infty$ such that for all $t \geq t_0$, the state-transition matrix $\Phi(t, t_0)$ satisfies the relation

$$\|\Phi(t, t_0)\| \leq \kappa(t_0) \tag{7.8}$$

If the constant κ does *not* depend on t_0, then the origin of the system described by (7.7) will be uniformly stable in the sense of Lyapunov.

PROOF: Recall that $x(t) = \Phi(t, t_0)x(t_0)$. Then by the Cauchy-Schwarz inequality (see Section 2.2.7), we have

$$\|x(t)\| \leq \|\Phi(t, t_0)\| \|x(t_0)\| \tag{7.9}$$

(Sufficiency) Suppose $\|\Phi(t, t_0)\| \leq \kappa(t_0)$. Given a value ε, we are free to choose the value $\delta(t_0, \varepsilon) = \varepsilon/\kappa(t_0)$. Then from (7.9),

$$\|x(t)\| \leq \|\Phi(t, t_0)\| \|x(t_0)\| \leq \kappa(t_0) \|x(t_0)\| < \kappa(t_0) \frac{\varepsilon}{\kappa(t_0)}$$

or

$$\|x(t)\| < \varepsilon \text{ for all } t \geq t_0$$

(Necessity) Assume that the system is stable and show that $\|\Phi(t, t_0)\|$ is bounded. Suppose that $\|\Phi(t, t_0)\|$ were *not* bounded. Then by definition, given $x(t_0)$ such that

$\|x(t_0)\| < \delta(t_0, \varepsilon)$ and an arbitrary value $\varepsilon < M < \infty$, there exists a time *t* such that

$$\|\Phi(t, t_0)x(t_0)\| > M$$

But since $x(t) = \Phi(t, t_0)x(t_0)$, this would imply

$$\|x(t)\| > M > \varepsilon$$

which violates the stability condition, a contradiction. Thus, necessity is proven.

For asymptotic stability, we give the following result:

THEOREM: The origin of the system (7.7) is uniformly asymptotically stable if and only if there exist two constants $\kappa_1 > 0$ and $\kappa_2 > 0$ such that

$$\|\Phi(t, t_0)\| \le \kappa_1 e^{-\kappa_2 (t - t_0)} \tag{7.10}$$

PROOF: (Sufficiency) If (7.10) holds and if $\|x(t_0)\| < \delta$, then

$$\|x(t)\| \le \|\Phi(t, t_0)\| \|x(t_0)\| < \kappa_1 e^{-\kappa_2 (t - t_0)} \delta$$

so clearly $\|x(t)\| \to 0$ as $t \to \infty$.

(Necessity) We will divide the time axis into equal steps of size τ. We will then look at the state of the system at a particular time *t*, considered to satisfy

$$t_0 + k\tau \le t < t_0 + (k+1)\tau \tag{7.11}$$

Knowing that the equilibrium is uniformly asymptotically stable (globally, because it is linear), then we know that for any starting point $\|x(t_0)\| \le \delta_1$, given an arbitrary value ε_1, there must exist a time τ such that $\|x(t_0 + \tau)\| < \varepsilon_1$. This being the case, we are free to choose $\|x(t_0)\| = 1$, and select $\varepsilon_1 = e^{-1}$. Then there will exist τ such that

$$\|x(t_0 + \tau)\| = \|\Phi(t_0 + \tau, t_0)x(t_0)\| < e^{-1}$$

By the definition of an operator norm,

$$\|\Phi(t_0 + \tau, t_0)\| = \sup_{\|x(t_0)\|=1} \|\Phi(t_0 + \tau, t_0)x(t_0)\| < e^{-1}$$

So now if we consider time $t_0 + k\tau$,

$$\begin{aligned}\|\Phi(t_0 + k\tau, t_0)\| &= \|\Phi(t_0 + k\tau, t_0 + (k-1)t_0)\\ &\quad \cdot \Phi(t_0 + (k-1)\tau, t_0 + (k-2)t_0)\cdots\\ &\quad \cdots\Phi(t_0 + 2\tau, t_0 + \tau)\Phi(t_0 + \tau, t_0)\|\\ &\le \|\Phi(t_0 + k\tau, t_0 + (k-1)t_0)\|\\ &\quad \cdot\|\Phi(t_0 + (k-1)\tau, t_0 + (k-2)t_0)\|\cdot\\ &\quad \cdots\|\Phi(t_0 + 2\tau, t_0 + \tau)\|\|\Phi(t_0 + \tau, t_0)\|\\ &< e^{-k}\end{aligned} \tag{7.12}$$

Now consider the quantity $\|\Phi(t, t_0 + k\tau)\|$. Because $\|\Phi(\cdot)\|$ is bounded, we know that $\|\Phi(t, t_0 + k\tau)\|$ is a finite number. Call it

$$\|\Phi(t, t_0 + k\tau)\| \overset{\Delta}{=} \kappa_1 e^{-1} \tag{7.13}$$

for some constant $\kappa_1 < \infty$. Now we have that

$$\|\Phi(t, t_0)\| = \|\Phi(t, t_0 + k\tau)\Phi(t_0 + k\tau, t_0)\|$$

so

$$\begin{aligned}\|\Phi(t, t_0)\| &\le \|\Phi(t, t_0 + k\tau)\| \cdot \|\Phi(t_0 + k\tau, t_0)\|\\ &< \kappa_1 e^{-1} e^{-k} = \kappa_1 e^{-(k+1)} = \kappa_1 e^{-(1/\tau)\ t_0 + (k+1)\tau - t_0}\end{aligned} \tag{7.14}$$

by (7.13) and (7.12). However, from (7.11) we have asserted that $t < t_0 + (k+1)\tau$, so

$$e^{-(t-t_0)} > e^{-\ t_0+(k+1)\tau-t_0}$$

and therefore,

$$\kappa_1 e^{-(1/\tau)\ t_0+(k+1)\tau-t_0} < \kappa_1 e^{-(1/\tau)(t-t_0)}$$

Now defining $\kappa_2 \overset{\Delta}{=} (1/\tau)$, we have proven necessity by showing that

$$\|\Phi(t,t_0)\| < \kappa_1 e^{-\kappa_2(t-t_0)}$$

As usual, we can do a little better than this for time-invariant systems. Given the proof above, the proofs of the following two theorems are relatively easy and are left as an exercise.

> THEOREM: The equilibrium of the LTI system $\dot{x} = Ax$ is Lyapunov stable if and only if all the eigenvalues of A have nonpositive real parts, and those that have zero real parts are nonrepeated. (7.15)

> THEOREM: The equilibrium of the LTI system $\dot{x} = Ax$ is asymptotically Lyapunov stable if and only if all the eigenvalues of A have negative real parts. (7.16)

It is these last two results that are the most commonly used to test for Lyapunov stability. However, as we will see in the next section, one must be careful not to apply the above results for time-invariant systems to time-varying systems.

Discrete-Time Systems

These results are directly extensible to discrete-time systems. In particular, all the boundedness conditions for the state-transition matrix $\Phi(t,t_0)$ extend as well to $\Psi(k,j)$. In the case of time-invariant systems, we can still derive eigenvalue conditions, but we replace all reference to the left half of the complex plane with the inside of the unit circle. Thus, the origin of a discrete-time LTI system is asymptotically stable if the eigenvalues of A_d are strictly inside the unit circle, and it is Lyapunov stable if the eigenvalues are inside or on the unit circle. Those eigenvalues that are on the unit circle, however, must not be repeated.

7.1.4 Eigenvalues of Time-Varying Systems

In general, one should avoid making conclusions about the stability of linear, time-varying systems based on the eigenvalues at any point in time. The following statements are true of time-varying systems:

- Time-varying systems whose "frozen-time" eigenvalues, i.e., eigenvalues computed at any single instant, all have negative real parts are not necessarily stable.
- If the eigenvalues of the matrix $A(t)+A^{\mathrm{T}}(t)$ have real parts that are always negative, the system is asymptotically stable (this is a sufficient but not necessary condition).
- If *all* the eigenvalues of $A(t)+A^{\mathrm{T}}(t)$ are *always* positive, the system is unstable [2].
- A time-varying system with one or more eigenvalues whose real part is positive is not necessarily unstable.
- If a time-varying system's eigenvalues have real parts such that $\mathrm{Re}(\lambda)<\gamma<0$ for all ℓ and all time *t*, and the system is sufficiently "slowly varying," then the system will be asymptotically stable. By slowly varying, we mean that there exists a value $\nu<\infty$ such that $\left\|\dot{A}(t)\right\|\le\nu$ that ensures this stability. Likewise, a slowly varying system with an eigenvalue in the right half plane that *never* crosses the imaginary axis can be shown to be unstable.

It is not easy to prove all these statements without the theory presented in the next section. We will demonstrate one of these results using an example.

Example 7.2: An Unstable System with Negative Eigenvalues

Determine the eigenvalues and closed-form solution to the system given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-\alpha & e^{2\alpha t}\\ 0 & -\alpha\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{7.17}$$

where $\alpha>0$.

Solution:

From its triangular form, we know that the eigenvalues are $\lambda_1=\lambda_2=-\alpha$. In order to solve the system, we can recognize that the variable $x_2(t)$ is decoupled from $x_1(t)$ and has the solution $x_2(t) = e^{-\alpha t} x_2(0)$. Then the first equation can be written as

$$\dot{x}_1(t) = -\alpha x_1(t) + e^{\alpha t} x_2(0)$$

This equation can be shown (how?) to have the solution

$$x_1(t) = \left[x_1(0) - \frac{1}{1+\alpha} x_2(0) \right] e^{-\alpha t} + \frac{1}{1+\alpha} x_2(0) e^{\alpha t}$$

which is clearly unbounded because of the second term. This system is therefore an example of an unstable time-varying system with negative eigenvalues.

7.2 Lyapunov's Direct Method

A different method for testing the stability of linear, zero-input systems is called Lyapunov's direct method. It is based on the concept of energy and dissipative systems. The physically intuitive explanation is that an isolated system will have a certain amount of a nonnegative abstract quantity that is similar to energy. If the system has no input and its energy is always decreasing, we think of it as being stable. If its energy increases, we think of it as being unstable. However, because not all state space systems are descriptions of physical phenomena normally endowed with real energy, we generalize the concept. This is the basis for Lyapunov's direct method.

First, we introduce the above physical model with a mechanical example. Consider the mass-spring-damper system pictured in Figure 7.3. The dynamic equation that describes this system, in the absence of an input force, is

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0 \tag{7.18}$$

The characteristic equation of such a system is therefore $ms^2 + bs + k = 0$. Knowledge of basic systems would indicate that if the three parameters *m*, *b*, and *k* are all of the same algebraic sign, then the roots of the characteristic polynomial will have negative real parts and the system will be stable in the sense of Lyapunov. Because the three parameters have physical interpretations of mass, damping, and restoring force constants, we know that all three will be positive. A state space version of (7.18) can be written by defining $x(t)$ and $\dot{x}(t)$ as the two state variables, giving

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \tag{7.19}$$

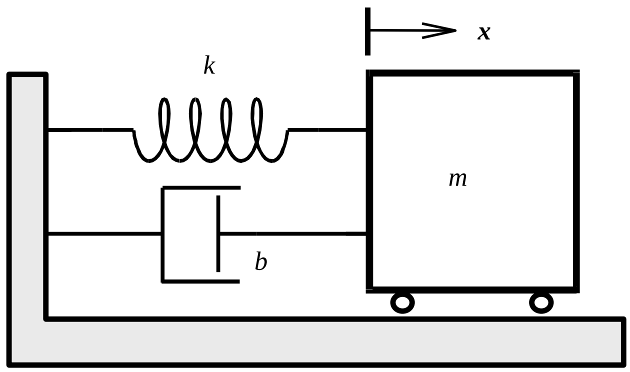

Figure 7.3 An unforced sliding system with a mass, spring, and damper. The position $x = 0$ is defined as the position of the mass when the string is unstretched and uncompressed.

Now consider the energy in the system, which we will call $V(x,\dot{x})$:

$$\begin{aligned} V(x,\dot{x}) &= V_{\text{kinetic}}(x,\dot{x}) + V_{\text{potential}}(x) \\ &= \frac{1}{2}\left(m\dot{x}^2 + kx^2\right) \end{aligned} \tag{7.20}$$

To determine how this energy changes over time, we compute

$$\dot{V}(x,\dot{x}) = m\dot{x}\ddot{x} + kx\dot{x}$$

Evaluating this energy change "along the trajectory," i.e., rearranging the original differential equation and substituting in for $\ddot{x}$,

$$\begin{aligned} \dot{V}(x,\dot{x}) &= \dot{x}\left(-b\dot{x} - kx\right) + kx\dot{x} \\ &= -b\dot{x}^2 \end{aligned} \tag{7.21}$$

If indeed $b > 0$, then the value of $\dot{V}(x,\dot{x})$ will be nonpositive, and will only be zero when $\dot{x} = 0$, i.e., when the system is stationary. If we should encounter a situation such as an initial condition where $\dot{x}_0 = 0$ and $x_0 \neq 0$, then we can check this condition against the invariant subspaces (eigenvectors) of the state space description (7.19). Because $\left[x(0) \quad \dot{x}(0)\right]^{\mathrm{T}} = \left[x_0 \quad 0\right]^{\mathrm{T}}$ is *not* an invariant subspace, the system will soon generate a nonzero velocity, and the energy will continue to decrease until $x(t) \to 0$ and $\dot{x}(t) \to 0$, i.e., move toward the equilibrium position.

Hence, the energy argument provides a valid means of testing this simple physical system for asymptotic stability. We can also argue that if, in this system, there were no damping, i.e., $b = 0$, then the energy is not decreasing but is instead constant. This situation would describe an oscillator whose energy is continually being exchanged between kinetic and potential forms (in the mass and spring, respectively). This would be a system that is Lyapunov stable. The electrical counterpart would be, of course, an RLC circuit in which the resistance is removed, leaving energy to bounce back and forth between the capacitor and the inductor.

7.2.1 Lyapunov Functions and Testing

In order to generalize this stability concept to more abstract, perhaps time-varying forms, we provide the following theorems. They are stated in terms of generic, possibly nonlinear systems, and they must apply within an open region surrounding the origin. We will examine the linear and LTI cases later:

> THEOREM: The origin of the linear time-varying system $\dot{x}(t) = f(x(t), t)$ is Lyapunov stable if there exists a time-varying function $V(x, t)$ such that the following conditions are satisfied:
>
> 1. $V(x,t) \geq \gamma_1(x) > 0$ for all $x \neq 0$ and for all $t \geq t_0$. $V(x,t) = 0$ only when $x = 0$. The function $\gamma_1(x)$ is any continuous, nondecreasing function of the state *x*, but not time *t* explicitly, such that $\gamma_1(0) = 0$.
>
> 2. $\dot{V}(x,t) \leq -\gamma_2(x) < 0$ for all $x \neq 0$ and for all $t \geq t_0$, where
>
> $$\dot{V}(x,t) = \frac{\partial V(x,t)}{\partial x} f(x,t) + \frac{\partial V(x,t)}{\partial t} \tag{7.22}$$
>
> The function $\gamma_2(x)$ is a continuous, nondecreasing function of the state *x*, but not time *t* explicitly, such that $\gamma_2(0) = 0$. (7.23)

> THEOREM: The origin of the system is uniformly Lyapunov stable if, in addition to the conditions above, $\gamma_3(x) \geq V(x,t)$ for all *x* and all $t \geq t_0$. The function $\gamma_3(x)$ is a continuous,

nondecreasing function of the state *x*, but not time *t* explicitly, such that $\gamma_3(0) = 0$. (7.24)

THEOREM: The origin of the system is globally Lyapunov stable if, in addition to conditions 1 and 2 above, $V(x,t)$ is unbounded as $\|x\| \to \infty$. (7.25)

If all three theorems hold, the system is globally, uniformly, Lyapunov stable.

THEOREM: If condition 2 is altered as follows, then stability in the above theorems is strengthened to *asymptotic* stability:

2a. $\dot{V}(x,t) < -\gamma_2(x) < 0$ for all $x \neq 0$ and for all $t \geq t_0$. (7.26)

In these theorems, condition 1 is equivalent to the function $V(x,t)$ being *positive definite*, condition 2 is equivalent to $\dot{V}(x,t)$ being *negative semidefinite*, and condition 2a is equivalent to $\dot{V}(x,t)$ being *negative definite*. For linear systems, these terms are used in the same sense as in Chapter 5, where we considered the positive- or negative-definiteness of quadratic forms only.

Although we present no proofs of the above theorems, we assert that they simplify for time-invariant systems. For time-invariant systems (where all stability is uniform), we have the considerably simpler statements given in the following theorem. For the purposes of this theorem, we say that any function $V(\xi)$ is positive (semi)definite if $V(\xi) > 0$ $[V(\xi) \geq 0]$ for all $\xi \neq 0$ and $V(0) = 0$. A function $V(\xi)$ is said to be negative (semi)definite if $-V(\xi)$ is positive (semi-)definite.

THEOREM: The origin of a time-invariant system $\dot{x}(t) = f(x(t))$ is Lyapunov stable if there exists a positive-definite function $V(x)$ such that its derivative

$$\dot{V}(x) = \frac{dV(x)}{dx}\frac{dx}{dt} = \frac{dV(x)}{dx} f(x)$$

is negative semidefinite. If $\dot{V}(x)$ is negative definite, the origin is asymptotically stable. (7.27)

In all these definitions and theorems, discrete-time systems can be easily accommodated. The only difference from continuous-time systems is that instead of speaking of the derivative $\dot{V}(x)$ or $\dot{V}(x,t)$, we speak instead of the first difference: $\Delta V(x,k) = V(x(k+1), k+1) - V(x(k), k)$.

If the functions $V(x,t)$, or $V(x)$, exist as defined above, they are called *Lyapunov functions*. As we shall see, although it can be shown that stable or asymptotically stable systems do indeed have Lyapunov functions, the theorems do nothing to help us find them. For nonlinear systems, in particular, finding such a function in order to prove stability can be a chore for which experience is the best assistance. The usual procedure is to guess at a Lyapunov function (called a *candidate* Lyapunov function) and test it to see if it satisfies the above conditions. However, for linear systems, Lyapunov functions generally take quadratic forms.

Example 7.3: Lyapunov Function for a Nonlinear System

Use the candidate Lyapunov function

$$V(x_1, x_2) = \frac{1}{2} x_1^2 + x_2^2$$

to test the stability of the system

$$\begin{aligned} \dot{x}_1 &= -x_1 - 2x_2^2 \\ \dot{x}_2 &= x_1 x_2 - x_2^3 \end{aligned} \tag{7.28}$$

Solution:

First we note that the system is time-invariant, so we need not go to the trouble of finding the functions γ_1, etc. Instead, we proceed to the simplified tests, the first of which is positive definiteness of $V(x)$. Clearly, $V(x) > 0$ and $V(0) = 0$, so V is positive definite. Now we compute

$$\begin{aligned} \dot{V}(x) &= x_1 \dot{x}_1 + 2x_2 \dot{x}_2 \\ &= x_1\left(-x_1 - 2x_2^2\right) + 2x_2\left(x_1 x_2 - x_2^3\right) \\ &= -x_1^2 - 2x_1 x_2^2 + 2x_1 x_2^2 - 2x_2^4 \\ &= -x_1^2 - 2x_2^4 \end{aligned}$$

The derivative $\dot{V}(x)$ is therefore negative definite, and we can claim that the system in (7.28) is asymptotically stable. Furthermore, because $V(x) \to \infty$ as $\|x\| \to \infty$, and the system is time-invariant, we can also say that the system is globally, uniformly asymptotically stable.

As the first step toward generating Lyapunov functions for linear systems, reconsider the linear system given in the mass-spring-damper example offered above. Recall that we chose as an "energy" function the true energy of the mechanical system

$$V(x,\dot{x}) = \frac{1}{2}\left(m\dot{x}^2 + kx^2\right)$$

This is a positive-definite function of the state of the system. Furthermore, we computed $\dot{V}(x,\dot{x}) = -b\dot{x}^2$, which, if $b > 0$, is strictly negative for all $\dot{x} \neq 0$, but may be zero for $x \neq 0$. We therefore do not make the claim that $\dot{V}$ is negative definite. Rather, we implicitly resorted in that section to LaSalle's theorem (see [12] and [13]), which is useful in many similar situations and which justifies our claim of asymptotic stability in this example.

> THEOREM (LaSalle's theorem): Within the region of the state space for which the derivative of the candidate Lyapunov function is such that $\dot{V}(x) \leq 0$, let *Z* denote the subset of the state space in which $\dot{V}(x) = 0$. Within *Z*, let *M* denote the largest *invariant* subset. Then every initial state approaches *M*, even if *V* is not positive-definite. (7.29)

As we apply this theorem to the mass-spring-damper problem, we note that $Z = \{(x,\dot{x}) \mid \dot{x} = 0\}$. The only invariant subspaces are the eigenvectors e_1 and e_2. These invariant sets intersect with *Z* only at the origin, so *M* is just the origin. Therefore, as we concluded, the origin is asymptotically stable.

7.2.2 Lyapunov Functions for LTI Systems

In the case of linear systems, it is sufficient to consider quadratic forms as Lyapunov functions. This is largely because for linear systems, stability implies global stability, and the parabolic shape of a quadratic function satisfies all the criteria of the above theorems. We will begin considering the special case of linear systems with the time-invariant case. LTI systems, when examined with Lyapunov's direct method as described here, result in some special tests and equations.

Consider the LTI system

$$\dot{x} = Ax \tag{7.30}$$

and the candidate Lyapunov function $V(x) = x^{\mathrm{T}} Px$, where matrix *P* is positive definite. Then testing the system (7.30), we compute

$$\begin{aligned}\dot{V}(x) &= \dot{x}^{\mathrm{T}} P x + x^{\mathrm{T}} P \dot{x} \\ &= (Ax)^{\mathrm{T}} P x + x^{\mathrm{T}} P (Ax) \\ &= x^{\mathrm{T}} A^{\mathrm{T}} P x + x^{\mathrm{T}} P A x \\ &= x^{\mathrm{T}} \left(A^{\mathrm{T}} P + P A \right) x\end{aligned} \tag{7.31}$$

To satisfy the theorem, then, it is necessary that the matrix $A^{\mathrm{T}} P + PA$ be negative definite for asymptotic stability, and negative semidefinite for Lyapunov stability. That is, for some positive (semi)definite matrix Q, it is sufficient to show that

$$A^{\mathrm{T}} P + PA = -Q \tag{7.32}$$

`lyap(A,C)`

to demonstrate stability of the homogeneous system in (7.30). Equation (7.32) is known as the Lyapunov equation.[M]

However, one must be cautious in the application of this test because if the quantity $A^{\mathrm{T}} P + PA$ is *not* negative (semi)definite, nothing should be inferred about the stability. Recall that Lyapunov's direct method asserts that if a Lyapunov function is found, then the system is stable. It does *not* say that if a candidate Lyapunov function such as $x^{\mathrm{T}} P x$ fails, the system is not stable. So if we choose a positive-definite matrix P and we compute a Q that is *in*definite, we have shown nothing. (If Q is negative definite, however, instability can be shown.)

Instead, we resort to the reverse process: We select a positive (semi) definite Q and compute the solution P to the Lyapunov equation (7.32). This result is stated as a theorem:

> THEOREM: The origin of the system in (7.30) described by matrix A is asymptotically stable if and only if, given a positive definite matrix Q, the matrix Lyapunov equation (7.32) has a solution P that is positive definite. (7.33)
>
> PROOF: (Sufficiency) If P and Q are both positive definite, then it has already been shown by Lyapunov's direct method that the system will be asymptotically stable, because $x^{\mathrm{T}} P x$ will have been demonstrated to be a Lyapunov function.
>
> (Necessity) If the origin is known to be a stable equilibrium and Q is positive definite, we will prove necessity by first proving that P is the unique, positive definite result of

$$P = \int_0^\infty e^{A^{\mathrm{T}}t} Q e^{At} \, dt \tag{7.34}$$

To demonstrate this, we substitute (7.34) into the left-hand side of (7.32):

$$\begin{aligned}
A^{\mathrm{T}}P + PA &= \int_0^\infty A^{\mathrm{T}} e^{A^{\mathrm{T}}t} Q e^{At} \, dt + \int_0^\infty e^{A^{\mathrm{T}}t} Q e^{At} A \, dt \\
&= \int_0^\infty \left[A^{\mathrm{T}} e^{A^{\mathrm{T}}t} Q e^{At} + e^{A^{\mathrm{T}}t} Q e^{At} A \right] dt \\
&= \int_0^\infty \frac{d\left[e^{A^{\mathrm{T}}t} Q e^{At} \right]}{dt} \, dt \\
&= e^{A^{\mathrm{T}}t} Q e^{At} \Big|_0^\infty \\
&= 0 - Q \\
&= -Q
\end{aligned} \tag{7.35}$$

where convergence of the integral is guaranteed by the fact that the eigenvalues of A are in the left half of the complex plane.

To show that this solution is unique, suppose that there were a second solution $\hat{P}$ such that $A^{\mathrm{T}}\hat{P} + \hat{P}A = -Q$. Then we would have

$$(A^{\mathrm{T}}P + PA) - (A^{\mathrm{T}}\hat{P} + \hat{P}A) = 0$$

implying

$$A^{\mathrm{T}}\left(P - \hat{P}\right) + \left(P - \hat{P}\right)A = 0$$

Multiplying this equation by $e^{A^{\mathrm{T}}t}$ from the left and by e^{At} from the right,

$$
\begin{aligned}
e^{A^{\mathrm{T}}t}A^{\mathrm{T}}\left(P-\hat{P}\right)e^{At}+e^{A^{\mathrm{T}}t}\left(P-\hat{P}\right)Ae^{At} &= \frac{d}{dt}\left[e^{A^{\mathrm{T}}t}\left(P-\hat{P}\right)e^{At}\right] \\
&= 0
\end{aligned}
$$

Now integrating this equation over $[0, \infty)$,

$$
\begin{aligned}
0 &= \int_0^\infty d\left[e^{A^{\mathrm{T}}t}\left(P-\hat{P}\right)e^{At}\right]dt = e^{A^{\mathrm{T}}t}\left(P-\hat{P}\right)e^{At}\Big|_0^\infty \\
&= 0-\left(P-\hat{P}\right) \\
&= \hat{P}-P
\end{aligned}
$$

so we must have $P=\hat{P}$.

Finally, to demonstrate that *P* is positive definite, consider the quadratic form in terms of an arbitrary vector ξ:

$$
\begin{aligned}
\xi^{\mathrm{T}}P\xi &= \xi^{\mathrm{T}}\left[\int_0^\infty e^{A^{\mathrm{T}}t}Qe^{At}\,dt\right]\xi \\
&= \int_0^\infty \xi^{\mathrm{T}}e^{A^{\mathrm{T}}t}Qe^{At}\xi\,dt
\end{aligned}
\tag{7.36}
$$

Because matrix *Q* is positive definite, the integrand of (7.36) is positive definite, so

$$
\int_0^\infty \xi^{\mathrm{T}}e^{A^{\mathrm{T}}t}Qe^{At}\xi\,dt > 0
$$

for all $\xi \neq 0$. Therefore, *P* will be positive definite and hence, will define a valid Lyapunov function. This therefore constitutes a constructive proof of the theorem.

We note that this theorem does not directly extend to Lyapunov stability; that is, choice of an arbitrary positive semidefinite *Q* will not necessarily generate a positive definite *P* for a system that is Lyapunov stable. However, by an extension of LaSalle's theorem above, we can provide the following generalization:

THEOREM: The origin of the system (7.30) described by matrix A is asymptotically stable if and only if, given a positive semidefinite matrix Q, such that $x^{\mathrm{T}}Qx$ is not identically zero when evaluated on a nonzero trajectory of (7.30), the matrix Lyapunov equation (7.32) has a solution P that is positive definite. (7.37)

The proof of this theorem, while not given explicitly here, results from the application of LaSalle's theorem.

Example 7.4: Stability Test for a Parameterized System

An LTI system is given by

$$\dot{x} = \begin{bmatrix} -3k & 3k \\ 2k & -5k \end{bmatrix} x \tag{7.38}$$

Use the Lyapunov stability theorem to find a bound on k such that the origin of this system is asymptotically stable.

Solution:

While the Lyapunov stability theorem is not commonly used for stability testing (because it is much easier to simply test the eigenvalues), it can be used to determine stability bounds on matrix parameters such as this one. Because the theorem says that *any* positive definite matrix Q will suffice, it is sufficient to choose $Q = I$. Then because we know that the required solution P will have to be positive definite and, hence, symmetric, we construct the equation

$$\begin{bmatrix} -3k & 2k \\ 3k & -5k \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -3k & 3k \\ 2k & -5k \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiplying out the left-hand side, we arrive at three independent scalar equations (the two off-diagonal scalar equations will be identical):

$$\begin{aligned} -3kp_{11} + 2kp_{12} - 3kp_{11} + 2kp_{12} &= -6kp_{11} + 4kp_{12} = -1 \\ -3kp_{12} + 2kp_{22} + 3kp_{11} - 5kp_{12} &= 3kp_{11} - 8kp_{12} + 2kp_{22} = 0 \\ 3kp_{12} - 5kp_{22} + 3kp_{12} - 5kp_{22} &= 6kp_{12} - 10kp_{22} = -1 \end{aligned}$$

As these constitute three linear equations in three unknowns, they can be readily solved as

$$p_{11} = \frac{19}{72k} \qquad p_{12} = \frac{7}{48k} \qquad p_{22} = \frac{15}{80k}$$

To verify the positive definiteness of this solution, we form the determinant

$$|P| = \begin{vmatrix} \frac{19}{72k} & \frac{7}{48k} \\ \frac{7}{48k} & \frac{15}{80k} \end{vmatrix} = \frac{1}{(4k)^2} \begin{bmatrix} \frac{19}{18} & \frac{7}{12} \\ \frac{7}{12} & \frac{15}{20} \end{bmatrix}$$

To ensure that both leading principal minors are positive, we can choose $k > 0$. Therefore, we may conclude that the origin of the system is asymptotically stable for all $k > 0$.

Example 7.5: Discrete-Time LTI Lyapunov Equation

Derive the Lyapunov equation for a discrete-time homogeneous LTI system:

$$x(k+1) = Ax(k) \tag{7.39}$$

Solution:

Assuming that we can again use a quadratic form as the candidate Lyapunov function:

$$V[x(k)] = x^{\mathrm{T}}(k)Px(k) \tag{7.40}$$

we can find the first difference as

$$\begin{aligned} \Delta V(x,k) &= V[x(k+1)] - V[x(k)] \\ &= x^{\mathrm{T}}(k+1)Px(k+1) - x^{\mathrm{T}}(k)Px(k) \\ &= x^{\mathrm{T}}(k)A^{\mathrm{T}}PAx(k) - x^{\mathrm{T}}(k)Px(k) \\ &= x^{\mathrm{T}}(k)\left(A^{\mathrm{T}}PA - P\right)x(k) \end{aligned}$$

So the term in parentheses must be a negative definite matrix if the quadratic form in (7.40) is to be a valid Lyapunov function. The discrete-time Lyapunov equation[M] is therefore

`dlyap(A,Q)`

$$A^{\mathrm{T}}PA - P = -Q \tag{7.41}$$

7.2.3 Unstable Systems

It is a common misconception that the theory of internal stability, and in particular Lyapunov stability, is limited in its ability to conclude that a system is unstable, i.e., not stable in the sense of Lyapunov. However, there are some results that one may use to label a system (actually, its equilibium) as unstable:

> THEOREM: The origin of the linear system in (7.7) is *unstable* if there exists a function $V(x,t)$ and a continuous nondecreasing function of the state only, $\gamma(x)$, such that $\gamma(x) > V(x,t)$ for all x and $t \geq t_0$, $\gamma(0) = 0$, and the following three conditions are satisfied:
>
> 1. $V(0,t) = 0$ for all $t \geq t_0$.
>
> 2. $V(x,t_0) > 0$ at any point arbitrarily close to the origin.
>
> 3. $\dot{V}(x,t)$ is positive definite in an arbitrary region around the origin. (7.42)

Note that the function $\gamma(x)$ is the same one that appears in the original theorems on Lyapunov stability, e.g., conditions 1 or 2. Its existence is unnecessary for time-invariant systems. We can illustrate the process on a trivial example.

Example 7.6: An Unstable System

Determine a function $V(x,t)$ that, according to the theorem above, demonstrates the instability of the following system:

$$\dot{x} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} x$$

Solution:

It is clear from the self-evident eigenvalues that the system will not be Lyapunov stable. To illustrate the existence of $V(x,t)$ consider

$$V(x,t) = x_1^2 - x_2^2 = x^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x \stackrel{\Delta}{=} x^{\mathrm{T}} P x \qquad (7.43)$$

One should be careful not to refer to such a function as a Lyapunov function, because it is not; it is clearly not positive definite. However, it is zero at the origin, and is positive arbitrarily near the origin [consider approaching the origin from the $(x_1, 0)$ direction]. Now compute

$$\begin{aligned}\dot{V}(x,t) &= 2x_1\dot{x}_1 - 2x_2\dot{x}_2 \\ &= 2x_1(2x_1) - 2x_2(-x_2) \\ &= 4x_1^2 + 2x_2^2\end{aligned}$$

which is clearly positive definite. The system has therefore been shown to be unstable.

7.3 External Stability

In each of the stability definitions, theorems, and examples above, linear systems were considered in the absence of any input. However, any student of systems and controls knows that the input generally has an effect on the qualitative behavior of the output. From this perspective on the system, i.e., the input/output or *external* behavior, we will need new definitions and tests for system stability. Such tests may disregard the behavior of the internal dynamics (i.e., the state vector) altogether.

7.3.1 Bounded Input, Bounded Output Stability

Our working definition for external stability will use the concept of bounded input, bounded output (BIBO) stability. We will consider the boundedness of the input function $u(t)$ and the output function $y(t)$. For a given function of time $f(t)$, we consider the function to be bounded if for all time t, there exists a finite constant M such that $\|f(t)\| \le M$. We only consider the stability of systems of the general form

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t)\end{aligned} \tag{7.44}$$

We further restrict our attention to systems wherein $A(t)$ and $B(t)$ are themselves bounded operators (otherwise instability might be obvious).

> ***BIBO Stability:*** The system in (7.44) is BIBO stable if for *any* bounded input $u(t)$, $\|u(t)\| \le M$, and for any initial condition $x_0 = x(t_0)$, there exists a finite constant $N(M, x_0, t_0)$ such that $\|y(t)\| \le N$ for all $t \ge t_0$. (7.45)

One must be careful when applying this definition. If *any* bounded input results in an unbounded output, then the system is considered not BIBO stable. Some texts will provide the definition of so-called "bounded input, bounded state" (BIBS) stability, which we give here for completeness, but which we will not discuss in much depth:

> ***BIBS Stability:*** The system in (7.44) is BIBS stable if for *any* bounded input $u(t)$, $\|u(t)\| \le M$, and for any initial condition $x_0 = x(t_0)$, there exists a finite constant $N_s(M, x_0, t_0)$ such that $\|x(t)\| \le N_s$ for all $t \ge t_0$. (7.46)

Testing for BIBO Stability

To investigate the conditions under which a system is BIBO stable, we first reconsider the expression for system output:

$$y(t) = C(t)\Phi(t,t_0)x(t_0) + \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t) \tag{7.47}$$

where, of course, $\Phi(t,\tau)$ is the state-transition matrix discussed in Chapter 6. In order to move the feedthrough term into the integrand, we can exploit the Dirac delta function to achieve

$$y(t) = C(t)\Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \left[C(t)\Phi(t,\tau)B(\tau) + \delta(t-\tau)D(\tau) \right] u(\tau)\, d\tau \tag{7.48}$$

This allows us to refer to the bracketed term

$$H(t,\tau) \overset{\Delta}{=} C(t)\Phi(t,\tau)B(\tau) + \delta(t-\tau)D(\tau) \tag{7.49}$$

as the *impulse-response matrix*. Now considering the initial condition $x(t_0)$ to be the result of an infinite history of past inputs, we use the expression

$$y(t) = \int_{-\infty}^{t} H(t,\tau)u(\tau)\, d\tau$$

in the following theorem:

THEOREM: The linear system in (7.44) is BIBO stable if and only if there exists a finite constant N_H such that

$$\int_{-\infty}^{t} \| H(t,\tau) \| \, d\tau \le N_H \tag{7.50}$$

PROOF: We will prove this theorem by initially considering the scalar case wherein we have a scalar impulse-response matrix denoted $h(t,\tau)$ and $\| h(t,\tau) \| = | h(t,\tau) |$.

(Sufficiency) Suppose input $u(t)$ is bounded by finite scalar *M*. Then we have

$$| y(t) | \le \int_{-\infty}^{t} | h(t,\tau) | | u(\tau) | \, d\tau \le M \int_{-\infty}^{t} | h(t,\tau) | \, d\tau$$

Then if scalar N_H exists such that

$$\int_{-\infty}^{t} | h(t,\tau) | \, d\tau \le N_H \tag{7.51}$$

then clearly

$$| y(t) | \le M N_H < \infty$$

and BIBO stability is demonstrated.

(Necessity) If the system is BIBO stable, then for any bounded input $u(t)$, there will exist a finite constant *N* such that

$$| y(t) | \le N \tag{7.52}$$

To construct an argument by contradiction, suppose that (7.50) were *not* true, i.e., for *any* finite scalar N_H

$$\int_{-\infty}^{t} |h(t,\tau)|\, d\tau > N_H \tag{7.53}$$

Because we have asserted BIBO stability, we can choose our input function, so long as it is bounded. We select

$$u(t) = \operatorname{sgn}\left[h(t,\tau) \right]$$

Then we have

$$|y(t)| = \left| \int_{-\infty}^{t} h(t,\tau) u(\tau)\, d\tau \right| = \int_{-\infty}^{t} |h(t,\tau)|\, d\tau \le N \tag{7.54}$$

from (7.52). Yet this result contradicts (7.53), so necessity is demonstrated.

The extension of this proof for sufficiency to the vector and multivariable case is straightforward. The necessity part is based on the argument that *if a matrix operator is unbounded, then at least one of its individual entries must also be unbounded*. This is used to justify a term-by-term application of the proof above to the impulse-response matrix [15].

When using a result such as this, we should remember that the matrix norm may be computed in a number of ways, as discussed in Section 3.2.1. One of the most common techniques is to use the singular value decomposition, recalling that $\|M\| = \sigma_1$, i.e., the largest eigenvalue of $M^{\mathrm{T}} M$.

BIBO Stability for Discrete-Time Systems

As one might expect from the development of the BIBO stability definition, test, and proof above, the test for discrete-time systems is analogous to that for continuous-time systems. In order to state this result, we must determine an expression for the impulse-response matrix that is similar to (7.49) for continuous-time. Recalling Equation (6.49), we can construct the output of the generic (i.e., possibly time-varying) discrete-time system as

$$y(k) = C_d(k)\left(\prod_{i=j}^{k-1} A_d(i)\right)x(j) + \sum_{i=j+1}^{k} C_d(k)\left[\prod_{q=i}^{k-1} A_d(q)\right]B(i-1)u(i-1) + D_d(k)u(k)$$

or, considering the initial condition $x(j)$ to have arisen from an infinite past history of inputs,

$$y(k) = \sum_{i=-\infty}^{k} C_d(k)\left[\prod_{q=i}^{k-1} A_d(q)\right]B(i-1)u(i-1) + D_d(k)u(k)$$

Performing the same trick of moving the feedthrough term into the summation by introducing the discrete-time Dirac delta function,

$$y(k) = \sum_{i=-\infty}^{k} \left\{ C_d(k)\left[\prod_{q=i}^{k-1} A_d(q)\right]B(i-1) + \delta(i-k-1)D_d(i-1) \right\} u(i-1) \tag{7.55}$$

Now the term in braces in (7.55) can be identified as the discrete-time impulse-response matrix, which we will denote $H_d(k,i)$.

$$H_d(k,i) = C_d(k)\left[\prod_{q=i}^{k-1} A_d(q)\right]B(i-1) + \delta(i-k-1)D_d(i-1) \tag{7.56}$$

With this notation, we can state the result as follows:

THEOREM: The discrete-time linear state space system

$$\begin{aligned} x(k+1) &= A_d(k)x(k) + B_d(k)u(k) \\ y(k) &= C_d(k)x(k) + D_d(k)u(k) \end{aligned}$$

with impulse-response matrix $H_d(k,i)$ as given in (7.56) is BIBO stable if and only if there exists a finite constant N_d such that

$$\sum_{i=-\infty}^{k} \left\| H_d(k,i) \right\| \le N_d \tag{7.57}$$

for all *k*.

7.3.2 BIBO Stability for Time-Invariant Systems

Consider a state space system described by (7.44) that is time-invariant, i.e., one for which the system matrices are not explicit functions of time. Then, recalling the definition of the impulse-response matrix (7.49), we have

$$H(t,\tau) \stackrel{\Delta}{=} Ce^{A(t-\tau)}B + \delta(t-\tau)D \tag{7.58}$$

Then, by the theorem giving the condition for BIBO stability, we see that the LTI system will be BIBO stable if and only if there exists a finite constant N_H such that

$$\int_{-\infty}^{t} \left\| H(t,\tau) \right\| d\tau = \int_{-\infty}^{t} \left\| Ce^{A(t-\tau)}B + \delta(t-\tau)D \right\| d\tau \le N_H$$

The first remark we must make about such a system is that for testing purposes, the *D*-matrix is often not included in the integrand. Its contribution to the integral is clearly additive, and its integral

$$\int_{-\infty}^{t} \delta(t-\tau)D\, d\tau = D$$

will clearly be unbounded if and only if *D* itself is infinite. Such infinite feedthrough–gain systems rarely occur, and if they do, the reason for the overall system instability would be obvious (recall that we are considering only bounded matrices anyway).

It remains therefore to test

$$\int_{-\infty}^{t} \left\| Ce^{A(t-\tau)}B \right\| d\tau = C\left(\int_{-\infty}^{t} \left\| e^{A(t-\tau)} \right\| d\tau \right) B \tag{7.59}$$

for boundedness. Realizing that $e^{A(t-\tau)} = \Phi(t,\tau)$ for an LTI system, the middle factor in (7.59) is reminiscent of the test for Lyapunov stability given in (7.8). However, Lyapunov stability and BIBO stability of LTI systems are *not* equivalent. The reason for this is that one cannot neglect the effects of premultiplication by C and postmultiplication by B in (7.59). Suppose, for example, that

$$\left(\int_{-\infty}^{t} \left\| e^{A(t-\tau)} \right\| d\tau \right) B \in \mathrm{N}(C) \tag{7.60}$$

where $\mathrm{N}(C)$ is the null space of C. Then no matter what the size of the integral, Equation (7.60) would imply the boundedness of (7.59). Therefore, it would be possible for a system that is not asymptotically (or even Lyapunov) stable to indeed be BIBO stable. This applies to time-varying systems as well. However, the reverse is true, as it is relatively easy to show that asymptotic stability implies BIBO stability, i.e., boundedness of the state does imply boundedness of the output (assuming bounded system matrices).

This leads to a relatively simple conclusion about the concept of BIBS stability. One can readily demonstrate that a linear system is BIBS stable if, for all bounded inputs, there exists a finite constant N_s such that

$$\int_{t_0}^{t} \left\| \Phi(t,\tau) B(\tau) \right\| d\tau \le N_s \tag{7.61}$$

for all t.

A better way to approach BIBO stability is in the frequency-domain. One can show that for an LTI system,

$$e^{At} = L^{-1}\left\{ (sI - A)^{-1} \right\}$$

(see Exercise 6.6). Therefore, demonstration of the boundedness of (7.59) can be achieved by considering the frequency-domain counterpart,

$$CL^{-1}\left\{ (sI - A)^{-1} \right\} B \tag{7.62}$$

Boundedness of (7.62) is therefore equivalent to the condition of having the poles of the expression in (7.62) in the closed left-half plane.* However, providing for the existence of a bounded input, which might have characteristic frequencies on the imaginary axis, we must restrict the poles of (7.62) to the *open* left-half plane.

---

* We are assuming here that the reader is familiar with the relationship between the characteristic frequencies of a signal and the qualitative behavior of its time-domain representation.

Otherwise, the characteristic frequencies of the output $y(t)$ might be repeated on the imaginary axis and therefore produce an unbounded result. By taking the LaPlace transform of the LTI system equations, it is relatively easy to demonstrate that the transfer function of a system (i.e., the LaPlace transform of its impulse-response) is given by

$$H(s) = C(sI - A)^{-1}B + D \tag{7.63}$$

(see Section 1.1.5). Therefore, if the matrix D is bounded, our condition for BIBO stability can be stated as follows: *An LTI continuous-time system is BIBO stable if and only if the poles of its (proper) transfer function are located in the open left-half complex plane*. In the case of an LTI discrete-time system, this statement should be adjusted so that the poles must lie inside the open unit circle. Of course, by speaking of the transfer function (singular), we are implying a SISO system. However, the above statements apply if we add the further condition that a MIMO system is BIBO stable if and only if each individual input-to-output subsystem transfer function is BIBO stable.

Readers familiar with the Routh-Hurwitz method for testing stability of a system may now consider this test in the context of the stability properties discussed in this chapter [4]. Conventional Routh-Hurwitz testing determines BIBO stability. Extended Routh-Hurwitz testing, i.e., with additional analysis of the "auxiliary equation" in the case that a row of the Routh-Hurwitz array is entirely zero, has the further ability to determine so-called "marginal stability." Marginal stability corresponds to stability that results when the output is guaranteed to be bounded whenever the input asymptotically approaches zero, i.e., when the input has no characteristic frequencies with zero real parts.

7.4 Relationship Between Stability Types

Given the discussions of the various kinds of stability in this chapter, the relationship between different types of stability can be summarized by the Venn diagram in Figure 7.4. Although it is easy to envision systems that are Lyapunov stable but not BIBO stable, e.g., systems with poles on the imaginary axis, it is sometimes not obvious how a system can be BIBO stable, but not asymptotically stable, or even Lyapunov stable. Such a system is given in the following example.

Example 7.7: A BIBO, but Not Asymptotically Stable System

Consider the system given by the differential equation

$$\ddot{y} + \dot{y} - 2y = \dot{u} - u \tag{7.64}$$

Show that this system is BIBO but not asymptotically stable.

Solution:

First we will put the system into state variable form. From the simulation diagram given in Figure 1.7, we can derive a state variable model as follows:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} -1 & 1 \end{bmatrix} x \tag{7.65}$$

Finding the eigenvalues of this system, we determine that $\sigma(A) = \{1, -2\}$. From these eigenvalues, we see that this system is neither asymptotically nor Lyapunov stable. We will now check to see whether it is BIBO stable. It is an easy exercise to check that

$$e^{At} = \begin{bmatrix} \frac{2}{3}e^{t} + \frac{1}{3}e^{-2t} & \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t} \\ \frac{2}{3}e^{t} - \frac{2}{3}e^{-2t} & \frac{1}{3}e^{t} + \frac{2}{3}e^{-2t} \end{bmatrix}$$

and that

$$Ce^{At}B = e^{-2t}$$

Therefore,

$$\int_{-\infty}^{t} \left\| Ce^{A(t-\tau)}B \right\| d\tau = \int_{-\infty}^{t} \left\| e^{-2(t-\tau)} \right\| d\tau = \frac{1}{2}$$

which is obviously bounded. The system is therefore BIBO stable.

As a further exercise, consider finding the transfer function from the original differential equation in (7.64). If the initial conditions are zero, we have

$$(s^2 + s - 2)Y(s) = (s-1)U(s)$$

giving

$$\frac{Y(s)}{U(s)} = \frac{(s-1)}{(s^2 + s - 2)} = \frac{(s-1)}{(s+2)(s-1)} = \frac{1}{(s+2)}$$

The transfer function shows poles in the open left-half of the complex plane, so again we have demonstrated that the system is BIBO stable.

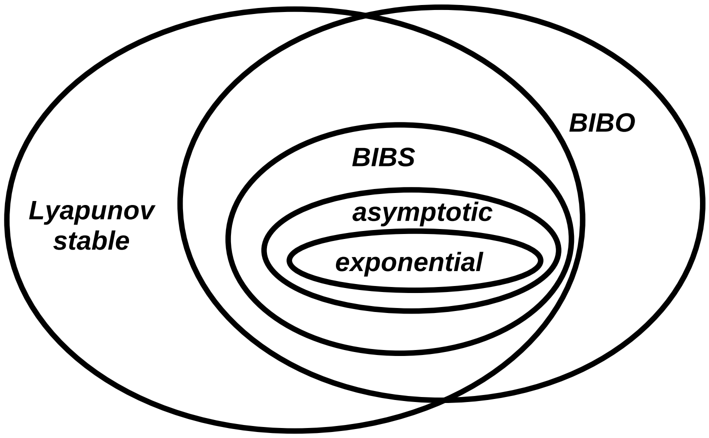

Figure 7.4 Venn diagram showing the relationship between stability types.

7.5 Summary

Before reading this chapter, one may already have formed an intuitive concept of system stability: A system is generally regarded as stable if its response remains finite. Although this interpretation often suffices in the sense that unstable systems are easy to recognize when they are encountered (we know one when we see one), the mathematically rigorous definitions for stability are more complex. Particularly, in the case of nonlinear or time-varying systems, we can encounter systems that are stable in the absence of inputs but unstable with them (Lyapunov but not BIBO stable); systems that are unstable when the internal dynamics (i.e., the states) are examined, but stable if only the output is of concern (BIBO but not Lyapunov stable); or systems that are stable for some initial conditions and/or initial times, but not from other initial states.

These distinctions are important for the study of newly modeled systems, for it is now clear that observation of one characteristic of a system or of one example of its behavior is not sufficient to characterize the stability of the system in general. A linear system that is designed and placed into a practical application should be assured of stability under all of its expected operating conditions. External stability, while usually desirable, is often insufficient for practical purposes. Usually, if internal states exceed their normal levels, the linear model upon which the analysis is performed will begin to fail, and all predictions of system performance will become invalid. Rarely, when internal states are conceptual abstractions, the fact that a system is internally not stable may not be important so long as the desired external behavior is stable.

The job of the engineer in system design is often predicated upon an analysis of the system's stability requirements. While undergraduate texts are fond of providing the many classical methods for achieving various performance specifications, real-life applications can be much more difficult. In complex systems, the goal of the designer is simply to achieve stability.

Tools for this purpose introduced in this chapter include the following:

- Stability in the sense of Lyapunov characterizes the system's state response in the absence of inputs, i.e., to initial conditions alone. For LTI systems, this type of stability is tested by examining the eigenvalues of a system to see that they are in the left-half plane (continuous-time systems) or inside the unit circle (discrete-time systems). For time-varying systems, as usual, the test is neither as simple to state nor to perform, and one should be careful not to reach erroneous conclusions based on the "frozen-time" eigenvalues.

- BIBO stability describes the behavior of the system in response to bounded inputs. Thus, BIBO stability is fundamentally different from Lyapunov stability, which neglects input. We have seen an example in which a system that is BIBO stable results from a state variable model whose zero-input counterpart is not Lyapunov stable. We will further explore this phenomenon after the following chapter on controllability and observability. As with Lyapunov stability, LTI systems prove to be much easier to test for BIBO stability than time-varying systems.

- Perhaps the most powerful tool introduced in the chapter is Lyapunov's direct, or *second* method.* This tool is particularly helpful in the study of nonlinear systems, for which there are relatively few other general stability analysis methods available. Lyapunov's method, wherein we construct so-called "Lyapunov functions," can be adapted to assist us in designing controllers, placing bounds on system parameters, and checking the sensitivity of a system to its initial conditions. We will provide more examples of its uses in the exercise problems.

7.6 Problems

7.1 Prove that the equilibrium of the LTI system $\dot{x} = Ax$ is asymptotically stable if and only if all the real parts of the eigenvalues of A are negative. Hint: Use Jordan forms.

7.2 Draw (or program a computer to draw) a phase portrait for the system in Example 7.3, which has the following nonlinear dynamic equations

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* Lyapunov's *first* method is known as the method of exponents. It and *Lyapunov's linearization method* [18] are used exclusively for nonlinear systems and thus are not explored in detail here.

$$\dot{x}_1 = -x_1 - 2{x_2}^2$$
$$\dot{x}_2 = x_1 x_2 - {x_2}^3$$

7.3 A certain real-valued nonlinear system is given by the equations

$$\dot{x}_1 = -x_1 + x_2 + x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -x_1 - x_2 + x_2(x_1^2 + x_2^2)$$

a) Determine the equilibrium points of the system.

b) Choose a candidate Lyapunov function and use it to determine the bounds of stability for the system. Sketch the region in the $x_1 - x_2$ plane in which stability is guaranteed by the Lyapunov function.

c) Draw a phase portrait for the system.

7.4 Determine whether the origin of the following system is Lyapunov stable: $\dot{x} = tx$.

7.5 An LTI system is described by the equations

$$\dot{x} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} x$$

Use Lyapunov's direct method to determine the range of variable a for which the origin is asymptotically stable.

7.6 An LTI system is described by the equations

$$x(k+1) = \begin{bmatrix} 1-a & -1 \\ 0 & -1 \end{bmatrix} x(k)$$

Use Lyapunov's direct method to determine the range of variable a for which the origin is asymptotically stable.

7.7 Show that a sufficient condition for a linear time-varying system to be asymptotically stable is for the matrix $A^{\mathrm{T}}(t) + A(t)$ to have all its eigenvalues in the open left-half of the complex plane.

7.8 Find a Lyapunov function that shows asymptotic stability for the system $\dot{x} = -(1+t)x$.

7.9 Show that matrix A has eigenvalues whose real parts are all less than $-\mu$ if and only if given an arbitrary positive definite matrix Q, there exists a unique, positive definite solution P to the matrix equation:

$$A^{\mathrm{T}} P + PA + 2\mu P = -Q$$

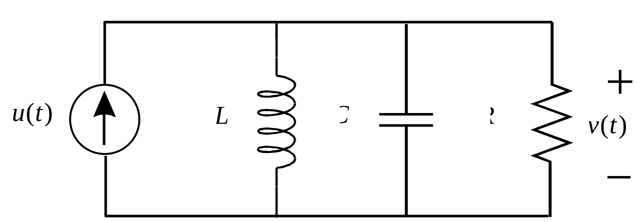
7.10 Determine whether the following system is BIBO stable: $\dot{x} = -x/t + u$.

7.11 Prove that an asymptotically stable system is by necessity also BIBO stable.

7.12 Prove the sufficiency of the condition given in Equation (7.45) for BIBS stability, i.e., that there exists a finite constant N_s such that

$$\int_{t_0}^{t} \left\| \Phi(t,\tau) B(\tau) \right\| d\tau \le N_s$$

7.13 Determine whether the system shown in Figure P7.13 is BIBO stable. Consider the input to be the current $u(t)$, and the output to be the voltage $v(t)$.



P7.13

Repeat if $R = \infty$ (i.e., remove the resistor).

7.14 Consider a system given by the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Determine all of the equilibrium points. Are they Lyapunov stable? If $b = [1 \quad 1]^T$, $c = [1 \quad 0]$ and $d = 0$, is the system BIBO stable? If instead $c = [0 \quad 0]$, is the system BIBO stable?

7.15 Consider the system given by

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} -1 & 2 \end{bmatrix} x$$

a) Find all equilibrium solutions x_e.

b) Determine which equilibria are asymptotically stable.

c) Determine if the equilibrium solutions are Lyapunov stable.

d) Determine if the system is BIBO stable.

e) Let $z_1 = x_1$, $z_2 = -x_1 + x_2$, and $u(t) = 0$. If we denote $z \stackrel{\Delta}{=} [z_1 \quad z_2]^T$ and $\dot{z} = \hat{A}z$, find the equilibrium solutions z_e and sketch them on the $z_1 - z_2$ plane.

f) Draw, on the same plane, a phase portrait.

7.16 For each of the following system matrices for systems of the form $\dot{x} = Ax + Bu$, determine a bounded input that *might* excite an unbounded output.

a) $A = \begin{bmatrix} 1 & 3 & 9 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -3 & 0 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 4 & 5 & 6 \end{bmatrix}$

(Hint: Very little computation is necessary. Note that the B-matrix is undetermined in each case.)

7.17 For the three systems given below, determine the stability (i.e., Lyapunov, asymptotic, or BIBO).

a)
$$x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 5 & 5 \end{bmatrix} x(k)$$

b)
$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x$$

c) $A = \begin{bmatrix} 2 & -5 \\ -4 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $c = \begin{bmatrix} 1 & 1 \end{bmatrix}$

7.18 Given the system

$$\dot{x} = \begin{bmatrix} -25 & -45 & -55 \\ -5 & -15 & -15 \\ 15 & 35 & 35 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0.2 \\ -0.8 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

determine whether the system is Lyapunov stable, asymptotically stable, and/or BIBO stable. If it is not BIBO stable, give an example of a bounded input that will result in an unbounded output.

7.7 References and Further Reading

Although stability for time-invariant systems is relatively easy to test, determining the stability of time-varying and nonlinear systems can be very difficult. General texts used to distill the material presented here include [3], [7], and [15]. An interesting analysis of stability in the sense of Lyapunov is given in [14], where the authors establish the equivalence between Lyapunov stability and the stability determined by the more familiar Routh-Hurwitz test. Further ties between stability types are established in [1] and [16], where the relationships between BIBO, BIBS, and asymptotic stability are established for time-varying systems, although the controllability and observability concepts introduced in Chapter 8 are required. BIBO stability tests based on the impulse-response matrix were introduced in [16].

The issue of slowly varying systems, as briefly mentioned in Section 7.1.4, is addressed in [5], [11], and [17].

Lyapunov's direct method is the topic of much study, owing partially to the difficulty with which Lyapunov functions are found for time-varying and nonlinear systems. A good introductory discussion is included in [7], [9], [12], [13], [18], and [19], with [10] and [20] concentrating on discrete-time systems. The issue of *decrescence*, i.e., the importance of the γ_i-functions described in the theorems of Section 7.2.1, is explicitly discussed in [7] and [18]. These functions are critical for generating Lyapunov functions for time-varying systems, but were not discussed in depth here.

Also useful for nonlinear and time-varying systems are other types of stability tests, such as the small gain theorems used to determine input/output stability, as discussed in [6], [12], and [19].

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