8

Controllability and Observability

We first encountered the concepts of controllability and observability in Examples 3.10 and 3.11. In those examples, we demonstrated the rank conditions for solving systems of simultaneous linear algebraic equations. At that time, the controllability problem was to ascertain the existence of a sequence of inputs to a discrete-time system that would transfer the system's initial state to the zero vector. The observability problem was to determine the existence of the solution of the state equations such that, if the input and output sequences were known for a certain duration, the system's initial state could be deduced. In those examples, we derived a test for such conditions, indicating whether the system considered was controllable and/or observable.

In this chapter, we will continue this discussion, investigating the controllability and observability properties of linear systems in more detail, and extending the results to continuous-time systems. This will mark the first step in the study of linear system controller *design*. We will see in Chapter 10 that the first stage of the design of a linear controller is often the investigation of controllability and observability, and that the process of testing a system for these properties can sometimes dictate the subsequent procedures for the design process.

8.1 Definitions

The descriptions of controllability and observability given in Examples 3.10 and 3.11 will serve as the foundation of the definitions we will use in this chapter. In these definitions, and throughout most of this entire chapter, we make no distinction between SISO and MIMO systems.

> ***Controllability:*** A linear system is *controllable* in an interval $[t_0, t_1]$ if there exists an input $u(t)$ that, when applied to the system from an initial state $x(t_0)$, transfers the system to the state $x(t_1) = 0$. If this property holds regardless of the initial time t_0 or the initial state $x(t_0)$, the system is said to be *completely controllable*. (8.1)

> ***Observability:*** A linear system is *observable* in an interval $[t_0, t_1]$ if, for an initial state $x(t_0)$, knowing two functions $u(t)$ and $y(t)$ over the same interval is sufficient information to uniquely solve for $x(t_0)$. If this property holds regardless of the initial time t_0 or the initial state $x(t_0)$, the system is said to be *completely observable*. (8.2)

We have seen from the examples in Chapter 3 that the property of controllability depends (at least for the discrete-time case) on the matrices *A* and *B*, and the property of observability depends on the matrices *A* and *C*. This is because controllability is dependent only on the input and the state, and observability is dependent only on the state and the output (provided that the input signal, whatever it may be, is known).

As with most issues concerning linear systems, the time-invariant situation is usually easier to develop than the time-varying case. In order to understand controllability and observability in their simplest contexts, we will at first restrict ourselves to LTI systems.

8.2 Controllability Tests for LTI Systems

The results of Examples 3.10 and 3.11 were derived only for the discrete-time case. However, we shall see that the same matrix "tests" hold for continuous-time systems as well.

8.2.1 The Fundamental Tests for Controllability and Observability

In Example 3.10, we derived the result that a discrete-time LTI system is controllable if and only if the matrix

$$P = \left[B \mid AB \mid \cdots \mid A^{k-1}B \right] \tag{8.3}$$

had rank greater than or equal to *n*, the order of the system. At the time, we made the note that because this *P* matrix is $n \times k$, this rank condition will in general require that $k \geq n$. We have since encountered the Cayley-Hamilton theorem, which eliminates the need to consider any power of matrix A greater than $n-1$. We can therefore restate the condition for controllability as follows: An n-dimensional discrete-time LTI system is controllable if and only if the matrix[M]

`ctrb(A,B)`

$$P \triangleq \left[B_d \;\vdots\; A_d B_d \;\vdots\; \cdots \;\vdots\; A_d{}^{n-1} B_d \right] \tag{8.4}$$

has rank n.

Exactly analogous reasoning leads to the additional statement: An n-dimensional discrete-time LTI system is observable if and only if the matrix[M]

`obsv(A,C)`

$$Q \triangleq \begin{bmatrix} C_d \\ \hdashline C_d A_d \\ \hdashline \vdots \\ \hdashline C_d A_d{}^{n-1} \end{bmatrix} \tag{8.5}$$

has rank n. The subscript d in each case refers to the fact that the system matrices come from a *discrete-time* system.

We will call these two matrix tests the *fundamental* tests for controllability and observability. They are by far the most common tests and are preprogrammed into many control system analysis and design software packages. Matrix P is often referred to as the *controllability matrix*, and Q is referred to as the *observability matrix*. We will now proceed to show that the exact same tests can be used to determine the controllability and observability properties of continuous-time LTI systems. Only the derivation and proof differ. Actually, we will present the (fairly lengthy) constructive proof only for controllability. Observability is a dual concept, and the proof of the observability test can be reconstructed by analogy to controllability.

THEOREM: A n-dimensional continuous-time LTI system is completely controllable if and only if the matrix

$$P \triangleq \left[B \;\vdots\; AB \;\vdots\; \cdots \;\vdots\; A^{n-1} B \right] \tag{8.6}$$

has rank n. It is observable if and only if the matrix

$$Q \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{8.7}$$

has rank *n*.

PROOF: We begin by recalling the solution of the LTI state equations

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau \tag{8.8}$$

Given an arbitrary initial condition, we would like to reach the zero state at some future (finite) time t_1. We desire $x(t_1) = 0$, so

$$\int_{t_0}^{t_1} e^{A(t_1-\tau)}Bu(\tau)\,d\tau = -e^{A(t_1-t_0)}x(t_0) \stackrel{\Delta}{=} \xi_1$$

where ξ_1 is an arbitrary $n \times 1$ vector, because we are allowing arbitrary initial conditions $x(t_0)$. Using the Cayley-Hamilton expansion

$$e^{A(t_1-\tau)} = \sum_{i=1}^{n} \gamma_i(\tau) A^{n-i}$$

we can express

$$\begin{aligned}
\xi_1 &= \int_{t_0}^{t_1} \left[\sum_{i=1}^{n} \gamma_i(\tau) A^{n-i} \right] Bu(\tau)\ d\tau \\
&= \int_{t_0}^{t_1} \left[\sum_{i=1}^{n} A^{n-i} B\gamma_i(\tau) \right] u(\tau)\ d\tau \\
&= \int_{t_0}^{t_1} \Big[A^{n-1}B\gamma_1(\tau)u(\tau) + A^{n-2}B\gamma_2(\tau)u(\tau) + \cdots \\
&\qquad \cdots + A^0 B\gamma_n(\tau)u(\tau) \Big]\ d\tau
\end{aligned}$$

$$= A^{n-1}B\int_{t_0}^{t_1}\gamma_1(\tau)u(\tau)\ d\tau + A^{n-2}B\int_{t_0}^{t_1}\gamma_2(\tau)u(\tau)\ d\tau + \cdots$$
$$\cdots + B\int_{t_0}^{t_1}\gamma_n(\tau)u(\tau)\ d\tau \qquad (8.9)$$

Defining

$$\Gamma_i \stackrel{\Delta}{=} \int_{t_0}^{t_1}\gamma_i(\tau)u(\tau)\ d\tau \qquad (8.10)$$

This gives

$$\xi_1 = \left[B \mid AB \mid \cdots \mid A^{n-1}B\right]\begin{bmatrix}\Gamma_n \\ \vdots \\ \Gamma_1\end{bmatrix} = P\begin{bmatrix}\Gamma_n \\ \vdots \\ \Gamma_1\end{bmatrix} \qquad (8.11)$$

Again, since ξ_1 is arbitrary, we must guarantee that this matrix equation always has a solution by requiring matrix P to have full rank. In the most general multivariable case where the system has m inputs, this matrix P will be $n \times nm$, so full rank means $r(P) = n$.

As mentioned, the observability test follows dual reasoning and results in the requirement that $r(Q) = n$, where Q is defined in (8.5).

Example 8.1: Controllability and Observability of a Circuit

For the circuit shown in Figure 8.1, formulate a state variable description using v_C and i_L as the state variables, with source voltage v_s as the input and v_x as the output. Then determine the conditions on the resistor that would make the system uncontrollable and unobservable.

Solution.

First, we notice that

$$v_C(t) = v_s(t) - v_x(t) = v_s(t) - L\frac{di_L(t)}{dt}$$

so

$$\frac{di_L(t)}{dt} = \frac{1}{L} v_s(t) - \frac{1}{L} v_C(t) \tag{8.12}$$

which is one of the necessary state equations. Furthermore, for the capacitor voltage,

$$\begin{aligned}
\frac{dv_C(t)}{dt} &= \frac{1}{C} i_C(t) \\
&= \frac{1}{C}\left(i_L(t) + i_R(t) - \frac{v_C(t)}{R} \right) \\
&= \frac{1}{C}\left(i_L(t) + \frac{v_s(t) - v_C(t)}{R} - \frac{v_C(t)}{R} \right) \\
&= \frac{i_L(t)}{C} - \frac{2v_C(t)}{RC} + \frac{v_s(t)}{RC}
\end{aligned} \tag{8.13}$$

Therefore the state equations are:

$$\begin{aligned}
\begin{bmatrix} \dot{v}_C(t) \\ \dot{i}_L(t) \end{bmatrix} &= \begin{bmatrix} -\frac{2}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{bmatrix} v_s(t) \\
v_x(t) &= v_s(t) - v_C(t) \\
&= \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + v_s(t)
\end{aligned} \tag{8.14}$$

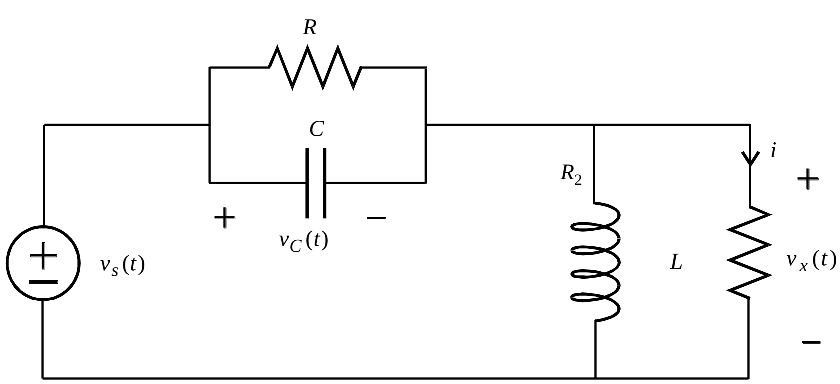


Figure 8.1 RLC circuit example.

Testing controllability using the fundamental controllability test,

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \frac{1}{RC} & -\frac{2}{R^2C^2} + \frac{1}{LC} \\ \frac{1}{L} & -\frac{1}{RLC} \end{bmatrix} \tag{8.15}$$

The rank of this matrix can be checked by determining the determinant:

$$\begin{aligned} \begin{vmatrix} \frac{1}{RC} & -\frac{2}{R^2C^2} + \frac{1}{LC} \\ \frac{1}{L} & -\frac{1}{RLC} \end{vmatrix} &= -\frac{1}{R^2LC^2} + \frac{2}{R^2LC^2} - \frac{1}{L^2C} \\ &= \frac{1}{R^2LC^2} - \frac{1}{L^2C} \end{aligned} \tag{8.16}$$

If we want the conditions under which this determinant is zero, we solve the equation

$$\frac{1}{R^2LC^2} - \frac{1}{L^2C} = 0 \tag{8.17}$$

which gives

$$R = \sqrt{\frac{L}{C}} \tag{8.18}$$

Similarly, the observability matrix is:

$$Q = \begin{bmatrix} -1 & 0 \\ \frac{2}{RC} & -\frac{1}{C} \end{bmatrix} \tag{8.19}$$

which obviously is always of full rank. Hence, the system is always observable, but becomes uncontrollable whenever $R = \sqrt{L/C}$.

It is useful to think in classical terms in this example to see the significance of controllability. If we were to calculate the transfer function between the output $V_x(s)$ and the input $V_s(s)$ (using conventional circuit analysis techniques), we might first find the differential equation:

$$\frac{d^2 v_x(t)}{dt^2} + \frac{2}{RC}\frac{dv_x(t)}{dt} + \frac{1}{LC}v_x(t) = \frac{d^2 v_s(t)}{dt^2} + \frac{1}{RC}\frac{dv_s(t)}{dt} \tag{8.20}$$

from which it should be apparent that the transfer function is:

$$\frac{V_x(s)}{V_s(s)} = \frac{s\left(s + \frac{1}{RC}\right)}{s^2 + \frac{2}{RC}s + \frac{1}{LC}} \tag{8.21}$$

If we wish to express this function in pole-zero form, we will need to compute the roots of the denominator quadratic:

$$s_{1,2} = \frac{-\frac{2}{RC} \pm \sqrt{\frac{4}{R^2C^2} - \frac{4}{LC}}}{2} = -\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{1}{LC}} \tag{8.22}$$

If $R = \sqrt{L/C}$ as computed from (8.18) above, then this becomes

$$s_{1,2} = -\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{1}{LC}} = -\frac{1}{RC} \pm \sqrt{\frac{1}{LC} - \frac{1}{LC}} = -\frac{1}{RC} \tag{8.23}$$

giving a system with repeated roots at $s = -1/RC$. Thus, in pole-zero form, (8.21) is

$$\frac{V_x(s)}{V_s(s)} = \frac{s\left(s + \frac{1}{RC}\right)}{\left(s + \frac{1}{RC}\right)\left(s + \frac{1}{RC}\right)} = \frac{s}{\left(s + \frac{1}{RC}\right)} \tag{8.24}$$

which is now a *first*-order system. What has apparently happened is that the energy storage elements, in conjunction with this particular resistance value, are interacting in such a way that their effects *combine*, giving an equivalent first-order system. In this case, the time constants due to these two elements are the same, making them equivalent to a single element. One might contrast this to a circuit with two capacitors in series or parallel, which can be combined by adding to give a single equivalent capacitance. In this case, the capacitor and inductor together result in an equivalent first-order system, but only for a specific value of *R*.

8.2.2 Popov-Belevitch-Hautus Tests

The Popov-Belevitch-Hautus (PBH) tests, also commonly known as simply the Hautus tests, are not nearly as common as the fundamental test above nor the Jordan form test below. However, they have some interesting geometric interpretations and help us prove the Jordan form test.

PBH Eigenvector Test

Before presenting this Hautus test, we define a vector known as the left eigenvector. The eigenvectors we have been considering up until now have been *right* eigenvectors. A *left* eigenvector corresponding to eigenvalue λ is a nonzero vector v such that

$$vA = \lambda v \tag{8.25}$$

where obviously, v must be a row vector. Given this definition, a new test for controllability may be stated as follows:

> LEMMA: The LTI system is *not* controllable if and only if there exists a nonzero left eigenvector v of A such that
>
> $$vB = 0 \tag{8.26}$$
>
> PROOF: (Sufficiency) Suppose there exists a nonzero left eigenvector v such that $vB = 0$. Then we could multiply (8.25) from the right by B to achieve
>
> $$vAB = \lambda vB = 0$$
>
> We could also multiply (8.25) from the right by AB to achieve
>
> $$vA^2B = \lambda vAB = \lambda^2 vB = 0$$
>
> and so on, until we demonstrate that
>
> $$vP = v\left[B \;\vdots\; AB \;\vdots\; \cdots \;\vdots\; A^{n-1}B\right] = 0$$
>
> implying that $r(P) < n$ and the system is not controllable.
>
> (Necessity) Conversely, suppose that the system is not controllable. Then we know that

$$r\left(\left[B \;\vdots\; AB \;\vdots\; \cdots \;\vdots\; A^{n-1}B\right]\right) < n$$

Then there exists a nonzero vector v such that

$$\begin{aligned} 0 &= v\left[B \;\vdots\; AB \;\vdots\; \cdots \;\vdots\; A^{n-1}B\right] \\ &= \left[vB \;\vdots\; vAB \;\vdots\; \cdots \;\vdots\; vA^{n-1}B\right] \end{aligned}$$

Restricting our attention to vectors v that are left eigenvectors, that is, for which (8.25) holds, then this implies

$$\begin{aligned} 0 &= \left[vB \;\vdots\; vAB \;\vdots\; \cdots \;\vdots\; vA^{n-1}B\right] \\ &= \left[vB \;\vdots\; \lambda vB \;\vdots\; \cdots \;\vdots\; \lambda^{n-1}vB\right] \end{aligned}$$

in turn implying that $vB = 0$ for the left eigenvector v.

A similar statement may be made about observability: A system that has a right eigenvector q ($Aq = \lambda q$) such that $Cq = 0$ will *not* be observable.

PBH Rank Test (Hautus Test)

Now we may state and prove the main PBH controllability test:

THEOREM (PBH rank test): An LTI system will be controllable if and only if $r\left(\left[sI - A \;\vdots\; B\right]\right) = n$ for all s. (8.27)

PROOF: (Sufficiency) If $r\left(\left[sI - A \;\vdots\; B\right]\right) = n$, then there can exist no nonzero vector v such that

$$v\left[sI - A \;\vdots\; B\right] = \left[v(sI - A) \quad vB\right] = 0$$

Consequently, there is *no* vector v such that $vs = vA$ and $vB = 0$. By the above lemma (i.e., the PBH eigenvector test), the system will therefore be controllable.

(Necessity) Now suppose the system is controllable. By reversing the above statements, we can demonstrate that $r\left(\left[sI - A \;\vdots\; B\right]\right) = n$.

Of course, the dual result for observability may also be stated:

> THEOREM (PBH rank test): An LTI system will be observable if and only if
>
> $$r\left(\begin{bmatrix} sI - A \\ \hdashline C \end{bmatrix}\right) = n \tag{8.28}$$
>
> for all s.

8.2.3 Controllability and Observability of Jordan Forms

When we first introduced the concept of the canonical form (e.g., the diagonal and Jordan forms), we indicated that canonical forms were useful for the purpose of exposing more saliently certain important properties of systems. Controllability and observability are foremost among these properties. By transforming systems to their Jordan forms, it is usually possible to ascertain controllability and observability by inspection [5].

Consider the standard-form of the linear LTI system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{8.29}$$

If we use the modal matrix M to perform the change of variables $x = M\overline{x}$ and thus compute the similarity transformation

$$\begin{aligned} \dot{\overline{x}} &= \overline{A}\overline{x} + \overline{B}u \\ y &= \overline{C}\overline{x} + \overline{D}u \end{aligned} \tag{8.30}$$

where $\overline{A} = M^{-1}AM$, $\overline{B} = M^{-1}B$, $\overline{C} = CM$, and $\overline{D} = D$, we arrive at the Jordan form. In the Jordan form, a simple test for controllability can be constructed.

Without loss of generality, we can arrange the Jordan form so that all Jordan blocks corresponding to the same eigenvalues are adjacent:

$$\bar{A} = \left[\begin{array}{ccc|c|ccc} \bar{A}_1^{\lambda_1} & & & & & & \\ & \ddots & & & & & \\ & & \bar{A}_{g_1}^{\lambda_1} & & & & \\ \hline & & & \ddots & & & \\ \hline & & & & \bar{A}_1^{\lambda_m} & & \\ & & & & & \ddots & \\ & & & & & & \bar{A}_{g_m}^{\lambda_m} \end{array}\right] \qquad \bar{B} = \left[\begin{array}{c} \bar{B}_1^{\lambda_1} \\ \vdots \\ \bar{B}_{g_1}^{\lambda_1} \\ \hline \vdots \\ \hline \bar{B}_m^{\lambda_1} \\ \vdots \\ \bar{B}_{g_m}^{\lambda_1} \end{array}\right] \tag{8.31}$$

where $\bar{A}_j^{\lambda_i}$ stands for the j^{th} Jordan block corresponding to eigenvalue λ_i. As we know, there will be a number of these blocks equal to the geometric multiplicity g_i of the eigenvalue. In this notation, *m* is the number of distinct eigenvalues. In the matrix $\bar{B}$, we have simply partitioned the rows to correspond to the partition of rows in the matrix $\bar{A}$.

Now consider performing the Hautus test with the matrices $\bar{A}$ and $\bar{B}$, i.e., determine the rank of the matrix $[sI - \bar{A} \quad \bar{B}]$. We will do this by counting the number of linearly independent rows in this matrix. First, if *s* is *not* an eigenvalue of $\bar{A}$, then each subblock of $sI - \bar{A}$ will itself be of full rank, so there will automatically be *n* linearly independent rows, and $r([sI - \bar{A} \quad \bar{B}]) = n$. If however *s is* an eigenvalue, say $s = \lambda_1$, then the Jordan blocks corresponding to eigenvalues other than λ_1, i.e., $\bar{A}_j^{\lambda_i}$ for $i \neq 1$, will each have full rank, but the blocks corresponding to $s = \lambda_1$ will be rank deficient. Suppose for the sake of argument that there are only two such blocks for λ_1, so that the rows of $\left[sI - \bar{A} \quad \bar{B}\right]$ corresponding to $s = \lambda_1$ appear as

$$
\left[\begin{array}{ccccc:ccccc:c}
0 & -1 & 0 & \cdots & 0 & & & & & & \bar{b}_{11}^{\lambda_1} \\
 & 0 & & \ddots & \vdots & & & & & & \\
\vdots & & & \ddots & 0 & & & 0 & & & \vdots \\
 & & & \ddots & -1 & & & & & & \\
0 & & \cdots & & 0 & & & & & & \bar{b}_{1\ell_1}^{\lambda_1} \\
\hdashline
 & & & & & 0 & -1 & 0 & \cdots & 0 & \bar{b}_{21}^{\lambda_1} \\
 & & & & & & 0 & & \ddots & \vdots & \\
 & & 0 & & & \vdots & & & \ddots & 0 & \vdots \\
 & & & & & & & & \ddots & -1 & \\
 & & & & & 0 & & \cdots & & 0 & \bar{b}_{2\ell_2}^{\lambda_1}
\end{array}\right] \tag{8.32}
$$

where the last column represents the appropriate rows of the blocks $\bar{B}_1^{\lambda_1}$ and $\bar{B}_2^{\lambda_1}$ and where $\bar{b}_{ij}^{\lambda_1}$ is the j^{th} row of block $\bar{B}_i^{\lambda_1}$. We are assuming that these two blocks have ℓ_1 and ℓ_2 rows, respectively. It is clear from this structure that the first $\ell_1 - 1$ rows of the first block are linearly independent and that the first $\ell_2 - 1$ rows of the second block are also linearly independent (of each other and of the first block). Therefore, if and only if the two rows $\bar{b}_{1\ell_1}^{\lambda_1}$ and $\bar{b}_{2\ell_2}^{\lambda_1}$ are independent will we have a complete set of n linearly independent rows. Of course, this applies to any number of Jordan blocks, for any number of repeated eigenvalues.

To state the results succinctly: Denote by $\bar{B}^{\lambda_i}$ the matrix consisting of the rows of the $\bar{B}$ -matrix that each corresponds to the last row of a different Jordan block for eigenvalue λ_i. That is,

$$
\bar{B}^{\lambda_i} \triangleq \begin{bmatrix} \bar{b}_{1\ell_1}^{\lambda_i} \\ \vdots \\ \bar{b}_{g_i \ell_{g_i}}^{\lambda_i} \end{bmatrix} \tag{8.33}
$$

A system in Jordan form is *controllable if and only if all the rows of* $\bar{B}^{\lambda_i}$ *are linearly independent for every* λ_i. Rows of the $\bar{B}$ -matrix that correspond to blocks from *different* eigenvalues do not necessarily have to be linearly independent of one another.

This statement has some obvious implications for simpler systems. For example, consider single-input systems. If a single-input system in Jordan form has a $\bar{B}$-matrix whose rows that correspond to the last rows of the Jordan blocks for any eigenvalue are linearly independent, then there must be only a single Jordan block for each eigenvalue. This is because $\bar{B}$ will consist of a single column. Therefore, if we gather all the rows as in $\bar{B}^{\lambda_i}$ above, we will again have a single column, which has at most rank 1. Two scalars are never linearly independent of one another. This extends to the case where the $\bar{A}$-matrix is diagonal, even if there are repeated eigenvalues (as is the case with symmetric matrices). There must be no more than one Jordan block for any eigenvalue, so any diagonal single-input system with repeated eigenvalues will inevitably be uncontrollable. In general, a system must have at least as many inputs (columns of *B*) as there are Jordan blocks for the eigenvalue that has the most Jordan blocks in order for the system to be controllable.

Furthermore, suppose a Jordan form consists of a diagonal $\bar{A}$-matrix with distinct eigenvalues. Then the diagonal form can be considered a collection of 1×1 blocks, each corresponding to a different eigenvalue. Therefore, for controllability, each row of the $\bar{B}$-matrix must simply be *nonzero*.

To extend these arguments to tests for observability, we change each reference from the *row*(s) of the $\bar{B}$*-matrix* corresponding to *last row*(s) of the Jordan form to the *column*(s) of the $\bar{C}$*-matrix* corresponding to *first column*(s) of the Jordan form. An example to illustrate this concept is presented here.

Example 8.2: Controllability and Observability of Multivariable Jordan Forms

A system is represented by the matrices given in Figure 8.2. Determine whether it is controllable and/or observable.

Solution:

Rather than redraw the system equations, we will mark them up as indicated to show the relevant rows and columns. Note that there are two Jordan blocks for the eigenvalues –5 and 0 while the other eigenvalues correspond to only a single block each. There are two inputs and one output, providing a two-column $\bar{B}$-matrix, and a one-row $\bar{C}$-matrix.

The arrows indicate the relevant rows and columns, with the arrows tied together for the rows that correspond to the Jordan blocks of a single eigenvalue. Examination of the $\bar{B}$-matrix reveals that the rows corresponding to Jordan blocks for $\lambda=-5$ are linearly independent of each other. However, the two rows corresponding to the two blocks for $\lambda=0$ are not linearly independent. Therefore the system is not controllable. As for observability, we know immediately that because the $\overline{C}$-matrix is a single row and some eigenvalues are providing multiple Jordan blocks, the system cannot possibly be observable.

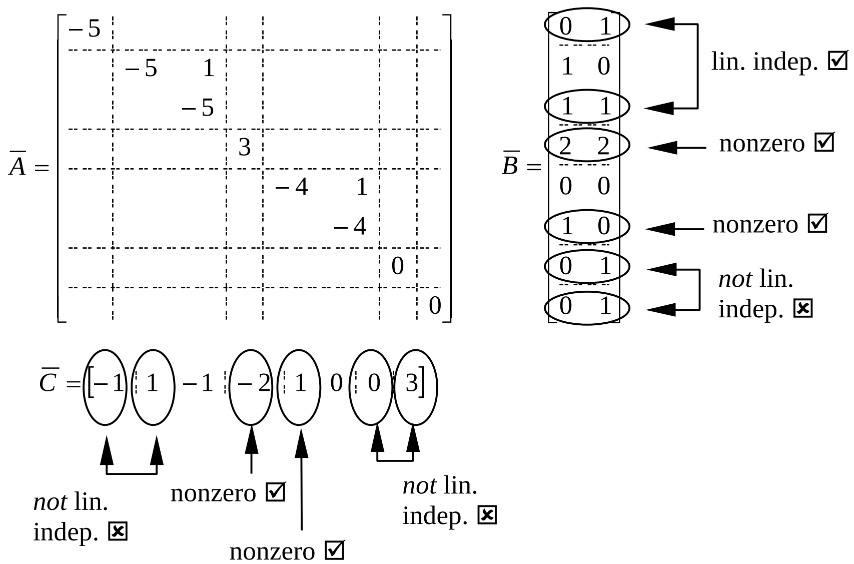


Figure 8.2 Marked-up Jordan form matrices showing criteria for controllability and observability.

One may reasonably ask the question "How can a system consisting of a single Jordan block be controllable if all the state variables must be controlled with a single input?" For example, the system

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \triangleq Ax + Bu$$

can easily be shown to be controllable, because

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

but this demonstration does not make clear how the first state variable, $x_1(t)$, can be made to go to zero when there is no direct input to the first differential equation of the system, $\dot{x}_1(t) = -2x_1(t) + x_2(t)$. The answer lies in the fact that the equations deriving from a Jordan block are *coupled*. Clearly, $x_2(t)$ is affected by the input $u(t)$ because it appears directly in the second equation, $\dot{x}_2(t) = -2x_2(t) + u(t)$. Essentially, $u(t)$ can make $x_2(t)$ behave as desired (i.e., approach zero), while at the same time $u(t)$ can force $x_2(t)$ to act as an input to the first equation, forcing $x_1(t)$ to zero. We say that $x_1(t)$ is controlled "through" $x_2(t)$.

8.2.4 Controllable and Observable Canonical Forms

`canon(sys, type)`

There are two other canonical forms[M] associated with controllability and observability. They are not strictly useful for testing for controllability and observability, because a system cannot be transformed into these canonical forms unless controllability and observability are already known. However, these forms will be important for future sections on controller and observer design, and the necessary transformations are derived from the controllability and observability matrices *P* and *Q*. It therefore seems appropriate to introduce these forms here.

Controllable Form

A system in controllable canonical form will appear as follows:

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{8.34}$$
$$C_c = [\text{arbitrary; no special structure}]$$

A system in observable canonical form has a similar structure:

$$A_o = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ -a_{n-2} & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ -a_1 & & & 0 & 1 \\ -a_0 & 0 & \cdots & 0 & 0 \end{bmatrix} \qquad B_o = \begin{bmatrix} \text{arbitrary;} \\ \text{no special} \\ \text{structure} \end{bmatrix} \tag{8.35}$$
$$C_o = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

The similarity in these forms is readily apparent. The common structure of the *A*-matrix, i.e., the superdiagonal of 1's and the single row or column of nonzero entries (in either the last row or the first column, respectively) is known as a *companion form*.[M] These two forms, the controllable canonical form and the observable canonical form, will be especially useful in the design of controllers and observers that we will introduce in Chapter 10. For now, we will show an algorithmic method for computing the transformation that changes the system into these forms.

`compan(P)`

Consider a single-input system

$$\dot{x} = Ax + bu$$

and denote the controllability matrix by

$$P = \left[b \; \vdots \; Ab \; \vdots \; \cdots \; \vdots \; A^{n-1}b \right]$$

If this matrix is full rank, it will be invertible (for a single-input system). Compute this inverse and call its last row p:

$$P^{-1} = \begin{bmatrix} \vdots \\ \cdots\cdots \\ p \end{bmatrix}$$

Knowing that $P^{-1}P = I$, we have

$$P^{-1}P = \begin{bmatrix} \cdot \\ \cdots \\ p \end{bmatrix} \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} = I \tag{8.36}$$

Considering only the bottom row of this matrix product, we see that

$$\begin{aligned} pb &= 0 \\ pAb &= 0 \\ &\vdots \\ pA^{n-2}b &= 0 \\ pA^{n-1}b &= 1 \end{aligned} \tag{8.37}$$

so that if we define the matrix transformation

$$U^{-1} \triangleq \begin{bmatrix} p \\ pA \\ \vdots \\ pA^{n-1} \end{bmatrix} \tag{8.38}$$

then it is clear from (8.37) that $U^{-1}b = b_c$, demonstrating that the state space transformation $x = Ux_c$ will accomplish the transformation necessary for the *b*-matrix.

Now consider what this transformation does for the *A*-matrix. Given U^{-1} in (8.38), define the columns of *U* as follows:

$$U \triangleq \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

Now, as we did before, we know that $U^{-1}U = I$, so that

$$U^{-1}U = \begin{bmatrix} p \\ pA \\ \vdots \\ pA^{n-1} \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

implying that

$$pA^i u_j = \begin{cases} 0 & \text{if} \quad i \neq j-1 \\ 1 & \text{if} \quad i = j-1 \end{cases}$$

This allows us to compute

$$\begin{aligned} U^{-1}AU &= \begin{bmatrix} p \\ pA \\ \vdots \\ pA^{n-1} \end{bmatrix} A \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \\ &= \begin{bmatrix} pA \\ pA^2 \\ \vdots \\ pA^n \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \\ pA^n u_1 & pA^n u_2 & & \cdots & pA^n u_n \end{bmatrix} \tag{8.39}$$

In order to simplify the bottom row of this matrix, recall that the Cayley-Hamilton theorem tells us that $A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$, or

$$A^n = -a_{n-1}A^{n-1} - \cdots - a_1 A - a_0 I$$

where the coefficients $a_i,\ i = 0, \ldots, n-1$ come from the characteristic polynomial of the system. Therefore (8.39) becomes

$$U^{-1}AU = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \\ -a_0 & -a_1 & & \cdots & -a_{n-1} \end{bmatrix} \tag{8.40}$$

We see therefore that the transformation $x(t) = Ux_c(t)$ does indeed perform the conversion to controllable canonical form. This results in

$$\begin{aligned} \dot{x}_c(t) &= U^{-1}AUx_c(t) + U^{-1}bu(t) \\ &\stackrel{\Delta}{=} A_c x_c + b_c u \\ y(t) &= cUx_c(t) + du(t) \\ &\stackrel{\Delta}{=} c_c x_c(t) + d_c u(t) \end{aligned} \tag{8.41}$$

where A_c and b_c have the forms given in (8.34).

Equation (8.40) also shows another interesting property of the controllable (and observable) canonical forms: For a system that is observable and controllable, the matrix elements $a_0, \ldots, a_{n-1}$ seen in A_c [or A_o, Equation (8.35)], are exactly the same scalar coefficients of the system's characteristic polynomial, i.e.,

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \tag{8.42}$$

In fact, if the c_c-matrix is given by $c_c = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$, then the transfer function of the system in (8.34) will be

$$G(s) = \frac{c_n s^{n-1} + \cdots + c_2 s + c_1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + d \tag{8.43}$$

The reason that the system must be both observable and controllable for this to work will become apparent in Chapter 9.

One may correctly guess that an analogous procedure is used for the observable canonical form, using the *first column* of the inverse observability matrix, etc. The result in Equation (8.43) pertaining to the transfer function also follows the analogy, using the b_c-matrix entries as the numerator of the transfer function instead of the c_c-matrix.

8.2.5 Similarity Transformations and Controllability

Before considering other aspects of controllability and observability, we should make an observation about the similarity transformations necessary to change the basis of a system, for example, to transform a system to a canonical form. It is perhaps not obvious that a system's controllability and observability properties are invariant under a full-rank transformation, but this is indeed the case. Consider a linear system with controllability matrix

$$P = \left[B \;\vdots\; AB \;\vdots\; \cdots \;\vdots\; A^{n-1}B \right] \tag{8.44}$$

Under a similarity transformation, say $x = M\tilde{x}$, we will have transformed matrices

$$\tilde{A} = M^{-1}AM \qquad\qquad \tilde{B} = M^{-1}B$$

Therefore, in this new basis, the controllability test will give

$$\begin{aligned}
\tilde{P} &= \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \tilde{A}^2\tilde{B} & \cdots \end{bmatrix} \\
&= \begin{bmatrix} M^{-1}B & M^{-1}AMM^{-1}B & M^{-1}AMM^{-1}AMM^{-1}B & \cdots \end{bmatrix} \\
&= \begin{bmatrix} M^{-1}B & M^{-1}AB & M^{-1}A^2B & \cdots \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} M^{-1}B & M^{-1}AB & \cdots & M^{-1}A^{n-1}B \end{bmatrix} \\
&= M^{-1}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \\
&= M^{-1}P
\end{aligned}
\tag{8.45}
$$

Because M is nonsingular, $r(\tilde{P}) = r(P)$. An analogous process shows that $r(\tilde{Q}) = r(Q)$. Thus, a system that is controllable (observable) will remain so after the application of a similarity transformation.

So far, we have spoken of systems as being only controllable (observable) or not. In the next section, we will consider uncontrollable (unobservable) systems and investigate *degrees* of controllability (observability). We will consider an uncontrollable (unobservable) system and ask: Is the system controllable (observable) *in part*? If so, what part?

8.3 Modal Controllability and Observability

Often control systems engineers are interested in the controllability and observability of the individual modes of a system, as defined and discussed in Section 6.3. Because in Jordan form the modes corresponding to the different eigenvalues are readily apparent and the system appears as a collection of decoupled blocks, it is sometimes useful to speak of controllable modes and observable modes (as well as uncontrollable modes and unobservable modes). In Example 8.2, we see that only the modes corresponding to eigenvalue $\lambda = 0$ are uncontrollable; all the other modes are controllable. Likewise, inspection of the columns of $\overline{C}$ shows that the modes corresponding to eigenvalues $\lambda = 3$ and $\lambda = -4$ are observable, but the modes corresponding to eigenvalues $\lambda = -5$ and $\lambda = 0$ are unobservable.

8.3.1 Geometric Interpretation of Modal Controllability

To understand how one part of a system can be controllable and/or observable and another part uncontrollable and/or unobservable, we consider in this section the concept of a controllable subspace and an observable subspace. It is helpful to first consider systems defined in discrete-time. From Examples 3.10 and 3.11, we may interpret the controllability matrix $P = [B \quad AB \quad \cdots \quad A^{n-1}B]$ as a mapping (operator) that takes the input sequence into the state space, by Equation (3.55). We have pictured this feedback relationship in Figure 8.3. From the feedback configuration, it can be seen that the state $x(k+1)$ is affected not only by $Bu(k)$, but, tracing through the feedback loop, it is also affected by $ABu(k)$, etc. By the Cayley-Hamilton theorem, there is no need to consider more than $n-1$ such passes around the feedback path.

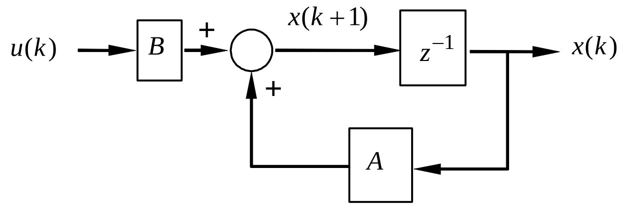


Figure 8.3 Feedback mechanism for discrete-time systems.

If we wish the system to be controllable and we do not know the initial state, then we surmise that the input signal might need to have some components, when mapped by B and traversing around the feedback loop up to $n-1$ passes, in every possible direction of the state space. Requiring that $\{u, Bu(t), ABu(t), \ldots, A^{n-1}Bu(t)\}$ span the entire state space is synonymous to the condition that $r(P) = n$. One might picture this on a phase plane as the ability to "push" the state vector around in all possible directions.

Now thinking about a similar system in continuous-time, we no longer have a matrix equation like (3.55) that maps discrete samples of $u(t)$ into the state space. Rather, the input feeds continuously into the state space and is integrated by the convolution integral (6.6) before we arrive at an expression for the state. This situation is pictured in Figure 8.4.

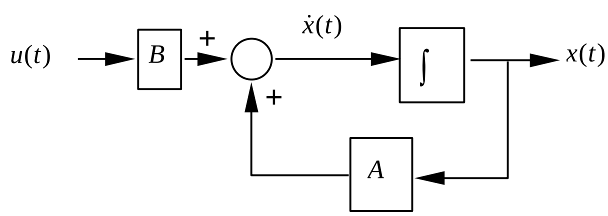


Figure 8.4 Feedback mechanism for continuous-time systems.

However, as we showed in the proof of the fundamental controllability test, the Cayley-Hamilton theorem may be used to reduce this continuous dependence on time into a dependence on the n values Γ_i, Equation (8.10), as mapped into the state space by the same controllability matrix P.

To state this interpretation in terms of subspaces, denote by β the range space of B: $\beta = \mathrm{R}(B)$. Defining the sum of two spaces X and Y as

$$\mathrm{X} + \mathrm{Y} = \{\boldsymbol{x} + \boldsymbol{y} \mid \boldsymbol{x} \in \mathrm{X}, \boldsymbol{y} \in \mathrm{Y}\}$$

we can define the controllable subspace of a system as

$$\mathrm{X}_c \overset{\Delta}{=} \beta + A\beta + \cdots + A^{n-1}\beta \tag{8.46}$$

Therefore, if the system is completely controllable, then $\mathrm{X}_c = \mathrm{X}$, or, the controllable subspace is the entire state space. This subspace X_c can also be interpreted as the smallest *A*-invariant subspace of X that contains β. This also leaves room to define the *uncontrollable* subspace as $\mathrm{X}_{\bar{c}} = \mathrm{X} - \mathrm{X}_c$. For systems that are not controllable, the span of the set of vectors $\{B, AB, \ldots, A^{n-1}B\}$ has a particular dimension, which we will call n_c. Obviously, $n_c = r([B \quad AB \quad \cdots \quad A^{n-1}B])$, and $n_{\bar{c}} = \dim(\mathrm{X}_{\bar{c}}) = n - n_c$.

We consider observable subspaces a little differently. We will define the *un*observable subspace as

$$\mathrm{X}_{\bar{o}} \overset{\Delta}{=} \bigcap_{i=1}^{n} \mathrm{N}(CA^{i-1}) \tag{8.47}$$

Therefore, the entire system will be observable if and only if

$$\bigcap_{i=1}^{n} \mathrm{N}(CA^{i-1}) = \varnothing \tag{8.48}$$

As with the controllable and uncontrollable subspaces, we define $\mathrm{X}_o \overset{\Delta}{=} \mathrm{X} - \mathrm{X}_{\bar{o}}$ and give the appropriate subspace dimensions as

$$n_{\bar{o}} \overset{\Delta}{=} \dim(\mathrm{X}_{\bar{o}}) = \dim\left(\bigcap_{i=1}^{n} \mathrm{N}(CA^{i-1})\right) \qquad n_o \overset{\Delta}{=} n - n_{\bar{o}}$$

It can be shown that $\mathrm{X}_{\bar{o}} = \mathrm{N}(Q)$, and naturally, $n_o = r(Q)$, where Q is the observability matrix. For a physical interpretation of this result, we envision the null spaces of *C*, *CA*, etc. If a state vector cannot be "observed" through the relationship $y = Cx + Du$ because it is in the null space of *C*, then, by the feedback diagrams above, it is perhaps possible to observe it through *CA*, etc. If a state vector cannot be observed through *any* of these mappings, then of course it cannot be observed at all. Geometrically, $\mathrm{X}_{\bar{o}}$ is the largest *A*-invariant subspace of X that is entirely contained in $\mathrm{N}(C)$.

8.3.2 Kalman Decompositions

In cases wherein some modes are controllable and/or observable and others are not, we may wish to group the controllable and observable modes together. Of course, there will be four possible groupings of modes: controllable and observable; controllable and unobservable; uncontrollable and observable; and uncontrollable and unobservable. It is possible to find a transformation matrix that accomplishes such groupings. We will present such a transformation based on the geometric interpretations introduced above.

Suppose it is known that for the SISO* LTI system (continuous or discrete)

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx + du \end{aligned} \tag{8.49}$$

that $r(P) = n_c < n$ from the controllability analysis and that $r(Q) = n_o < n$ from observability analysis, i.e., the system is neither controllable nor observable. What we seek in this section is a transformation that rearranges the state variables in such a way that the modes that are both controllable and observable are grouped together, the modes that are neither controllable nor observable are grouped together, the modes that are controllable but not observable are grouped together, and the modes that are observable but not controllable are grouped together. Such a transformation is called a *Kalman decomposition* [6].

First we consider separating only the controllable subspace from the rest of the state space.[M] To do this, we construct a change of basis matrix whose first n_c columns are a maximal set of n_c linearly independent columns from controllability matrix P. The remaining $n - n_c$ columns do not matter, except, of course, that they need to be linearly independent of the first n_c columns that span the controllable subspace and linearly independent themselves. We therefore construct the change of basis as the nonsingular $n \times n$ operator

ctrbf(A,B,C)

$$V = \left[v_1 \quad \cdots \quad v_{n_c} \;\middle|\; v_{n_c+1} \quad \cdots \quad v_n \right] \tag{8.50}$$

By creating a new state vector $x = Vw$, we will get the transformed system with the form

$$\begin{aligned} \dot{w} &= V^{-1}AVw + V^{-1}bu \\ y &= CVw + du \end{aligned}$$

which has the decomposed structure

---

* MIMO systems present special cases, because different inputs (outputs) may be more or less useful in controlling (observing) the system. Multivariable systems are not discussed here, but are in Chapter 10.

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} A_c & A_{12} \\ 0 & A_{\bar{c}} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} b_c \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} c_c & c_{\bar{c}} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + du \tag{8.51}$$

In this form, we use the notation (A_c, b_c) to denote the $n_c \times n_c$ subsystem consisting of the first n_c state variables w_1. Because of the way we constructed the transformation *V*, Equation (8.50), we know that this subsystem (A_c, b_c) is controllable, and the remainder of the system is not controllable. The uncontrollability of the second subsystem, corresponding to the state variables w_2, is obvious from the structures of the system and input matrices in (8.51). Not only is there no direct mapping of the input $u(t)$ into the state space, but there is also no coupling into this second subsystem from the controllable subsystem.

A similar procedure may be used to decompose a system into observable and unobservable parts[M]: First find the n_o-dimensional observability matrix *Q*. Gather n_o linearly independent *rows* from *Q* and append to these any $n - n_o$ other rows that are independent of the first n_o in order to create a nonsingular operator:

`obsvf(A,B,C)`

$$U = \begin{bmatrix} u_1 \\ \vdots \\ u_{n_o} \\ \hline u_{n_o+1} \\ \vdots \\ u_n \end{bmatrix}$$

With the change of basis $z = Ux$ (note the distinction between this transformation and the previous one $x = Vw$), a new system may be derived with the form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_o & 0 \\ A_{21} & A_{\bar{o}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} b_o \\ b_{\bar{o}} \end{bmatrix} u$$
$$y = \begin{bmatrix} c_o & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + du \tag{8.52}$$

Here again, it is clear that the subsystem consisting of only the state variable components z_2 is in the null space of the observability matrix in the new basis and will therefore be unobservable.

By carefully applying a sequence of such controllability and observability transformations,* the complete Kalman decomposition will result in a system of the block-form:

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \\ \dot{x}_{\bar{c}o} \\ \dot{x}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} b_{co} \\ b_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_{co} & 0 & c_{\bar{c}o} & 0 \end{bmatrix} \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \\ x_{\bar{c}o} \\ x_{\bar{c}\bar{o}} \end{bmatrix} + du$$

In this form, the four distinct subsystems are appropriately grouped and denoted, e.g., $(A_{\bar{c}o}, c_{\bar{c}o})$ represents the subsystem that is not controllable but is observable. There are also coupling terms, e.g., A_{43}, but these do not affect the controllability and observability properties of the subsystems. These dependencies are illustrated in the block diagram of Figure 8.5. In this figure, the dependencies of the subsystems on the input and the dependency of the output on the subsystems are all apparent. We will see the further significance of the Kalman decomposition in Chapter 9.

Example 8.3: Decomposition of an Uncontrollable System

Given the system

$$\dot{x} = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 3 & 6 \\ -5 & -1 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} x$$

determine whether the system is controllable and/or observable. If it is not controllable, perform a Kalman decomposition to separate controllable and uncontrollable subsystems.

* It is not simply a matter of applying one transformation after another. The full procedure is left as an exercise.

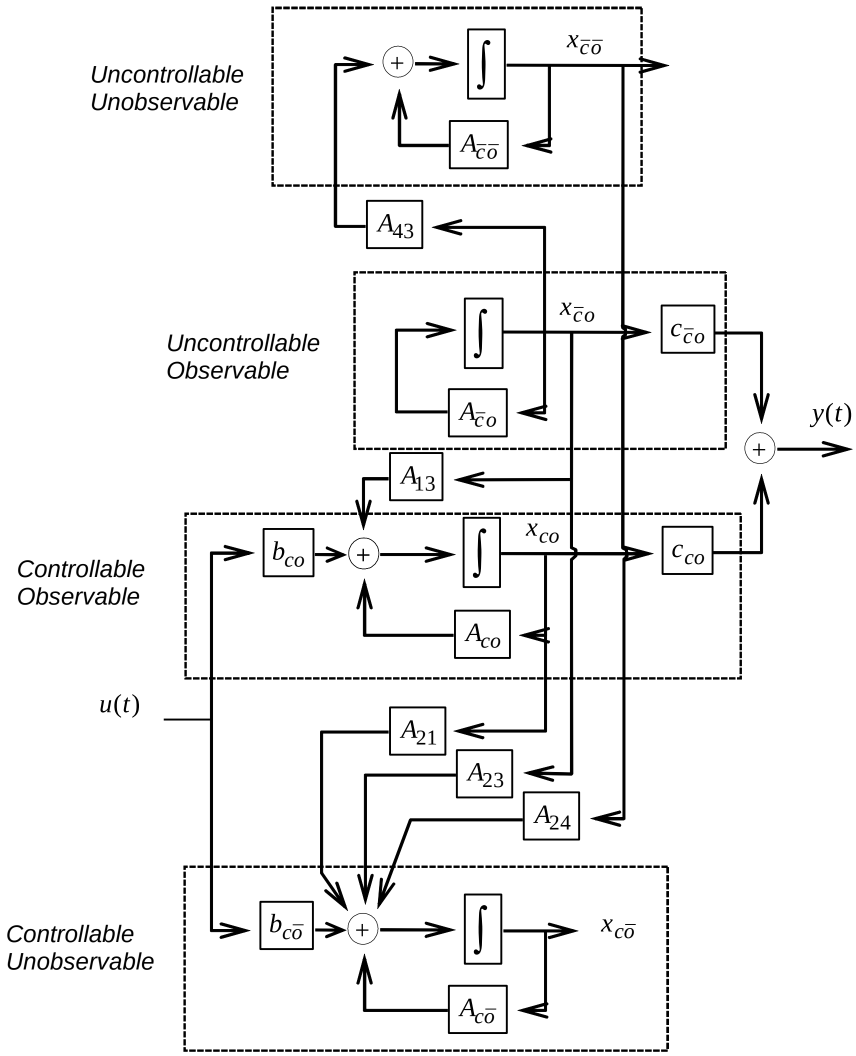


Figure 8.5 Block diagram of a system in Kalman decomposition.

Solution:

To first determine controllability and observability, we compute

$$P = \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & -5 \\ 0 & -5 & 5 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} c \\ cA \\ cA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -3 & 2 & -1 \\ 9 & 4 & 13 \end{bmatrix}$$

It is then a simple matter to determine that $r(P) = 2$ and $r(Q) = 2$, so this system is neither controllable nor observable.

To decompose into controllable and uncontrollable parts, we note that the first two columns of P are linearly independent, so by appending a third linearly independent column, we can construct a suitable similarity transformation:

$$V = \left[\begin{array}{cc:c} 1 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & -5 & 1 \end{array}\right]$$

By now performing the similarity transformation $x = Vw$, we compute

$$A_w = V^{-1}AV = \left[\begin{array}{cc:c} 0 & 6 & -1.4 \\ 1 & -1 & 1.2 \\ \hdashline 0 & 0 & 2 \end{array}\right] \qquad b_w = V^{-1}b = \left[\begin{array}{c} 1 \\ 0 \\ \hdashline 0 \end{array}\right] \qquad c_w = cV = \left[\begin{array}{cc:c} 1 & -3 & 2 \end{array}\right]$$

From this structure, we see that the two-dimensional controllable subsystem is grouped into the first two state variables and that the third state variable, represented by the '2' block, has neither an input signal nor a coupling from the other subsystem. It is therefore not controllable.

8.3.3 Stabilizability and Detectability

Although we have not yet considered the problems of controller and observer design, we have noted in Chapter 7 that the primary goal in control system design is stability. We will see in Chapter 10, as we might suspect from the terminology, that controllability of a mode is a prerequisite for successful controller design and observability of a mode is a prerequisite for observer design. If indeed the designer's primary design criterion is to achieve system stability, however, it is sometimes not necessary for all modes to be controllable and/or observable. This may be the case if the uncontrollable modes are already

Lyapunov stable and the unobservable modes are also Lyapunov stable, thus leading to the following two simple definitions:

> ***Stabilizability:*** A system is *stabilizable* if its uncontrollable modes, if any, are stable. Its controllable modes may be stable or unstable. (8.53)

> ***Detectability:*** A system is *detectable* if its unobservable modes, if any, are stable. Its observable modes may be stable or unstable. (8.54)

8.4 Controllability and Observability of Time-Varying Systems

Controllability and observability studies of time-varying systems are sufficiently different from those of time-invariant systems that they are treated separately in this section. Although the definitions of controllability and observability are basically the same as for time-invariant systems, the tests are considerably different. As was the case for stability, we must be careful not to conclude anything about controllability and observability properties from the "frozen-time" equations of the system. The "duality" of controllability and observability is preserved for the time-varying case, so again we will omit explicit derivations and proofs for observability when they are sufficiently parallel to those for controllability.

8.4.1 Continuous-Time Systems

Consider the time-varying system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{8.55}$$

(Because feedthrough affects neither controllability nor observability, we will simplify our equations by dropping the *Du* term.) We have seen that in the time-invariant case, the fundamental controllability test was derived from a linear independence test on the columns of a matrix that was a function of *A* and *B*. For our first test of controllability for time-varying systems, we draw from the same concept; however, the similarities end soon thereafter.

> THEOREM: The state equations in (8.55) are *controllable* in an interval $[t_0, t_1]$ if the rows of the matrix product

$\Phi(t_0,\tau)B(\tau)$, where $\Phi(t_0,\tau)$ is the state-transition matrix, are linearly independent. (8.56)

PROOF: We know from the results of Chapter 2 that linear independence of time functions is established through the testing of a grammian matrix. In this case, the statement that the rows of $\Phi(t_0,\tau)B(\tau)$ are linearly independent is equivalent to saying that the so-called "controllability grammian"[M]

`gram(sys,'c')`

$$G_c(t_0,t_1) \stackrel{\Delta}{=} \int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t_0,\tau)\, d\tau \qquad (8.57)$$

is nonsingular, i.e., invertible. To begin the proof, we suppose (8.57) is invertible. Then we may consider an input to the system of the form

$$u(t) = -B^{\mathrm{T}}(t)\Phi^{\mathrm{T}}(t_0,t)G_c^{-1}(t_0,t_1)x(t_0) \qquad (8.58)$$

assuming, as we must, that $x(t_0)$ is known. This being the input, we can use the convolution integral solution to determine the state at the end of the interval, $x(t_1)$:

$$\begin{aligned} x(t_1) &= \Phi(t_1,t_0)x(t_0) \\ &\quad -\int_{t_0}^{t_1} \Phi(t_1,\tau)B(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t_0,\tau)\,G_c^{-1}(t_0,t_1)x(t_0)\,d\tau \end{aligned}$$

Factoring $\Phi(t_1,t_0)$ from the left side of this expression (recalling the properties of state-transition matrices),

$$\begin{aligned} x(t_1) &= \Phi(t_1,t_0)\left[x(t_0) - \int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t_0,\tau)\,d\tau\, G_c^{-1}(t_0,t_1)x(t_0)\right] \\ &= \Phi(t_1,t_0)\left[x(t_0) - G_c(t_0,t_1)G_c^{-1}(t_0,t_1)x(t_0)\right] \\ &= \Phi(t_1,t_0)\left[x(t_0) - x(t_0)\right] \\ &= 0 \end{aligned}$$

This implies that the (cleverly chosen) control signal in (8.58) can force the system to reach the zero state in finite time.

Now we suppose that the system is controllable and show that the controllability grammian $G_c(t_0,t_1)$ is invertible. We do this by contradiction. If $G_c(t_0,t_1)$ were *not* invertible, then it would, by definition, be rank-deficient, and would therefore have a nontrivial null space. Therefore, there would exist a nonzero vector z such that $G_c(t_0,t_1)z=0$ and likewise $z^{\mathrm{T}}G_c(t_0,t_1)z=0$. Explicitly,

$$\begin{aligned} z^{\mathrm{T}}G_c(t_0,t_1)z &= \int_{t_0}^{t_1} z^{\mathrm{T}}\Phi(t_0,\tau)B(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t_0,\tau)z\ d\tau \\ &= \left\| B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t_0,\tau)z \right\|^2 \\ &= 0 \end{aligned}$$

implying

$$z^{\mathrm{T}}\Phi(t_0,\tau)B(\tau)=0 \tag{8.59}$$

If the system is, as we assumed, controllable, then it is possible to find an input $u_z(t)$ that, when applied from an arbitrary initial state, transfers the system to the zero state at time t_1. Selecting $x(t_0)=z$, we have

$$0=\Phi(t_1,t_0)z+\int_{t_0}^{t_1}\Phi(t_1,\tau)B(\tau)u_z(\tau)\ d\tau$$

Solving this equation for z:

$$\begin{aligned} z &= -\Phi^{-1}(t_1,t_0)\int_{t_0}^{t_1}\Phi(t_1,\tau)B(\tau)u_z(\tau)\ d\tau \\ &= -\int_{t_0}^{t_1}\Phi(t_0,\tau)B(\tau)u_z(\tau)\ d\tau \end{aligned} \tag{8.60}$$

Multiplying this equation through from the left by z^{T} and applying Equation (8.59) results in

$$z^{\mathrm{T}}z=-\int_{t_0}^{t_1}z^{\mathrm{T}}\Phi(t_0,\tau)B(\tau)u_z(\tau)\ d\tau=0$$

However, this can only be true if $z \equiv 0$. Therefore a nonzero z such that $G_c(t_0, t_1)z = 0$ does not exist, and it follows that $G_c(t_0, t_1)$ must be invertible.

As has been repeatedly noted, any operation involving the state-transition matrix $\Phi(t_1, t_0)$ is difficult. Therefore, the computation of the controllability grammian $G_c(t_0, t_1)$ from its definition in (8.57) is not recommended and in fact, is not always possible. Instead, it is a straightforward application of calculus to find a differential equation for which $G_c(t_0, t_1)$ is a solution:

$$
\begin{aligned}
\frac{d}{dt} G_c(t, t_1) &= \frac{d}{dt}\left[\int_t^{t_1} \Phi(t,\tau)B(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t,\tau)\, d\tau\right] \\
&= \int_t^{t_1} \dot{\Phi}(t,\tau)B(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t,\tau)\, d\tau \\
&\quad + \int_t^{t_1} \Phi(t,\tau)B(\tau)B^{\mathrm{T}}(\tau)\dot{\Phi}^{\mathrm{T}}(t,\tau)\, d\tau - \Phi(t,\tau)B(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t,\tau)\Big|_{\tau=t} \\
&= A(t)\int_t^{t_1} \Phi(t,\tau)B(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t,\tau)\, d\tau \\
&\quad + \int_t^{t_1} \Phi(t,\tau)B(\tau)B^{\mathrm{T}}(\tau)\Phi^{\mathrm{T}}(t,\tau)\, d\tau\, A^{\mathrm{T}}(t) - B(t)B^{\mathrm{T}}(t) \\
&= A(t)G_c(t,t_1) + G_c(t,t_1)A^{\mathrm{T}}(t) - B(t)B^{\mathrm{T}}(t)
\end{aligned}
$$

so

$$
\frac{d}{dt} G_c(t, t_1) = A(t)G_c(t,t_1) + G_c(t,t_1)A^{\mathrm{T}}(t) - B(t)B^{\mathrm{T}}(t) \tag{8.61}
$$

for which numerical solutions may be sought.

Clearly, Equation (8.61) is more convenient for the computation of $G_c(t_0, t_1)$ than is (8.57) because (8.61) does not require knowledge of the state-transition matrix. However, solution of (8.61), especially in closed form, is not easy either. Another method for testing controllability of time-varying systems is based on the following definitions [8]: Let $M_0(t) \overset{\Delta}{=} B(t)$ and

$$
M_j(t) \overset{\Delta}{=} -A(t)M_{j-1}(t) + \dot{M}_{j-1}(t) \tag{8.62}
$$

for $j = 1, \ldots, n$. [This definition requires that $A(t)$ and $B(t)$ be continuously differentiable at least $n-1$ times.] Then we can state the following result:

THEOREM: The system (8.55) is controllable in the interval $[t_0, t_1]$ if there exists a $\tau \in [t_0, t_1]$ such that

$$r\left(\left[\, M_0(\tau) \mid M_1(\tau) \mid \cdots \mid M_{n-1}(\tau)\right]\right) = n \qquad (8.63)$$

PROOF: (By contradiction) Suppose (8.63) holds for some τ yet the system is not controllable. Then the controllability grammian $G_c(t_0, t_1)$ is not invertible, and there will exist as a consequence a nonzero vector z such that

$$z^{\mathrm{T}} \Phi(t_0, t) B(t) = 0 \qquad (8.64)$$

for $t \in [t_0, t_1]$. Define another vector $\bar{z} \stackrel{\Delta}{=} \Phi^{\mathrm{T}}(t_0, \tau) z$. Then (8.64) gives

$$\begin{aligned} z^{\mathrm{T}} \Phi(t_0, t) B(t) &= \bar{z}^{\mathrm{T}} \Phi(\tau, t_0) \Phi(t_0, t) B(t) \\ &= \bar{z}^{\mathrm{T}} \Phi(\tau, t) B(t) \\ &= 0 \end{aligned}$$

for any $t \in [t_0, t_1]$. Supposing that $t = \tau$, this gives

$$\begin{aligned} \bar{z}^{\mathrm{T}} \Phi(\tau, \tau) B(\tau) &= \bar{z}^{\mathrm{T}} B(\tau) \\ &= \bar{z}^{\mathrm{T}} M_0(\tau) \\ &= 0 \end{aligned} \qquad (8.65)$$

Now differentiating (8.64) and evaluating at $t = \tau$,

$$\begin{aligned} \frac{d}{dt}\left(z^{\mathrm{T}} \Phi(t_0, t) B(t)\right)\Big|_{t=\tau} &= z^{\mathrm{T}}\left[\dot{\Phi}(t_0, t) B(t) + \Phi(t_0, t)\dot{B}(t)\right]\Big|_{t=\tau} \\ &= z^{\mathrm{T}}\left[-\Phi(t_0, t) A(t) B(t) + \Phi(t_0, t)\dot{B}(t)\right]\Big|_{t=\tau} \\ &= z^{\mathrm{T}} \Phi(t_0, t)\left[-A(t)B(t) + \dot{B}(t)\right]\Big|_{t=\tau} \\ &= z^{\mathrm{T}} \Phi(t_0, \tau) M_1(\tau) \\ &= \bar{z}^{\mathrm{T}} M_1(\tau) \\ &= 0 \end{aligned} \qquad (8.66)$$

where the fact that $\dot{\Phi}(t_0,t) = -\Phi(t_0,t)A(t)$ is a result of applying what we learned in Exercise 3.2 to find $\frac{d}{dt}\left(\Phi^{-1}(t,t_0)\right)$. We can take a second derivative of (8.64), which is most easily accomplished by taking the first derivative of the fourth line of (8.66) above. This will lead to

$$\begin{aligned}\frac{d^2}{dt^2}\left(z^{\mathrm{T}}\Phi(t_0,t)B(t)\right)\Big|_{t=\tau} &= \frac{d}{dt}\left(z^{\mathrm{T}}\Phi(t_0,t)M_1(t)\right)\Big|_{t=\tau} \\ &= \bar{z}^{\mathrm{T}}M_2(\tau) \\ &= 0\end{aligned}$$

by a similar process. Continuing a total of $n-1$ times, we will develop the expression

$$\bar{z}^{\mathrm{T}}\left[\,M_0(\tau) \quad M_1(\tau) \quad \cdots \quad M_{n-1}(\tau)\right] = 0$$

which contradicts the linear independence implied by the assertion of (8.63). Therefore, if (8.63) holds, then the system is controllable.

We might note at this point the parallel alternative route this theorem provides to the fundamental controllability test. If the system in question is time-invariant, then in (8.62), $\dot{M}_j(t) \equiv 0$, and the rank test will reduce to

$$r\left[B \;\vdots\; -AB \;\vdots\; A^2B \;\vdots\; \cdots \;\vdots\; (-1)^{n-1}A^{n-1}B\right] \overset{?}{=} n \tag{8.67}$$

However, if the matrices A and B are constant, then the rank of the above matrix is not affected by the alternating negative signs, and the controllability matrix P is apparent. Likewise, there is a route to the fundamental controllability test from the controllability grammian condition, which exploits the series expansion for

$$\Phi(t,t_0) = e^{A(t-t_0)}$$

but we will not belabor the point by presenting that derivation here.

Example 8.4: Controllability for a Time-Varying System

Test the system below for controllability over the interval $t \in [0,1]$:

$$\dot{x}(t) = \begin{bmatrix} 0 & t \\ 1 & t \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ t \end{bmatrix} u(t)$$

Solution:

Using the test in (8.63), we generate the matrix

$$\begin{bmatrix} M_0(t) & M_1(t) \end{bmatrix} = \begin{bmatrix} 2 & -t^2 \\ t & -(t^2+1) \end{bmatrix}$$

To test the rank of this matrix, we can check the determinant:

$$\det \begin{bmatrix} 2 & -t^2 \\ t & -(t^2+1) \end{bmatrix} = t^3 - 2t^2 - 2$$

Because the equation $t^3 - 2t^2 - 2 = 0$ does not hold identically in $t \in [0,1]$, we can conclude that the system is controllable in that interval.

Analogous Observability Results

In the development of the controllability results for time-varying systems above, we neglected to point out the corresponding tests for observability. This was so that the sometimes algebraically involved derivations and proofs did not get interrupted by such dual issues. We have instead gathered them here, where with one exception, we present them without detailed derivation or proof. Once again, the definition of observability is the same for time-varying systems as it is for time-invariant systems, and the results generated here apply equally well for MIMO systems as they do for SISO systems.

That exception is in the definition and use of the *observability grammian*[M] to show observability. In our proof that an invertible controllability grammian implies controllability, we relied on a cleverly chosen input, Equation (8.58), that drives the system to the origin. Naturally, in the case of observability, the control input is irrelevant, so the system's output is not a variable that we are free to specify. Therefore, the analogous observability result is obtained a little differently.

`gram(sys,'o')`

THEOREM: The linear, time-varying system in Equation (8.55) is observable in the interval $t \in [t_0, t_1]$ if the columns of the matrix $C(t)\Phi(t, t_0)$ are linearly independent. (8.68)

PROOF: As before, this linear independence condition is equivalent to the invertibility of the matrix we call the observability Grammian:

$$G_o(t_0,t_1) \stackrel{\Delta}{=} \int_{t_0}^{t_1} \Phi^{\mathrm{T}}(\tau,t_0)C^{\mathrm{T}}(\tau)C(\tau)\Phi(\tau,t_0)\, d\tau \qquad (8.69)$$

In the absence of an input, the output of a system is given by

$$y(t) = C(t)\Phi(t,t_0)x(t_0)$$

Multiplying both sides of this equation by $\Phi^{\mathrm{T}}(t,t_0)C^{\mathrm{T}}(t)$ and integrating from $t = t_0$ to $t = t_1$ give

$$\begin{aligned}\int_{t_0}^{t_1} \Phi^{\mathrm{T}}(t,t_0)C^{\mathrm{T}}(t)y(t)\, dt &= \int_{t_0}^{t_1} \Phi^{\mathrm{T}}(t,t_0)C^{\mathrm{T}}(t)C(t)\Phi(t,t_0)x(t_0)\, dt \\ &= G_o(t_0,t_1)x(t_0)\end{aligned} \qquad (8.70)$$

Considering the left-hand side of (8.70) to be an arbitrary vector ξ in the state space, we are left with the linear algebraic system to solve:

$$\xi = G_o(t_0,t_1)x(t_0)$$

This equation is familiar to us, being quite similar to the condition for observability of the discrete-time systems given originally in Chapter 3. For a unique solution to exist in general, we therefore require the observability grammian $G_o(t_0,t_1)$ to be invertible. This proves the sufficiency of the grammian condition. Proof of necessity is by contradiction, and is omitted here.

The observability grammian $G_o(t_0,t_1)$ satisfies the matrix differential equation

$$\frac{d}{dt}G_o(t,t_1) = -A^{\mathrm{T}}(t)G_o(t,t_1) - G_o(t,t_1)A(t) - C^{\mathrm{T}}(t)C(t) \qquad (8.71)$$

For an observability test analogous to that culminating in the rank test of Equation (8.63), define $N_0(t) \triangleq C(t)$, and

$$N_j(t) \triangleq N_{j-1}(t)A(t) + \dot{N}_{j-1}(t) \tag{8.72}$$

for $j = 1, \ldots, n$.

THEOREM: The system (8.55) is observable in the interval $[t_0, t_1]$ if there exists a $\tau \in [t_0, t_1]$ such that

$$r\begin{bmatrix} N_0(\tau) \\ \hdashline N_1(\tau) \\ \hdashline \vdots \\ \hdashline N_{n-1}(\tau) \end{bmatrix} = n \tag{8.73}$$

As usual, the duality between controllability and observability is readily apparent. As a practical matter, one can get away with only a single set of tests, i.e., for controllability or observability. Observability of the pair (A, C) is equivalent to controllability of the pair $(A^{\mathrm{T}}, C^{\mathrm{T}})$, as direct substitution into any of the above formulas will reveal. For example, if we wish to write a set of subroutines that test for controllability, a single routine suffices for both observability and controllability.

8.4.2 Reachability and Reconstructibility

Depending on the nature of the system to be controlled, it may be desirable that the final state of the system be something other than zero. For example, in nonzero set-point control, the goal is to take a system *from* the zero state *to* a nonzero state. Note that this is the opposite of the problem implied by the definition of controllability, which says that a system can be transferred from a nonzero state to the zero state. If a system can be moved from the zero initial state to an arbitrary nonzero final state, the system is said to be *reachable*. For continuous-time systems, a system is reachable if and only if it is controllable, so there is no pressing need for this distinction. However, this is not the case for discrete-time systems. We will consider discrete-time systems in the next section of this chapter.

Likewise, it may sometimes be desired not to deduce the starting state from a sequence of inputs and outputs, as specified in the observability problem, but rather to “reconstruct” the final state from a knowledge of only the output. That is, we may want to deduce the final state rather than the initial state. A system for which this is possible is called *reconstructible*. As with reachability, reconstructibility is equivalent to observability in continuous-time systems but is somewhat different in discrete-time systems. We will revisit this distinction in the next section as well.

8.5 Discrete-Time Systems

Throughout this part of the book, we have often pointed to the dualities between controllability and observability, and used these similarities as excuses for omitting some of the detailed derivations and proofs of certain properties. At times, such as in Chapter 7 on stability, we have made similar observations about the analogies between discrete-time and continuous-time systems. For most situations, this is sufficient, as the distinctions between the two domains do not warrant the time spent illustrating algebraic differences in the derivations. However, we have devoted this final section of this chapter to just such discrepancies because, in the case of controllability and observability, they are fundamentally different.

We should first point out that the examples we presented in Chapter 3 were from a somewhat simplified viewpoint because their purpose at the time was to illustrate an example of solving simultaneous equations. Here, we will revisit the discrete-time controllability and observability properties in more depth.

8.5.1 Controllability and Reachability

Recall that in Chapter 6, the solution for the state of the set of discrete-time state equations

$$\begin{aligned} x(k+1) &= A_d(k)x(k) + B_d(k)u(k) \\ y(k) &= C_d(k)x(k) + D_d(k)u(k) \end{aligned} \tag{8.74}$$

beginning with initial state $x(j_0) = x_0$ was

$$x(j) = \Psi(j, j_0)x_0 + \sum_{i=j_0+1}^{j} \Psi(j,i)B(i-1)u(i-1) \tag{8.75}$$

which has a slight change in notation from Equation (6.52). Also recall that

$$\Psi(j,k) \overset{\Delta}{=} \prod_{i=k}^{j-1} A_d(i))$$

To first consider the problem of *controllability* on the time interval $[j_0, j_1]$, we take the final value to be $x(j_1)=0$ and seek the circumstances under which the equations

$$\begin{aligned}
-\Psi(j_1, j_0)x_0 &= \sum_{i=j_0+1}^{j}\Psi(j_1, i)B(i-1)u(i-1) \\
&= \left[B(j_1-1) \;\vdots\; \Psi(j_1, j_1-1)B(j_1-2) \;\vdots\; \cdots \right. \\
&\qquad \left. \cdots \;\vdots\; \Psi(j_1, j_0+1)B(j_0)\right]\begin{bmatrix} u(j_1-1) \\ u(j_1-2) \\ \vdots \\ u(j_0) \end{bmatrix} \\
&\stackrel{\Delta}{=} R_c(j_0, j_1)\begin{bmatrix} u(j_1-1) \\ u(j_1-2) \\ \vdots \\ u(j_0) \end{bmatrix}
\end{aligned} \tag{8.76}$$

have a solution for the vector of inputs (which is not necessarily unique). We call the matrix $R_c(j_0, j_1)$ defined in (8.76) the *reachability* matrix. In the case of a time-invariant system, as we have seen, the matrix becomes

$$R_c(j_0, j_1) = \begin{bmatrix} B_d & A_d B_d & \cdots & A_d^{j_1-1}B_d \end{bmatrix}$$

The first interesting point to illustrate is that in Chapter 3, we assumed that the left-hand side of Equation (8.76) is a generic vector in $\Re^n$, so that the requirement we reached was that $r\big(R_c(j_0, j_1)\big)=n$. In the sense that we must allow for an arbitrary initial condition x_0, this is true. However, in the discrete-time case we find situations in which $r\big(\Psi(j_1, j_0)\big)<n$, implying that the left-hand side of (8.76) may be restricted to only a proper subspace of the entire state space. This means that in such cases, the system may be controllable, in its strict definition, with $r\big(R_c(j_0, j_1)\big)<n$. In such a situation, controllability would result if $\mathrm{R}\big(R_c(j_0, j_1)\big)=\mathrm{R}\big(\Psi(j_1, j_0)\big)$. Physically, this situation is possible because a rank-deficient matrix $A_d(k)$ might result in a system that is controllable to the zero state with zero input. For example, in the extreme case wherein $A_d(k)=[0]_{n\times n}$, then the state of the system will go immediately to zero with a zero input regardless of the matrix $B_d(k)$. Such a situation does not

happen with rank deficient *A*-matrices in continuous-time because rank deficiency in continuous-time does not take the state itself to zero; rather, it makes the state's *derivative* zero, resulting in an initial state that does not change. Thus, controllability conditions in discrete-time are fundamentally different from those in continuous-time.

Another distinction between discrete-time and continuous-time controllability lies in the fact that the number of terms in the matrix $R_c(j_0, j_1)$ depends on the number of time steps in the interval $[j_0, j_1]$. Suppose that $r\left(\Psi(j_1, j_0)\right) = n_1$ so that we require $\mathrm{R}\left(R_c(j_0, j_1)\right) = \mathrm{R}\left(\Psi(j_1, j_0)\right)$ for controllability. Suppose also that the system is single-input. Then the number of columns of $R_c(j_0, j_1)$ in (8.76) is equal to the number of time steps $j_1 - j_0$. Naturally, if $j_1 - j_0 < n_1$, we have no hope for controllability whatsoever. The Cayley-Hamilton theorem tells us that there is no reason to ever consider *more* than *n* columns for $R_c(j_0, j_1)$ [or, more generally, powers of $A_d(k)$ higher than $n-1$], just as it did in continuous-time. However, in continuous-time, the controllability matrix never has *fewer* than *n* columns.

Consider now the discrete-time reachability problem. In this problem, we consider that $x(j_0) = 0$ and the desired final state is $x(j_1) = x_f$. When that is the case, Equation (8.75) reduces to

$$
\begin{aligned}
x_f &= \sum_{i=j_0+1}^{j} \Psi(j, i) B(i-1) u(i-1) \\
&= R_c(j_0, j_1) \begin{bmatrix} u(j_1 - 1) \\ u(j_1 - 2) \\ \vdots \\ u(j_0) \end{bmatrix}
\end{aligned}
\tag{8.77}
$$

It is clear that in this case, regarding x_f as an arbitrary vector in $\Re^n$, we *must* have

$$
r\left(R_c(j_0, j_1)\right) = n \tag{8.78}
$$

for reachability, hence the name *reachability matrix*. This is true regardless of the number of system inputs. [For multi-input systems, the solution of (8.76) may require a pseudoinverse, as discussed in Chapter 3.] For this reason, the property of reachability is more desirable than controllability for discrete-time systems. *A discrete-time system can be controllable without being reachable.* Note that the observation about the minimum number of time steps required still applies, just as it did for controllability. Thus, although singularity of the state-transition matrix is no longer an issue, the length of the control interval still is.

For the time-invariant case, with $A_d(k) = A_d$ and $B_d(k) = B_d$ the reachability matrix condition is

$$r(R_c) \overset{?}{=} r\left(\begin{bmatrix} B_d & A_d B_d & \cdots & A_d^{n-1} B_d \end{bmatrix}\right) = n \tag{8.79}$$

which would imply that for time-invariant systems, reachability and controllability are equivalent. Even though the concept of reachability is fundamentally distinct from controllability, this condition indicates that the test is much the same.

Reachability Grammians

As with controllability, there are alternate tests for reachability although, as in continuous-time, (8.6) or (8.7) is by far the most common. Fortunately, the discrete-time reachability grammian test is considerably easier to prove than it was in continuous-time.

Define the *reachability grammian* as the matrix[M] `gram(sys,'c')`

$$H_c(j_0, j_1) \overset{\Delta}{=} \sum_{i=j_0}^{j_1-1} \Psi(j_1, i+1) B(i) B^{\mathrm{T}}(i) \Psi^{\mathrm{T}}(j_1, i+1) \tag{8.80}$$

We will state the reachability grammian test in the form of a theorem.

> THEOREM: The system in (8.74) is reachable on the interval $[j_0, j_1]$ if and only if the reachability grammian in (8.80) is nonsingular. (8.81)

Rather than presenting a detailed proof of this theorem, which can be constructed parallel to the proof given for continuous-time, we need merely point out that $H_c(j_0, j_1) = R_c(j_0, j_1) R_c^{\mathrm{T}}(j_0, j_1)$. Our knowledge of the rank of operators makes it immediately clear that $r(H_c(j_0, j_1)) = n$ if and only if $r(R_c(j_0, j_1)) = n$, which has already been proven a valid reachability test [7].

Example 8.5: A Discrete-Time Control Sequence

Determine whether the state equation given below is controllable. Then determine the *minimum* number of time steps necessary to take the system from

$x(0) = \begin{bmatrix} -2 & -2 & 2 & -4 & -3 \end{bmatrix}^{\mathrm{T}}$ to the zero state, and find an input sequence that does so.

$$x(k+1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -0.5 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -0.5 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} u(k) \tag{8.82}$$

Solution:

First, it is a trivial computation (on a computer) to determine that

$$r(P) = r\left(\begin{bmatrix} b_d & A_d b_d & A_d^2 b_d & A_d^3 b_d & A_d^4 b_d \end{bmatrix}\right) = 5$$

so that the system is clearly controllable (and reachable). This guarantees that after five time steps, the equation

$$-A_d^j x(0) = P \begin{bmatrix} u(j-1) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix}$$

with $j = 5$ has a unique solution for any possible $x(0)$.

However, for the given $x(0)$ in particular, we need only satisfy the requirement that

$$-A_d^j x(0) \in \mathrm{R}\left(\begin{bmatrix} b_d & A_d b_d & \cdots & A_d^{j-1} b_d \end{bmatrix}\right)$$

which might occur even if $j < n$. If we try at first *one* time step, we have the equation

$$-A_d x(0) = b_d u(0) \tag{8.83}$$

from which we find that $r([b_d]) \neq r\left(\begin{bmatrix} -A_d x(0) & b_d \end{bmatrix}\right)$, so (8.83) cannot have a solution. After two time steps, we must solve

$$-A_d^2 x(0) = \begin{bmatrix} b_d & A_d b_d \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \tag{8.84}$$

This time, we can find that

$$r\left([b_d \quad A_d b_d]\right) = r\left(\begin{bmatrix} -A_d^2 x(0) & b_d & A_d b_d \end{bmatrix}\right) \quad (=2) \tag{8.85}$$

so that (8.85) must have a solution. That solution is easily found as

$$\begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} b_d & A_d b_d \end{bmatrix}^{+} \left(-A_d^2 x(0)\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

using the pseudoinverse. If we apply this sequence to the original equations in (8.82) starting at $x(0) = \begin{bmatrix} -2 & -2 & 2 & -4 & -3 \end{bmatrix}^T$, we get

$$\begin{aligned} x(1) &= A_d x(0) + b_d u(0) = \begin{bmatrix} -1 & 2 & 0 & 2 & 1 \end{bmatrix}^T \\ x(2) &= A_d x(1) + b_d u(1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \end{aligned}$$

Thus, although the system requires at least five steps to ensure a solution for an *arbitrary* initial condition, there are certain initial conditions which can be "controlled" with fewer steps.

8.5.2 Observability and Reconstructibility

We will see in this section that although the concepts of observability and reconstructibility are closely related, there is an important difference between the two, just as we found with controllability and reachability. First, we must clarify the definition of observability to suit the discrete-time interval:

> ***Observability:*** The linear discrete-time system in (8.74) is said to be *observable* in the time interval $[j_0, j_1]$ if an arbitrary initial state $x(j_0)$ can be uniquely determined given knowledge of the sequence $y(j)$ for $j = j_0, \cdots, j_1 - 1$. (8.86)

The distinction between this definition and that for continuous-time is primarily the specification of the output sequence at the discrete-time instants rather than in a continuous interval. This definition is sometimes given with the requirement that the input sequence $u(j)$ be known as well. While this is true, we can show that the conditions for the existence of the actual solution for the initial state $x(j_0)$ are the same for the zero-state response as for the complete response. We will thereafter simplify our algebra by considering only the zero-state response.

First we consider the general solution for output $y(j)$ as derived in Chapter 6:

$$\begin{aligned} y(j) &= C_d(j)\Psi(j, j_0)x(j_0) \\ &\quad + C_d(j)\sum_{i=j_0+1}^{j}\Psi(j,i)B_d(i-1)u(i-1) + D_d(j_0)u(j_0) \end{aligned} \tag{8.87}$$

Beginning with the initial condition $x(j_0) = x_0$, we can write the first few terms of this solution, similar to what we did for the time-invariant case in Chapter 3:

$$\begin{aligned} y(j_0) &= C_d(j_0)x_0 + D_d(j_0)u(j_0) \\ y(j_0+1) &= C_d(j_0+1)\Psi(j_0+1, j_0)x_0 \\ &\quad + C(j_0)B_d(j_0)u(j_0) + D_d(j_0+1)u(j_0+1) \\ &\vdots \end{aligned} \tag{8.88}$$

To justify considering the zero-input response only, consider that in order to solve the above equations in (8.88) for the required x_0, we rewrite the sequence of equations as

$$\begin{bmatrix} y(j_0) \\ y(j_0+1) \\ \vdots \\ y(j_1-1) \end{bmatrix} = \begin{bmatrix} C_d(j_0) \\ C_d(j_0+1)\Psi(j_0+1, j_0) \\ \vdots \\ C_d(j_1-1)\Psi(j_1-1, j_0) \end{bmatrix} x_0 + \begin{bmatrix} y_{zs}(j_0) \\ y_{zs}(j_0+1) \\ \vdots \\ y_{zs}(j_0-1) \end{bmatrix} \tag{8.89}$$

where $y_{zs}(j)$ stands for the zero-state response at time *j*, which includes all the terms of (8.88) that include input $u(j)$. In order to determine x_0, we can subtract this zero-state response vector from the left-hand side, but this will not change the rank condition that then becomes apparent for the coefficient matrix of x_0. It is therefore not important that we consider the zero-input portion of (8.88) at all, and for simplicity we simply omit it.

Therefore, because $x_0 \in \Re^n$, and we require a unique solution, (8.89) implies the observability condition that

$$r\left(\begin{bmatrix} C_d(j_0) \\ C_d(j_0+1)\Psi(j_0+1, j_0) \\ \vdots \\ C_d(j_1-1)\Psi(j_1-1, j_0) \end{bmatrix}\right) \stackrel{\Delta}{=} r\big(R_0(j_0, j_1)\big) = n \tag{8.90}$$

We will refer to the matrix $R_0(j_0, j_1)$ as the *observability matrix*. If the system has p outputs, this matrix has dimensions $j_1 p \times n$. As with controllability and reachability, observability is possible only if $j_1 p \geq n$, implying a minimum number of observations (specifically, n of them for a single-input system). When the system is time-invariant, the condition in (8.90) reduces to the fundamental observability condition, Equation (8.5). In both cases, the likelihood of observability of a specific state is improved if more time instants are considered.

As for reconstructibility, we first give the formal definition from which we will derive a test.

> ***Reconstructibility:*** A linear discrete-time system, Equation (8.74), is *reconstructible* in the time interval $[j_0, j_1]$ if given an arbitrary initial condition $x(j_0)$, the final state $x(j_1)$ can be uniquely determined from the output response sequence $y(j)$ for $j = j_0, \cdots, j_1 - 1$. (8.91)

Again, we will work with (8.89) in such a way that the zero-state responses are irrelevant and will therefore be neglected. To determine our reconstructibility test, we note that without this zero-input part, (8.89) becomes

$$\begin{bmatrix} y(j_0) \\ y(j_0+1) \\ \vdots \\ y(j_1-1) \end{bmatrix} = \begin{bmatrix} C_d(j_0) \\ C_d(j_0+1)\Psi(j_0+1, j_0) \\ \vdots \\ C_d(j_1-1)\Psi(j_1-1, j_0) \end{bmatrix} x_0 \tag{8.92}$$

However, we also know from the inversion property of the state-transition matrix that if $\Psi^{-1}(j_1, j_0)$ exists [it will exist only if *all* of its component factors $A_d^{-1}(k)$, for $k = j_0, \ldots, j_1 - 1$, exist], then $\Psi^{-1}(j_1, j_0) = \Psi(j_0, j_1)$ and $x(j_0) = \Psi(j_0, j_1)x(j_1)$. If this is the case, (8.92) can be rewritten as

$$\begin{bmatrix} y(j_0) \\ y(j_0+1) \\ \vdots \\ y(j_1-1) \end{bmatrix} = \begin{bmatrix} C_d(j_0)\Psi(j_0, j_1) \\ C_d(j_0+1)\Psi(j_0+1, j_0)\Psi(j_0, j_1) \\ \vdots \\ C_d(j_1-1)\Psi(j_1-1, j_0)\Psi(j_0, j_1) \end{bmatrix} x(j_1)$$

$$= \begin{bmatrix} C_d(j_0)\Psi(j_0, j_1) \\ C_d(j_0+1)\Psi(j_0+1, j_1) \\ \vdots \\ C_d(j_1-1)\Psi(j_1-1, j_1) \end{bmatrix} x(j_1) \tag{8.93}$$

From (8.93), the obvious condition for the solution for $x(j_1)$ is the *reconstructibility* test:

$$r\left(\begin{bmatrix} C_d(j_0)\Psi(j_0, j_1) \\ C_d(j_0+1)\Psi(j_0+1, j_1) \\ \vdots \\ C_d(j_1-1)\Psi(j_1-1, j_1) \end{bmatrix} \right) \stackrel{\Delta}{=} r\left(\overline{R}_0(j_0, j_1)\right) = n \tag{8.94}$$

We call the matrix $\overline{R}_0(j_0, j_1)$ thus defined the *reconstructibility matrix*.

Recall now that controllability of a discrete-time system did not imply reachability. Can we make an analogous statement about observability and reconstructibility? The answer is yes but in the opposite sense. *In discrete-time, observability implies reconstructibility, but not vice versa*. This is apparent if we consider the relationship between the observability and reconstructibility matrices. Although $\Psi^{-1}(j_1, j_0)$ may not exist because of singularity of one of the $A_d(k)$ factors, we are assured from its construction that $\Psi(j_1, j_0)$ itself exists. Therefore, the relationship between (8.92) and (8.93) can be expressed as

$$\overline{R}_0(j_0, j_1)\Psi(j_1, j_0) = R_0(j_0, j_1)$$

Therefore, we have

$$r\left(\overline{R}_0(j_0, j_1)\right) \geq r\left(R_0(j_0, j_1)\right)$$

If the system is observable and $r\left(R_0(j_0, j_1)\right) = n$, then certainly $r\left(\overline{R}_0(j_0, j_1)\right) = n$. For this reason, we normally desire to ascertain the *observability* of a system more often than the reconstructibility.

Observability Grammians

With the observability test in (8.90), we can construct a grammian test by defining the *observability grammian* as follows:

$$H_o(j_0, j_1) \stackrel{\Delta}{=} \sum_{i=j_0}^{j_1-1} \Psi^{\mathrm{T}}(i, j_0) C^{\mathrm{T}}(i) C(i) \Psi(i, j_0) \tag{8.95}$$

Now, as we did with reachability, a grammian test can be given in the form of a theorem:

> THEOREM: The system (8.74) is observable on the interval $[j_0, j_1]$ if and only if the observability grammian in (8.95) is nonsingular. (8.96)

Again, the detailed proof of this test is not provided since once again it is clear that $H_o(j_0, j_1) = R_o^{\mathrm{T}}(j_0, j_1) R_o(j_0, j_1)$ and the full-rank test, Equation (8.90), for $R_o(j_0, j_1)$ suffices to establish invertibility of (8.95).

8.6 Controllability and Observability Under Sampling

As a final note, we should remember that many discrete-time systems are derived as discretizations of continuous-time systems. When considering stability, we discovered that the stability of the discretization of a system is not necessarily preserved. It is an important fact that the same phenomenon can occur with the controllability (or reachability) and observability (or reconstructibility) properties. When a controllable and observable continuous-time system is sampled and discretized, such as shown in Section 6.5.1, it may not remain controllable or observable [6]. Usually this degradation occurs at specific sample periods *T*. Nevertheless, such occurrences should not be allowed to go unnoticed.

8.7 Summary

It is unfortunate that the material presented in this chapter must necessarily come before Chapter 10 on controller and observer design. The concepts of controllability and observability provide answers to the questions “Can this system be controlled?” and “Can this system be observed?” Until we know exactly what a “controller” is and what an “observer” is, it may be difficult to appreciate the implications of controllability and observability. However, we will soon find that the processes of designing a controller and observer will be completely precluded if a system is not controllable or observable, so we must investigate these properties first.

As with any good modeling paradigm, such as the state space system, we hope to find that a single technique will suffice for the most commonly encountered problems and that more sophisticated methods will be needed only for exceptional situations. This is true with controllability and observability. We have already seen in previous chapters that time-invariant systems, whether continuous-time or discrete-time, are much easier to analyze than time-varying systems. We therefore apply the time-invariant state space model whenever it sufficiently captures the physical phenomena we seek to study. If this is the case, the two tests we have called the fundamental controllability test and the fundamental observability test [Equations (8.6) and (8.7)] will suffice to ascertain controllability and observability. It is these tests and the matrices P and Q that will immediately spring to the control engineer's mind upon hearing the words "controllability" and "observability." Other tests are rarely used for time-invariant systems, and the average controls engineer will steer away from time-varying systems as well as nonlinear systems.

Nevertheless, systems of various characteristics will inevitably arise so we have presented here a relatively comprehensive study of controllability and observability determination. Among our findings:

- As mentioned above, most controllability and observability tests are performed with the matrices P and Q. If these matrices have rank n, the systems are controllable and observable, respectively.

- Various other tests, such as the Hautus eigenvector and rank tests, are useful mostly for their value in proofs or in generating still further tests, such as the Jordan form test. The Jordan form test is convenient because it can usually be applied by inspection, provided that the system already appears in Jordan form.

- The controllable and observable canonical forms were derived in preparation for their use in controller and observer design, respectively. Their special structure will facilitate the computation of the gain matrices that are integral components of controllers and observers. Thus, they are not *tests* per se because we cannot even find them unless the system is already known to be controllable or observable.

- The concept of modal controllability was used as a vehicle to introduce the geometric interpretation of controllability and observability. These interpretations are the topic of a large body of literature (for example, [10]), and are particularly helpful when generalizations to nonlinear systems are developed. Furthermore, by accepting the fact that *parts* of systems may be controllable and/or observable while other parts are not, we were able to derive the Kalman decomposition, which groups subsystems with comparable properties. Such decompositions are used mostly to aid in the understanding of system behaviors, as depicted in Figure 8.5. We will refer back to the Kalman decomposition in the next chapter as well.

- We have shown that although time-varying systems are more difficult to analyze than time-invariant systems, controllability and observability tests are nevertheless available. Many texts will derive these tests at the outset and later treat time-invariant systems as a special case, for which the time-varying results simplify. We have not taken this approach, preferring instead to build on the two linear equation solving examples from Chapter 3.

- In the special case of time-varying discrete-time systems, we distinguished the property of reachability from controllability, and the property of reconstructibility from observability. Of these two new properties, we concentrated on reachability over controllability because reachability implies controllability. We concentrate on observability over reconstructibility for the same reason.

Although we can lament the sometimes tedious study of controllability and observability before the study of controllers and observers, we will nevertheless not directly proceed at this point with their introduction. That will occur in Chapter 10. The next chapter presents yet another property of state space systems, realizability. This is also an important property that will allow us to better understand the relationship between controllability, observability, and stability. It will also lead to a practical understanding of issues relevant to *implementing* state space systems. Finally, in Chapter 10 we begin to *design* in earnest.

8.8 Problems

8.1 Determine whether each of the systems below is controllable and/or observable

a)
$$x(k+1) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 5 & 1 \end{bmatrix} x(k)$$

b)
$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x$$

c)
$$A = \begin{bmatrix} 2 & -5 \\ -4 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad c = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

d)
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

8.2 For the system given by the state equations

$$\dot{x} = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 3 & 6 \\ -5 & -1 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} x$$

a) Determine whether the system is controllable and/or observable by using the fundamental controllability and observability tests.

b) Transform the system into its Jordan canonical form, then determine which individual modes are controllable and/or observable.

c) Find the transfer function of this system and observe which modes appear as system poles.

8.3 Consider the single-input system $\dot{x} = Ax + bu$. Suppose that the system is controllable. Suppose we select a new basis as the n linearly independent columns of the controllability matrix $P = \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}$. Find the structure of the system in this new basis.

8.4 Consider the p-input state equations $\dot{x} = Ax + Bu$. Suppose there exists a transformation matrix T such that

$$TB = \begin{bmatrix} B_1 \\ \hdashline 0 \end{bmatrix} \qquad \text{and} \qquad TAT^{-1} = \left[\begin{array}{c:c} A_{11} & A_{12} \\ \hdashline A_{21} & A_{22} \end{array}\right]$$

where matrix B_1 has n_1 rows and $rank(B_1) = n_1$, and matrix A_{11} is $n_1 \times n_1$. The other submatrices are correspondingly dimensioned. Suppose also that $r(B) = r(B_1)$. Show that the pair (A, B) is controllable if and only if the pair (A_{22}, A_{21}) is controllable.

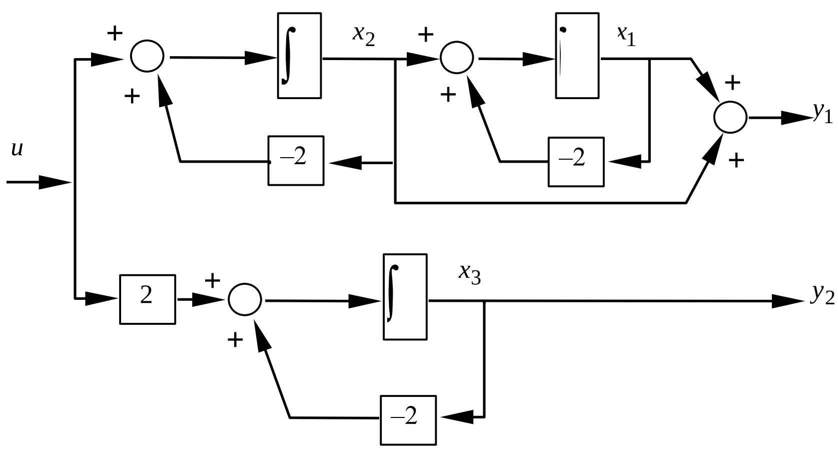
8.5 Determine whether the system described by the following matrices is controllable and/or observable.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & -1 & 1 & 0 \end{bmatrix}$$

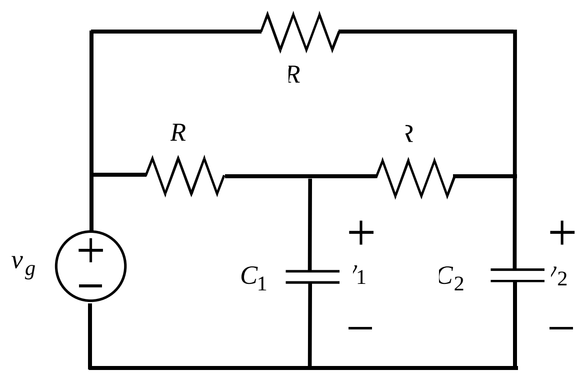
8.6 Determine whether the system in Problem 8.5 is stabilizable and/or detectable.

8.7 For the system shown in Figure P8.7, use the indicated state variables to write complete state equations and determine which modes are controllable and which are observable.



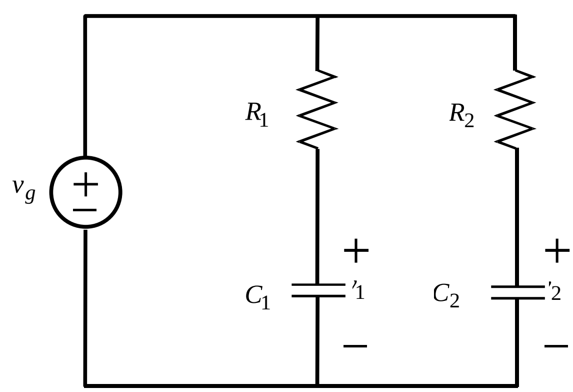
P8.7

8.8 For the electrical circuit shown in Figure P8.8, find conditions on C_1 and C_2 that will make the system uncontrollable. Consider v_g to be the input, and v_1 and v_2 to be the state variables.



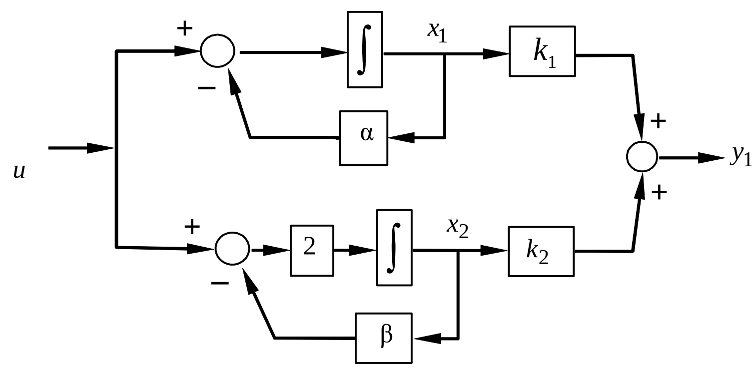
P8.8

8.9 For the circuit shown in Figure 8.9, find conditions on the system components $R_1, R_2, C_1,$ and C_2 that result in an uncontrollable system. Consider v_g to be the input, and v_1 and v_2 to be the state variables.



P8.9

8.10 For the system given in the block diagram in Figure 8.10, find necessary and sufficient conditions for the values of α, β, k_1, and k_2 such that the system will be both controllable and observable.



P8.10

8.11 Consider a linear, time-invariant system $\dot{x} = Ax + Bu$, $y = Cx$, that is also asymptotically stable. Define a matrix as

$$G_o = \int_0^\infty e^{A^\mathrm{T} t} C^\mathrm{T} C e^{At}\, dt$$

Show that the matrix G_o is positive definite if and only if the system is observable.

8.12 Consider a linear, time-invariant system $\dot{x} = Ax + Bu$, $y = Cx$. Suppose it is known that this system is controllable. Show that the system is asymptotically stable if and only if the equation

$$AN + NA^\mathrm{T} = -BB^\mathrm{T}$$

has a positive-definite solution *N*.

8.13 Consider a linear, time-invariant system $x(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k)$. Suppose it is known that this system is controllable. Show that the system is asymptotically stable if and only if the equation

$$ANA^{\mathrm{T}} - N = -BB^{\mathrm{T}}$$

has a positive-definite solution N.

8.14 Devise a "reconstructibility grammian" test corresponding to the observability grammian test [Equation (8.69)].

8.15 Determine if the following systems are controllable.

a) $$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix} u$$

b) $$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ e^{-2t} \end{bmatrix} u$$

c) $$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

8.16 A p input, q output system $\dot{x} = A(t)x + B(t)u$, $y = C(t)x$ is said to be *output controllable* at an initial time t_0 if there exists an input $u(t)$ defined over $[t_0, t_1]$ such that with the arbitrary initial state $x(t_0) = x_0$, the output can be transferred to value $y(t_1) = 0$, with t_1 finite.

a) Show that a system with impulse-response matrix $H(t,\tau) = C(t)\Phi(t,\tau)B(\tau)$ (see *Testing for BIBO Stability* in Section 7.3.1) is output controllable if the rows of $H(t_1,\tau)$ are linearly independent (over the field of complex numbers).

b) Show that if the system is time-invariant, then it is output controllable if and only if

$$r\left(\begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \end{bmatrix}\right) = q$$

c) Is there a condition on matrix C such that state controllability can then imply output controllability?

8.9 References and Further Reading

The topics of controllability and observability can be relatively simple, as for time-invariant linear systems, or they can be very difficult to understand and apply, such as with time-varying and nonlinear systems. The treatments given here originate with the work of Kalman [6] (including, of course, the Kalman decomposition), and are completely discussed in [2], [3], [4], and [7]. The PBH tests can be found in a number of sources, but are applied to circuit theory by Belevitch in [1].

For time-varying systems, the tests for controllability and observability are due to [8], and are also specifically discussed in [4], [9], the latter concentrating on discrete-time systems.

The geometric interpretation is, for controllability and observability, one of the best conceptual tools for understanding the basic concepts, and is best explained in by Wonham in [10].

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